Homework set 6 – Solutions

Math 5047 – Renato Feres

1. Read Section 16, Chapter 2, pages 128 to 135 of the textbook.

2. Isometries of the unit sphere. (30 points.) Let $S^{n-1} = S^{n-1}(1)$ be the sphere of radius 1 centered at the origin of $\mathbb{R}^n$. We give $S^{n-1}$ the Riemannian metric that makes the inclusion map an isometric embedding. This means that the metric on the sphere is the restriction to its tangent spaces of the standard dot product: $\langle u, v \rangle_x = u \cdot v$. Note the natural identification $T_x S^{n-1} = \{ u \in \mathbb{R}^n : x \cdot u = 0 \}$.

It is not difficult to show that the restriction to the sphere of an orthogonal transformation $A : \mathbb{R}^n \to \mathbb{R}^n$ is a Riemannian isometry. Conversely, show that every isometry $f : S^{n-1} \to S^{n-1}$ is the restriction to $S^{n-1}$ of an orthogonal transformation. Thus the isometry group of the sphere is the orthogonal group $O(n) = \{ A : \text{an } n \times n \text{ real matrix such that } A^T A = I \}$.

Suggestion: By composing $f$ with an orthogonal map, we can assume that there is a point $x \in S^{n-1}$ such that $f(x) = x$ and $df_x$ is the identity map on the tangent space at $x$. Now use Theorem 15.2, page 116, to argue that $f$ must be the identity map on a neighborhood of $x$ and, consequently, (why?) the identity map.

Solution. The orthogonal group acts transitively on the sphere. So, given an isometry $f$ and a point $x$, there is an orthogonal transformation $A : \mathbb{R}^n \to \mathbb{R}^n$ such that $A^{-1}f(x) = x$ and $df_x$ is the identity map on the tangent space at $x$. Now use Theorem 15.2, page 116, to argue that $f$ must be the identity map on a neighborhood of $x$ and, consequently, (why?) the identity map.

Solution. The orthogonal group acts transitively on the sphere. So, given an isometry $f$ and a point $x$, there is an orthogonal transformation $A$ such that $A^{-1}f(x) = x$. Thus $A^{-1}f$ fixes $x$ and acts as the identity on $T_x S^{n-1}$. Let $B \in O(n)$ be a matrix that fixes $x$ and acts on the orthogonal complement of $x$ as $A^{-1}d f_{x_0}$. Thus $g := B^{-1} \circ A^{-1} \circ f$ is an isometry of the sphere that fixes $x$ and acts as the identity on $T_x S^{n-1}$.

Now observe that

$$g(\text{Exp}_x(v)) = \text{Exp}_x(dg_x v) = \text{Exp}_x v.$$

This means that $g$ is the identity on a neighborhood of $x$. By this argument, the set of fixed points of $g$ is open. But it is clearly also closed. Since the sphere is connected, this set (which, as seen, is not empty) must be the whole sphere. Thus $f = AB \in O(n)$.

3. Problem 16.4, page 135 of textbook, modified. Volume form of a sphere in Cartesian coordinates. (40 points.) Let $S^{n-1}(\rho)$ be the sphere of radius $\rho$ centered at the origin of $\mathbb{R}^n$, where $\rho(x) = \sqrt{x_1^2 + \cdots + x_n^2}$ and $x_1, \ldots, x_n$ are the Cartesian coordinates on $\mathbb{R}^n$. Orient $S^{n-1}(\rho)$ as the boundary of the solid ball of radius $\rho$. 

$\diamondsuit$
(a) Prove that the volume form on $S^{n-1}(\rho)$ is (the pull-back to the sphere under the inclusion map of)

$$\text{vol}_{S^{n-1}(\rho)} = \frac{1}{\rho} \sum_{i=1}^{n} (-1)^{i-1} x_i^j \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n.$$ 

Here the hat $\widehat{dx_i}$ indicates that the term $dx_i$ is absent. (Hint: Use Theorem 16.11.)

(b) Let $\text{vol}_{\mathbb{R}^n} = dx_1 \wedge \cdots \wedge dx_n$ be the standard volume form on $\mathbb{R}^n$. Show that $\text{vol}_{\mathbb{R}^n} = d\rho \wedge \text{vol}_{S^{n-1}(\rho)}$.

