Homework set 5 - due 10/30/20

Math 5047 – Renato Feres

1. Read Chapter 3 of the textbook, from page 95 to 125, with special attention to the sections on Lie groups and Lie algebras (pages 119-125). Most of this assignment is about basic Lie theory. Here’s an overview of the main concepts and facts from those sections:

(a) Lie groups, left- and right-translations. A Lie group $G$ is simultaneously a smooth manifold and a group, in which group multiplication $m(g_1, g_2) = g_1g_2$ and inverse $i(g) = g^{-1}$ are smooth maps. It follows that, for each $g \in G$, the maps $l_g : G \to G$ and $r_g : G \to G$ defined by $l_g(h) = gh$ (left-translation) and $r_g(h) = hg$ (right-translation) are diffeomorphisms.

(b) The Lie algebra $\mathfrak{g}$ of a Lie group $G$. In the vector space of all smooth vector fields $\mathfrak{X}(G)$ we consider the subspace of left-invariant vector fields. The vector field $X$ is said to be left-invariant if for all $g, h \in G$

\[(dl_g)_h X_h = X_{gh}.\]

I’m using here the notation $(dl_g)_h : T_h G \to T_{gh} G$ for the derivative map of $l_g$ at $h$, also denoted $(l_g)_* h$ or $(l_g)_*$. In the textbook. Because $G$ acts simply transitively on itself by left (or right) translations, we have a bijective correspondence between left-invariant vector fields and $T_e G$, where $e$ is the identity element. For general vector fields $X, Y$ we know that $(l_g)^* [X, Y] = [(l_g)_* X, (l_g)_* Y]$. (Here, $(l_g)_* X$ is the push-forward of $X$ under $l_g$ defined by $(l_g)_* X)_h = (dl_g)_g^{-1}h X_{g^{-1}h}$. So the Lie bracket of left-invariant vector fields is also left-invariant.

The vector space of left-invariant vector fields thus inherits from $(\mathfrak{X}(G), [\cdot, \cdot])$ the structure of Lie algebra. It is called the Lie algebra of $G$, normally denoted $\mathfrak{g}$. It has dimension equal to the manifold dimension of $G$.

Note that $\mathfrak{g}$ is linearly isomorphic to $T_e G$. The latter acquires a Lie bracket operation by setting

\[[u, v] := [U, V]_e,\]

where $U, V$ are the left-invariant vector fields satisfying $U_e = u$ and $V_e = v$, making $\mathfrak{g}$ and $T_e G$ also isomorphic as Lie algebras.

(c) One-parameter subgroups and the exponential map. Let $X$ be a left-invariant vector field on $G$. It turns out that integral curves of $X$ are defined for all time. (This is not hard to prove.) Thus they give rise to a (global) flow $\varphi_t : G \to G$ (i.e., a one-parameter group of diffeomorphisms). This means that $\varphi_{t+s} = \varphi_t \circ \varphi_s$ and $\varphi_0 = \text{Id}_G$ (the identity map on $G$). The Lie group exponential map is defined by

\[\exp(u) := \varphi_1(u)\]

where $\varphi_t$ is the flow of the left-invariant vector field $X$ such that $X_e = u \in \mathfrak{g}$. The exponential map is thus a smooth map, $\exp : \mathfrak{g} \to G$. For a general Lie group, $\exp$ need not be injective or surjective, but it is not
difficult to show that its derivative map at $e$ is the identity on $\mathfrak{g}$, so by the inverse function theorem $\exp$ is a diffeomorphism from a neighborhood of $0$ in $\mathfrak{g}$ to a neighborhood of $e$ in $G$.

If $f : H \to G$ is a smooth homomorphism of Lie groups, its derivative map $d f_e : \mathfrak{h} \to \mathfrak{g}$ at the identity element of $H$ can be shown to be a Lie algebra homomorphisms, due to the fact that $f_*$ respects the Lie bracket of vector fields and $f(e) = e$. An important property of the exponential map is stated in Theorem 15.12 in the textbook: $\exp \circ (d f_e) = f \circ \exp$.

