1. **The Pfaffian.** Let $\mathfrak{so}(2m)$ denote the vector space of $2m \times 2m$ skew-symmetric real matrices (the Lie algebra of the special orthogonal group $SO(2m)$). We define a map

$$\text{Pf}: \mathfrak{so}(2m) \to \mathbb{R}$$

as follows. Let $e_1, \ldots, e_{2m}$ be a basis of $\mathbb{R}^{2m}$ and consider for each $A = (a_{ij}) \in \mathfrak{so}(2m)$ the alternating tensor

$$\alpha = \sum_{i < j} a_{ij} e_i \wedge e_j = \frac{1}{2} \sum_{i,j=1}^{2m} a_{ij} e_i \wedge e_j.$$ 

Then the Pfaffian $\text{Pf}(A)$ is the real number such that

$$\alpha^m = \alpha \wedge \cdots \wedge \alpha = m! \text{Pf}(A) e_1 \wedge e_2 \wedge \cdots \wedge e_{2m}.$$

(a) Using the definition, show that the Pfaffian of the $4 \times 4$ matrix

$$A = \begin{pmatrix}
0 & a_{12} & a_{13} & a_{14} \\
-a_{12} & 0 & a_{23} & a_{24} \\
-a_{13} & -a_{23} & 0 & a_{34} \\
-a_{14} & -a_{24} & -a_{34} & 0
\end{pmatrix}$$

is given by

$$\text{Pf}(A) = a_{12}a_{34} - a_{13}a_{24} + a_{23}a_{14}.$$ 

Observe that the value of the Pfaffian does not depend on the choice of basis of $\mathbb{R}^{2m}$.

(b) Let $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and consider the block-diagonal matrix

$$A = \text{diag}(a_1 J, \ldots, a_m J).$$ 

Find the associated alternating 2-vector $\alpha$ and show that

$$\text{Pf}(A) = a_1 \cdots a_m.$$ 

(c) If $A \in \mathfrak{so}(2m)$ and $B$ is any $2m \times 2m$ matrix, show that

$$\text{Pf}(B^T AB) = \text{Pf}(A) \det(B).$$

It follows that the Pfaffian is an $SO(2m)$-invariant polynomial on $\mathfrak{so}(2m)$. 
(d) Show that if \( A \in \text{so}(2m) \), then
\[
(Pf(A))^2 = \det(A).
\]
You may take for granted the following fact from matrix algebra (which I encourage you to try to prove for yourself). There exists an orthogonal matrix \( B \in O(2n) \) such that
\[
B^\top A B = \text{diag}(a_1, \ldots, a_m).
\]

(e) Convince yourself (no need to write it down) that
\[
Pf(A) = \frac{1}{2^m m!} \sum_{\sigma \in S_{2m}} \text{sign}(\sigma) a_{\sigma(1)} a_{\sigma(2)} \cdots a_{\sigma(2m-1)} a_{\sigma(2m)}.
\]

Solution.

(a) The associated form \( \alpha \) is
\[
\alpha = a_{12} e_1 \wedge e_2 + a_{13} e_1 \wedge e_3 + a_{14} e_1 \wedge e_4 + a_{23} e_2 \wedge e_3 + a_{24} e_2 \wedge e_4 + a_{34} e_3 \wedge e_4.
\]
The Pfaffian is then
\[
Pf(A) e_1 \wedge e_2 \wedge e_3 \wedge e_4 = \frac{1}{2} \alpha \wedge \alpha = (a_{12} a_{34} - a_{13} a_{24} + a_{23} a_{14}) e_1 \wedge e_2 \wedge e_3 \wedge e_4.
\]
So
\[
Pf(A) = a_{12} a_{34} - a_{13} a_{24} + a_{23} a_{14}.
\]

(b) The 2-form \( \alpha \) for this \( A \) is
\[
\alpha = a_1 e_1 \wedge e_2 + a_2 e_3 \wedge e_4 + \cdots + a_m e_{2m-1} \wedge e_{2m}.
\]
It follows that
\[
a_m = m! a_1 \cdots a_m e_1 \wedge \cdots \wedge e_{2m}.
\]
Therefore \( Pf(A) = a_1 \cdots a_m \).

(c) We may assume that \( B \) is invertible. Since invertible matrices are dense in the set of \( 2m \times 2m \) matrices, we will be able to conclude that \( Pf(B^\top A B) = 0 \) when \( B \) is not invertible (so that the identity still holds). Let
\[
u_k = \sum_l b_{k,l}.
\]
Then
\[
\sum_{i,j} (B^\top A B)_{ij} e_i \wedge e_j = \sum_{i,j,k,\ell} b_{k,i} a_{k,\ell} b_{\ell,j} e_i \wedge e_j = \sum_{k,\ell} a_{k,\ell} u_k \wedge u_\ell.
\]
Then, as \( u_1 \wedge \cdots \wedge u_{2m} = \det(B) e_1 \wedge \cdots \wedge e_{2m} \), we obtain
\[
Pf(B^\top A B) e_1 \wedge \cdots \wedge e_{2m} = Pf(A) u_1 \wedge \cdots \wedge u_{2m} = Pf(A) \det(B) e_1 \wedge \cdots \wedge e_{2m}.
\]
The desired identity follows.