(c) Let $Z$ be a vector field on an orientable Riemannian manifold $M$ with volume form $\omega$. The divergence of $Z$, denoted $\text{div}Z$ or $\text{div}_M Z$, is the unique function on $M$ such that

$$\mathcal{L}_Z \omega = (\text{div}Z)\omega.$$ 

Thus the divergence measures the rate at which the flow of $Z$ changes the volume in $M$ at the linear level. Now suppose $Z$ is a vector field in $\mathbb{R}^n$ such that $Z \cdot x = 0$; that is, $Z$ is, at each point, tangent to the sphere containing that point. Show that $\text{div}_{S^{n-1}} Z = \text{div}_{\mathbb{R}^n} Z$. In words: the restriction to the sphere of the divergence of $Z$ regarded as a vector field on $\mathbb{R}^n$ is the divergence of $Z$ regarded as a vector field on the sphere. (Suggestion: Apply the Lie derivative along $Z$ to $d\rho \wedge \text{vol}_{S^{n-1}}$.) Also check that the divergence of a vector field $Z = \sum_i f_i \partial/\partial x_i$ in $\mathbb{R}^n$ is $\text{div}Z = \sum_i \partial f_i/\partial x_i$.

(d) A skew-symmetric $n \times n$ real matrix $z$ (an element of the Lie algebra $\mathfrak{so}(n)$) induces a vector field $Z$ on $\mathbb{R}^n$ such that at each $x \in \mathbb{R}^n$,

$$Z_x = \left. \frac{d}{dt} e^{tz} x \right|_{t=0} = z x = \sum_{i,j} z_{ij} x_j \frac{\partial}{\partial x_i}.$$ 

Show that the restriction of $Z$ to the unit sphere has 0 divergence.

(e) For this final item, recall that if $M$ is an $n$-dimensional manifold and $\omega$ is a smooth $n$-form with compact support (in particular, if $\omega$ is any $n$-form on a compact $M$) and $F : M \to M$ is a diffeomorphism then, for every open set $U \subseteq M$,

$$\int_{F(U)} \omega = \int_U F^\ast \omega.$$ 

Show that if $Z$ is the vector field on the sphere defined in the previous item and $\Phi_t : S^{n-1} \to S^{n-1}$ is the flow of $Z$ then, for every open set $U \subseteq S^{n-1}$,

$$\text{Vol}(\Phi_t(U)) := \int_{\Phi_t(U)} \text{vol}_{S^{n-1}}$$ 

is constant in $t$.

Solution.

(a) The volume form on $\mathbb{R}^n$ and on the interior of the ball with boundary $S^{n-1}(\rho)$ is $\text{vol}_{\mathbb{R}^n} = dx_1 \wedge \cdots \wedge dx_n$. The unit normal vector field on $S^{n-1}(\rho)$ is given at each $x$ on the sphere by

$$v(x) = \frac{x}{\rho} = \frac{1}{\rho} \left\{ x_1 \frac{\partial}{\partial x_1} + \cdots + x_n \frac{\partial}{\partial x_n} \right\}.$$ 

By Theorem 16.11, page 133 of the textbook, the hypersurface area form on the sphere is given by $i_v \text{vol}_{\mathbb{R}^n}$. We know from homework 4 (problem 4) that the interior product satisfies $i_v (\theta \wedge \eta) = i_v \theta \wedge \eta + (-1)^k \theta \wedge i_v \eta$, when $\theta$ is a $k$-form. Also observe that $dx_i(v_x) = x_i/\rho$. Putting these remarks together results in the form of $\text{vol}_{S^{n-1}(\rho)}$ given above.
(b) This is a calculation. First observe that
\[ dp = \frac{1}{\rho} \sum_j x_j \, dx_j. \]
Therefore
\[ dp \wedge \text{vol}_{S^{n-1}(\rho)} = \left( \frac{1}{\rho} \sum_j x_j \, dx_j \right) \wedge \left( \frac{1}{\rho} \sum_{j=1}^n (-1)^j x_j \, dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n \right) \]
\[ = \frac{1}{\rho^2} \sum_{i,j} (-1)^{i-1} x_i x_j \, dx_i \wedge dx_j \wedge \cdots \wedge dx_n \]
\[ = \frac{1}{\rho^2} \sum_{i=1}^n (-1)^{i-1} x_i^2 \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n \]
\[ = \frac{1}{\rho^2} \left( \sum_{i=1}^n x_i^2 \right) dx_1 \wedge \cdots \wedge dx_n \]
\[ = dx_1 \wedge \cdots \wedge dx_n = \text{vol}_{S^n}. \]

(c) Applying the Lie derivative with respect to \( Z \) to both sides of the identity \( \text{vol}_{R^n} = dp \wedge \text{vol}_{S^{n-1}(\rho)} \) gives
\[ (\text{div}_{R^n})Z \text{vol}_{R^n} = (L_Z dp) \wedge \text{vol}_{S^{n-1}} + dp \wedge L_Z \text{vol}_{S^{n-1}} = d(Z \rho) \wedge \text{vol}_{S^{n-1}} + (\text{div}_{S^{n-1}} Z) dp \wedge \text{vol}_{S^{n-1}}. \]
But \( Z \rho = 0 \) because \( Z \) is tangent to the sphere at every point on the sphere. This shows that the two divergences agree.