(d) The adjoint representation. The map $h \mapsto c_g(h) = g h g^{-1} = (f_g \circ \iota_{g^{-1}})(h)$ is a homomorphism from $G$ to itself whose derivative map at $e$ is denoted $\text{Ad}(g) : \mathfrak{g} \to \mathfrak{g}$ and called the adjoint representation of $G$ on $\mathfrak{g}$. Note that $\text{Ad}(g_1 g_2) = \text{Ad}(g_1) \circ \text{Ad}(g_2)$. It can be shown (Proposition 15.15) that

$$\text{ad}_u v := \frac{d}{dt} \text{Ad}(\exp(tu))u \bigg|_{t=0} = [u, v].$$

2. Left-invariant Riemannian metrics and their Levi-Civita connection. (40 points.) Let $\langle \cdot, \cdot \rangle$ be a left-invariant Riemannian metric on a Lie group $G$. This means that

$$\langle (d_f g)_h u, (d_f g)_h v \rangle_{gh} = \langle u, v \rangle_h$$

for all $g, h \in G$ and $u, v \in T_h G$. Notice that a left-invariant metric is uniquely determined by the positive inner product $\langle \cdot, \cdot \rangle_e$ on $\mathfrak{g} = T_e G$. For the given metric, define the bilinear map $B : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ by

$$\langle B(u, v), w \rangle := \langle [v, w], u \rangle.$$

Note that $u \mapsto B(u, v)$ is the adjoint relative to $\langle \cdot, \cdot \rangle$ of the linear transformation $w \mapsto [v, w] = \text{ad}_v w$ on $\mathfrak{g}$. Thus we may write $B(u, v) = \text{ad}_v^* u$. See Section 15.7, page 123 of the textbook for further information on $\text{ad}$. We assume here that $G$ is connected. It is not difficult to show that if the left-invariant metric on $G$ is also right-invariant (in which case we say that the metric is bi-invariant), then

$$\langle [x, y], z \rangle + \langle y, [x, z] \rangle = 0$$

for all $x, y, z \in \mathfrak{g}$. This is the content of Proposition 15.16. The converse is also true when $G$ is connected. You may take this fact for granted. Another fact that you may take for granted (which is not difficult to show) is that the Levi-Civita connection is left-invariant (respectively, bi-invariant) if the Riemannian metric is left-invariant (respectively, bi-invariant).

(a) If $\nabla$ denotes the Levi-Civita connection associated to the given left-invariant Riemannian metric and $X, Y$ are left-invariant vector fields such that $X_e = x, Y_e = y$, show that

$$(\nabla_X Y)_e = \frac{1}{2} \left\{ [x, y] - B(x, y) - B(y, x) \right\}.$$   

A left-invariant vector field $X$ is a geodesic vector field if $\nabla_X X = 0$. Conclude that $X$ is a geodesic vector field if and only if $B(x, x) = 0$ where $x = X_e$.

(b) Show that the following are equivalent:

i. The Riemannian metric is bi-invariant.
ii. $B$ is skew-symmetric, so that $(\nabla_X Y)_e = \frac{1}{2} [u, v]$.
iii. Left-invariant vector fields are geodesic vector fields.
iv. Flow lines of left-invariant vector fields are geodesics. (Thus geodesics are left-translates of 1-parameter subgroups. In particular, the Lie theoretic exponential map exp and the Riemannian exponential map Exp coincide.)

(c) We adopt here the following notation (which is especially natural for matrix Lie groups such as $SO(n)$ as we’ll see later in this assignment): if $v \in T_g G$, then

$$g^{-1} v := (d\phi^-_g)_v \quad v \in T_e G = \mathfrak{g};$$

and $\dot{g}(t) = g'(t)$ is the tangent, or velocity, vector field along the curve $g(t)$.

Now let $g(t)$ be any smooth curve in $G$ and $X$ a vector field along $g(t)$, not necessarily left-invariant. Define $z(t) = g(t)^{-1} \dot{g}(t)$ and $w(t) = g(t)^{-1} X(t)$. Show that

$$g(t)^{-1} \left( \frac{\nabla X}{dt} \right)_{g(t)} = \dot{w} + \frac{1}{2} (\langle z, w \rangle - B(z, w) - B(w, z)).$$

As a suggestion, introduce a (global) frame $E_1, \ldots, E_n$ of left-invariant vector fields and use the idea in the proof of existence of the covariant derivative (Theorem 13.1, page 96). Notice that I’m using $\nabla/dt$ instead of $D/dt$ as in the textbook.