(d) By the previous part of this exercise, if \( B \) is special orthogonal, then \( Pf(B^\top A B) = Pf(A) \). From the second part, we conclude \( Pf(A) = a_1 \cdots a_m \). But it is a simple verification to check that \( \det(B^\top A B) = a_1^2 \cdots a_m^2 \). So we conclude that \( (Pf(A))^2 = \det(A) \).

\[ \diamond \]
2. **Exterior derivative expressed in terms of a connection.** If $\omega$ is a differential $k$-form on a smooth manifold $M$ equipped with a torsion-free (i.e., symmetric) connection $\nabla$, show that

$$d\omega(X_0,\ldots,X_k) = \sum_{i=0}^{k} (-1)^i \{ \nabla_{X_i} \omega \} \{ X_0,\ldots,\hat{X}_i,\ldots,X_k \},$$

where $X_0, X_1,\ldots, X_k \in \mathfrak{X}(M)$ and the hat symbol $\hat{X}$ indicates that the corresponding vector field is dropped. Note: We have seen in an earlier homework assignment special cases of the following general formula:

$$(d\omega)(X_0,\ldots,X_k) = \sum_{i=0}^{k} (-1)^i X_i \{ \omega \{ X_0,\ldots,\hat{X}_i,\ldots,X_k \} \}$$

$$+ \sum_{i<j} \omega \{ [X_i,X_j],X_0,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots,X_k \},$$

which you may take for granted. You may also need to recall from a previous assignment the covariant derivative of a tensor field.

**Solution.** First observe that

$$D_i := (\nabla_{X_i} \omega) \{ X_0,\ldots,\hat{X}_i,\ldots,X_k \} = X_i \{ \omega \{ X_0,\ldots,\hat{X}_i,\ldots,X_k \} \}$$

$$- \sum_{j=0}^{i} \omega \{ X_0,\ldots,\nabla_{X_j} X_j,\ldots,\hat{X}_i,\ldots,X_k \},$$

$$- \sum_{j>i}^{k} \omega \{ X_0, X_j,\ldots,\hat{X}_i,\ldots,\nabla_{X_j} X_j,\ldots,X_k \}.$$

So

$$\sum_{i=0}^{k} (-1)^i D_i = \sum_{i=0}^{k} (-1)^i X_i \{ \omega \{ X_0,\ldots,\hat{X}_i,\ldots,X_k \} \}$$

$$- \sum_{j<i} (-1)^i \omega \{ X_0,\ldots,\nabla_{X_j} X_j,\ldots,\hat{X}_i,\ldots,X_k \}$$

$$- \sum_{j>i} (-1)^j \omega \{ X_0, X_j,\ldots,\hat{X}_i,\ldots,\nabla_{X_j} X_j,\ldots,X_k \}$$

$$= \sum_{i=0}^{k} (-1)^i X_i \{ \omega \{ X_0,\ldots,\hat{X}_i,\ldots,X_k \} \}$$

$$- \sum_{j<i} (-1)^i+j \omega \{ \nabla_{X_j} X_j, X_0,\ldots,\hat{X}_i,\ldots,X_k \}$$

$$+ \sum_{j>i} (-1)^j+j \omega \{ \nabla_{X_j} X_j, X_0,\ldots,\hat{X}_i,\ldots,X_k \}$$

$$= \sum_{i=0}^{k} (-1)^i X_i \{ \omega \{ X_0,\ldots,\hat{X}_i,\ldots,X_k \} \}$$

$$+ \sum_{i<j} (-1)^{j} \omega \{ \nabla_{X_j} X_j - \nabla_{X_j} X_i, X_0,\ldots,\hat{X}_i,\ldots,X_k \}$$

$$= \sum_{i=0}^{k} (-1)^i X_i \{ \omega \{ X_0,\ldots,\hat{X}_i,\ldots,X_k \} \}$$

$$+ \sum_{i<j} (-1)^{j} \omega \{ [X_i,X_j], X_0,\ldots,\hat{X}_i,\ldots,X_k \}$$

$$= d\omega(X_0,\ldots,X_k).$$

$\diamondsuit$
3. Differential forms with coefficients in a vector bundle. Let $M$ be a smooth manifold and $\pi : E \to M$ a smooth vector bundle. We consider the vector bundle of alternating forms with values in $E$: $\bigwedge^k(T^*M) \otimes E$ and its smooth sections

$$
\Omega^k_M(E) := \Gamma\left(\bigwedge^k(T^*M) \otimes E\right) = \Omega^k(M) \otimes \Gamma(E).
$$

Note that the second symbol $\otimes$ is the tensor product of modules. Elements of $\Omega^k_M(E)$ are, locally, linear combinations of terms of the form $\omega \otimes \xi$ where $\omega$ is a smooth $k$-form on $M$ and $\xi$ is a local section of $E$. An example of an element of $\Omega^2_M(E)$ is given by the curvature tensor $R(\cdot, \cdot)$ of a connection on a vector bundle $F$, where $E = F \otimes F^* \cong \text{End}(F)$.