If \( Z = \sum_i f_i \partial / \partial x_i \), then
\[ (\text{div} Z) dx_1 \wedge \cdots \wedge dx_n = L_Z dx_1 \wedge \cdots \wedge dx_n = \sum_i dx_1 \wedge \cdots \wedge L_Z dx_i \wedge \cdots \wedge dx_n. \]

Now \( L_Z dx_i = df_i = \sum_j (\partial f_i / \partial x_j) dx_j \). Inserting this into the above expression and noting that \( (\partial f_i / \partial x_j) dx_j \) contribute 0 to the sum if \( j \neq i \), we obtain
\[ (\text{div} Z) dx_1 \wedge \cdots \wedge dx_n = \sum_i \frac{\partial f_i}{\partial x_i} dx_1 \wedge \cdots \wedge dx_n. \]

Therefore \( \text{div} Z = \sum_i \partial f_i / \partial x_i \).

(d) Note that \((zx) \cdot x = x \cdot (z^T x) = -x \cdot (zx)\), so \( zx \) is perpendicular to \( x \) at each point \( x \). This means that \( x \rightarrow Z_x \) corresponds to a vector field on any sphere centered at the origin. By the previous item, it is sufficient to check that \( Z \) has zero divergence as a vector field on \( R^n \):
\[ (\text{div}_{R^n})Z = \sum_i \frac{\partial}{\partial x_i} \left( \sum_j z_{ij} x_j \right) = \sum_i \delta_{ij} z_{ij} = \text{Tr}(z) = 0. \]

Here I have used the fact that the trace of a skew-symmetric matrix is 0.

(e) It suffices to show that the derivative of \( \text{Vol}(\Phi_t(U)) \) is 0. But
\[ \frac{d}{dt} \text{Vol}(\Phi_t(U)) = \frac{d}{dt} \int_U \Phi_t^* \text{vol}_{S^{n-1}} = \int_U \frac{d}{dt} \Phi_t^* \text{vol}_{S^{n-1}}. \]
Observe that
\[
\frac{d}{dt} \Phi^*_t \text{vol}_{S^{n-1}} = \frac{d}{ds} \Phi^*_s \text{vol}_{S^{n-1}} \bigg|_{s=0} = \Phi^*_0 L Z \text{vol}_{S^{n-1}} = 0.
\]
Therefore \( \frac{d}{dt} \text{Vol}(\Phi_t(U)) = 0. \)

4. Problems 16.6 and 16.7, page 136 of textbook. Volume form of an \( n \)-sphere in spherical coordinates and hypersurface area. (30 points.) For \( n \geq 3 \) we define the spherical coordinates on \( \mathbb{R}^n \) as follows. (See Figure 1.)

Let \( r_k \) be the distance of the point \((x_1, \ldots, x_k)\) from the origin in \( \mathbb{R}^k \):
\[
r_k = \sqrt{x_1^2 + \cdots + x_k^2}.
\]
The spherical coordinates on \( \mathbb{R}^2 \) are the usual polar coordinates \( r = r_2, \theta, 0 \leq \theta < 2\pi \).

![Figure 1: Spherical coordinates.](image)

For \( n \geq 3 \), if \( x = (x_1, \ldots, x_n) \), the angle \( \varphi_n \) is the angle the vector \( x \) makes relative to the \( x_n \)-axis; it is determined uniquely by the formula
\[
\cos \varphi_n = \frac{x_n}{r_n}, \quad 0 \leq \varphi_n < \pi.
\]
Project \( x \) to \( \mathbb{R}^{n-1} \) along the \( x_n \)-axis. By induction, the spherical coordinates \( r_{n-1}, \theta, \varphi_3, \ldots, \varphi_{n-1}, \varphi_n \) of the projection \((x_1, \ldots, x_{n-1})\) in \( \mathbb{R}^{n-1} \) are defined. Then the spherical coordinates of \((x_1, \ldots, x_n)\) in \( \mathbb{R}^n \) are defined to be \( r_n, \theta, \varphi_3, \ldots, \varphi_{n-1}, \varphi_n \). Thus for \( k = 3, \ldots, n \) we have \( \cos \varphi_k = x_k/r_k \). More explicitly, setting \( \rho = r_n \),
\[
\begin{align*}
x_n &= \rho \cos \varphi_n \\
x_{n-1} &= \rho \sin \varphi_n \cos \varphi_{n-1} \\
x_{n-2} &= \rho \sin \varphi_n \sin \varphi_{n-1} \cos \varphi_{n-2} \\
& \vdots \\
x_3 &= \rho \sin \varphi_n \cdots \sin \varphi_4 \cos \varphi_3 \\
x_2 &= \rho \sin \varphi_n \cdots \sin \varphi_4 \sin \varphi_3 \sin \theta \\
x_1 &= \rho \sin \varphi_n \cdots \sin \varphi_4 \sin \varphi_3 \cos \theta.
\end{align*}
\]
Note that \( r_k = \rho \sin \varphi_n \cdots \sin \varphi_{k+1} \) (or \( r_k = \sin \varphi_{k+1} r_{k+1} \)) for \( k < n \). Setting \( \rho = a \), the above defines a parametrization of \( S^{n-1}(a) \), which we may write as \( x = F(\theta, \varphi_3, \varphi_4, \ldots, \varphi_n) \). (A parametrization of a manifold is the inverse of
constitutes an orthonormal frame along the sphere, where \( e_1 = \nu \) is a unit normal vector field.