(d) Conclude that $g(t)$ is a geodesic if and only if $\dot{z} = B(z, z)$. Also conclude that $X(t)$ is a parallel vector field along a smooth curve $g(t)$ in $G$ if and only if $X_t = g(t) w(t)$ where $w(t)$ is the unique solution curve in $\mathfrak{g}$ of the initial value problem $\dot{w} = \frac{1}{2} [w, z] + B(z, w) + B(w, z)$ and $w(0) = g(0)^{-1} X_0$.

Note that $\dot{z} = B(z, z)$ and $\dot{w} = \frac{1}{2} [w, z] + B(z, w) + B(w, z)$ are differential equations on the vector space $\mathfrak{g}$. Once a solution $z(t)$ is found, the actual geodesic $g(t)$ becomes the solution of the (linear) equation $\dot{g}(t) = g(t) z(t)$. If the metric is bi-invariant, $g(t)$ is a geodesic if and only if $z(t) = z$ is constant and since $\dot{g}(t) = g(t) z$, $\dot{g}(t)$ is an integral curve of the left-invariant vector field $Z$ such that $Z_e = z$. Therefore $g(t) = g_0 \exp(tz)$. This means that geodesics are the flow lines of the left-invariant vector field associated to $z \in \mathfrak{g}$. In particular, geodesics through the identity element of $G$ ($g_0 = e$) are the 1-parameter subgroups of $G$ when $G$ has a bi-invariant Riemannian metric. This, again, shows that the Lie theoretic exponential map agrees with the Riemannian exponential map for bi-invariant metrics. For more details see Section 15.6, page 122, in the textbook. For bi-invariant metrics, the equation in $\mathfrak{g}$ for parallel transport reduces to $\dot{w} = \frac{1}{2} [w, z]$.

(e) Show that the Riemannian tensor of a bi-invariant metric can be written at the identity element of $G$ as

$$R(x, y)z = -\frac{1}{4} [\langle x, y \rangle, z]$$

for $x, y, z \in \mathfrak{g}$. The curvature tensor for a general left-invariant metric can also be easily written in terms of the bilinear map $B$, although the expression is not as simple. For this part of the exercise, don’t forget the Jacobi identity: $\langle x, [y, z] \rangle = \langle [x, y], z \rangle + \langle y, [x, z] \rangle$. This identity means that the map $ad_x$ is a derivation of the Lie algebra: $ad_x [y, z] = [ad_x y, z] + [y, ad_x z]$.

3. The special orthogonal group $SO(n)$. $\quad$ (30 points.) Recall that the Lie group $SO(n)$ consists of the $n \times n$ real matrices $A$ such that $A^T A = I$, where $I$ is the identity matrix, and $\det A = 1$.

(a) (8 points.) If $z$ is an $n \times n$ real matrix, show that $g(t) = e^{t z}$ (the matrix exponential) is a curve in $SO(n)$ if and only if $z^T = -z$, where $^T$ indicates matrix transpose.
(b) (8 points.) Define for each skew-symmetric matrix $z$ a vector field $Z$ such that, at $A \in SO(n),$ 

$$Z_A = \frac{d}{dt} A e^{tZ} \bigg|_{t=0} = A z \in T_A SO(n).$$

Show that $Z$ is a left-invariant vector field. Conversely, if $Z$ is a left-invariant vector field on $SO(n),$ show that $Z_A = A z$ for a skew-symmetric $n \times n$ real matrix $z.$ Thus we identify the Lie algebra $\mathfrak{so}(n)$ of $SO(n),$ as a vector space, with the space of skew-symmetric $n$-by-$n$ real matrices. As we'll see shortly, this is also a Lie algebra isomorphism, where the latter vector space is given the matrix commutator bracket $[x, y] = xy - yx.$

(c) (7 points.) Show that the flow $\Phi^Z_t$ associated to the left-invariant vector field $Z$ is given by 

$$\Phi^Z_t (A) = A e^{tZ}$$

where $z$ is the skew-symmetric matrix generating $Z.$ (Here we only need to recall the basic fact from linear differential equations that the solution $c(t)$ to the linear differential equation $\dot{c}(t) = c(t) z$ is $c(t) = c(0) e^{tZ}.$)