(a) Give the sphere \( S^{n-1}(a) \) the boundary orientation of the closed solid ball of radius \( a \). (This is the orientation defined by the form \( \text{vol}_{S^{n-1}(a)} \).) Show that the volume form on \( S^{n-1}(a) \) in spherical coordinates is, up to sign,

\[
\omega = a^{n-1} (\sin^{n-2} \varphi_1) (\sin^{n-3} \varphi_{n-1}) \cdots (\sin \varphi_3) \, d\theta \wedge d\varphi_3 \wedge \cdots \wedge d\varphi_n.
\]

Note: A smooth \( n \)-form \( \omega \) on an oriented \( n \)-dimensional Riemannian manifold \( M \) is the volume form of \( M \) if \( \omega_p(u_1, \ldots, u_n) = 1 \) for any positive orthonormal basis \( \{u_i\} \) at \( p \), for all \( p \in M \).

(b) By integration by parts, show that

\[
\int_0^\pi \sin^n \varphi \, d\varphi = \frac{n-1}{n} \int_0^\pi \sin^{n-2} \varphi \, d\varphi.
\]

(c) Give a numerical expression for \( \int_0^\pi \sin^2 \varphi \, d\varphi \) and \( \int_0^\pi \sin^{2k-1} \varphi \, d\varphi \).

(d) Show that

\[
\frac{\text{Vol}(S^{n+1}(1))}{\text{Vol}(S^{n-1}(1))} = \int_0^\pi \sin^n \varphi \, d\varphi \int_0^\pi \sin^{n-1} \varphi \, d\varphi.
\]

Although this wasn’t asked in the original version of the problem, it is not difficult to conclude that

\[
\frac{\text{Vol}(S^{n+1}(1))}{\text{Vol}(S^{n-1}(1))} = \frac{2\pi}{n}
\]

for \( n \) both even and odd.

**Solution.**

(a) To show that \( \omega \) is, up to sign, the volume form of \( S^{n-1}(a) \), suffices to check \( \omega(e_2, \ldots, e_n) = \pm 1 \). Note that

\[
\omega = r_n \cdots r_3 r_2 \, d\theta \wedge d\varphi_3 \wedge \cdots \wedge d\varphi_n.
\]

Therefore

\[
\omega(e_2, \ldots, e_n) = \frac{1}{r_n \cdots r_3 r_2} \omega \left( \frac{\partial}{\partial \rho}, \frac{\partial}{\partial \varphi_3}, \ldots, \frac{\partial}{\partial \varphi_n} \right) = d\theta \wedge d\varphi_3 \wedge \cdots \wedge d\varphi_n \left( \frac{\partial}{\partial \rho}, \frac{\partial}{\partial \varphi_3}, \ldots, \frac{\partial}{\partial \varphi_n} \right) = 1.
\]

(b) This is a standard application of integration by parts.

(c) If \( n \) is even,

\[
\int_0^\pi \sin^{2k} \varphi \, d\varphi = \frac{2k-1}{2k} \times \frac{2k-3}{2k-2} \times \cdots \times \frac{3}{4} \times \frac{1}{2} \pi.
\]

And if \( n \) is odd,

\[
\int_0^\pi \sin^{2k-1} \varphi \, d\varphi = \frac{2k-2}{2k-1} \times \frac{2k-4}{2k-3} \times \cdots \times \frac{2}{3} \times 2.
\]
(d) We now have
\[ \int_{S^{n-1}(a)} \omega = a^{n-1} 2\pi \int_{0}^{\pi} \sin \varphi_3 \, d\varphi_3 \int_{0}^{\pi} \sin^2 \varphi_4 \, d\varphi_4 \cdots \int_{0}^{\pi} \sin^{n-2} \varphi_n \, d\varphi_n. \]

The value of the ratio of volumes now follows from this. The explicit value of the quotient is
\[ \frac{\text{Vol}(S^{n+1}(1))}{\text{Vol}(S^{n-1}(1))} = \frac{2\pi}{n} \]

for \( n \) both even and odd.