(d) (7 points.) Show that if $X, Y$ are left-invariant vector fields on $SO(n)$ associated to the skew-symmetric matrices $x, y,$ then the Lie bracket $[X, Y]$ is the left-invariant vector field associated to the skew-symmetric matrix $xy - yx.$ (This is the matrix commutator, which we also denote $[x, y].$) As a suggestion, you may begin with the identity $[X, Y] = \mathcal{L}_X Y,$ where the Lie derivative of vector fields was defined in the previous homework assignment as 

$$\mathcal{L}_X Y = \frac{d}{dt} (\Phi^{-t}_X)_* Y \bigg|_{t=0}.$$ 

Now, if you unwind the definition of the push-forward, you should get 

$$\left[ (\Phi^{-t}_X)_* Y \right]_X = \frac{d}{ds} e^{tx} e^{sy} e^{-tx} \bigg|_{s=0} = e^{tx} y e^{-tx}.$$ 

4. **Left-invariant and bi-invariant metrics on $SO(n).** (30 points.) Here we apply the results of Exercise 2 to the Lie group $SO(n).$ Let $\langle \cdot, \cdot \rangle_0$ denote the positive inner product on the Lie algebra $\mathfrak{so}(n)$ of the special orthogonal group defined by 

$$\langle x, y \rangle_0 := \frac{1}{2} \text{Tr} \{ xy^\top \}.$$ 

A general positive inner product on $\mathfrak{so}(n)$ would then take the form 

$$\langle x, y \rangle := \langle M(x), y \rangle_0$$

where $M : \mathfrak{so}(n) \to \mathfrak{so}(n)$ be a positive symmetric linear map relative to $\langle \cdot, \cdot \rangle_0.$ A case of particular interest is $M(x) = Mx + xM$ where $M$ is a positive symmetric $n \times n$ matrix. We assume this form of $M$ here. In this case, a simple algebraic manipulation gives 

$$\langle x, y \rangle = \text{Tr} \{ xMy^\top \}.$$ 

(a) Show that the left-invariant Riemannian metric on $SO(n)$ obtained from $\langle \cdot, \cdot \rangle_0$ is also right-invariant. (Note: it can be shown that $SO(n)$ is connected. As pointed out earlier in this assignment, being both left- and right-invariant is then equivalent to the inner product on $\mathfrak{so}(n)$ being invariant under ad. This amounts to the condition: $\langle [x, y], z \rangle_0 + \langle y, [x, z] \rangle_0 = 0$ for all $x, y, z \in \mathfrak{so}(n).$ See the argument in the proof of Proposition 15.16, page 125 of the textbook.)

(b) Can you express $B(x, y)$ in terms of $x,$ $y,$ and $M?$ (This part is optional. My original answer does not seem to be correct.)
(c) Given any two vectors $a, b \in \mathbb{R}^n$, define $a \times b \in \mathfrak{so}(n)$ as the linear transformation of $\mathbb{R}^n$ given by

$$(a \times b)c = (a \cdot c)b - (b \cdot c)a.$$  

(The notation is meant to suggest the cross-product from vector calculus.) Show that for $a, b, c, d \in \mathbb{R}^n$,

$$\langle a \times b, c \times d \rangle_0 = (a \cdot c)(b \cdot d) - (a \cdot d)(b \cdot c).$$

(d) Let $V$ be a 2-dimensional subspace of $\mathfrak{so}(n)$. The sectional curvature of $G$ along $V$ is defined by

$$K_e(V) = \langle R(x, y)y, x \rangle_0$$

where $x, y$ is any orthonormal basis of $V$. (By the symmetries of the curvature tensor studied in a previous homework set, it is not difficult to show that $K_e(V)$ does not depend on the choice of orthonormal basis.) Show that

$$K_e(V) = \frac{1}{4} \| [x, y] \|_0^2 = -\frac{1}{8} \text{Tr}([x, y]^2).$$

Here $\| \cdot \|_0$ is the norm associated to $\langle \cdot, \cdot \rangle_0$.

(e) Let $a, b, c, d$ be orthonormal vectors in $\mathbb{R}^n$. Find the sectional curvature along the plane $V \subseteq \mathfrak{so}(n)$ for the bi-invariant metric $\langle \cdot, \cdot \rangle_0$, where

i. $V$ is linearly spanned by $x = a \times b$ and $y = c \times d$.

ii. $V$ is linearly spanned by $x = a \times b$ and $y = a \times d$.  

5