Given a connection $\nabla$ on $E$, we define the covariant exterior derivative on $\Omega^*_M(E)$ as the map

$$
d^\nabla : \Omega^k_M(E) \to \Omega^{k+1}_M(E)
$$

that satisfies

(a) $d^\nabla = \nabla$ for $k = 0$; thus $(d^\nabla \xi)(X) = \nabla_X \xi$ for $X \in \mathfrak{X}(M)$.

(b) $d^\nabla (\omega \otimes \xi) = d \omega \otimes \xi + (-1)^k \omega \wedge d^\nabla \xi$.

Show the following:

(a) If $\psi \in \Omega^1_M(E)$ and $X_0, \ldots, X_k \in \mathfrak{X}(M)$, then

$$
(d^\nabla \psi)(X_0, \ldots, X_k) = \sum_t ((-1)^t \nabla_{X_t} \psi (X_0, \ldots, \hat{X}_t, \ldots, X_k))
$$

$$
+ \sum_{i<j} (-1)^{i+j} \psi ([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_k).
$$

(b) If $\psi \in \Omega^0_M(E)$ (that is, a smooth section of $E$), then

$$
d^\nabla \circ d^\nabla \xi = R(\cdot, \cdot) \xi
$$

where $R$ is the curvature tensor of $\nabla$.

Solution.

(a) It suffices to check the identity for separable forms of the type $\psi = \omega \otimes \xi$, where $\omega$ is an ordinary $k$-form and $\xi$ is a section of $E$. In this case, the expression to be proved takes the following form:

$$
I := \sum_t ((-1)^t \nabla_{X_t} \omega (X_0, \ldots, \hat{X}_t, \ldots, X_k) \xi) + \sum_{i<j} (-1)^{i+j} \omega ([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_k) \xi
$$

$$
= \left( \sum_t ((-1)^t X_t \omega (X_0, \ldots, \hat{X}_t, \ldots, X_k) + \sum_{i<j} (-1)^{i+j} \omega ([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_k) \right) \xi
$$

$$
+ \sum_t (-1)^t \omega (X_0, \ldots, \hat{X}_t, \ldots, X_k) \nabla_{X_t} \xi
$$

$$
= (d \omega \otimes \xi)(X_0, \ldots, X_k) + \sum_t (-1)^t \omega (X_0, \ldots, \hat{X}_t, \ldots, X_k) \nabla_{X_t} \xi
$$

Now observe the following. If $\theta$ is a 1-form,

$$
(\omega \wedge \theta)(X_0, \ldots, X_k) = \sum_t (-1)^{i+k} \omega (X_0, \ldots, \hat{X}_i, \ldots, X_k) \theta(X_t).
$$
This identity is easily obtained by using the expression for the wedge product in terms of shuffles, as in Proposition 19.15 of the textbook.

Thus we obtain

\[ I = ((d\omega \otimes \xi)(X_0, \ldots, X_k) + (-1)^k (\omega \wedge d^V \xi)(X_0, \ldots, X_k)). \]

From the defining two properties of \( d^V \) we obtain the following.

\[
\left(d^V (\omega \otimes \xi)\right)(X_0, \ldots, X_k) = \left( d\omega \otimes \xi + (-1)^k \omega \wedge d^V \xi \right)(X_0, \ldots, X_k) = d\omega (X_0, \ldots, X_k) \xi + (-1)^k (\omega \wedge d^V \xi)(X_0, \ldots, X_k).
\]

Therefore we can conclude that \( d^V \psi \) can be written as claimed.

(b) It is convenient to express \( d^V \xi \) in terms of a local frame for \( TM \): let \( \{X_1, \ldots, X_n\} \) be such a frame and \( \{\theta_1, \ldots, \theta_n\} \) its dual frame. Then

\[ d^V \xi = \nabla \xi = \sum_i \theta_i \otimes \nabla_{X_i} \xi. \]

From this and the properties of \( d^V \) we obtain

\[ d^V \circ d^V \xi = \sum_i d\theta_i \otimes \nabla_{X_i} \xi - \sum_{i,j} \theta_i \wedge \nabla_{X_i} \xi = \sum_i d\theta_i \otimes \nabla_{X_i} \xi - \sum_{i,j} \theta_i \wedge \theta_j \otimes \nabla_{X_i} \nabla_{X_j} \xi. \]

Therefore, given vector fields \( X, Y \) on \( M \),

\[
\left(d^V \circ d^V \xi\right)(X, Y) = \sum_i d\theta_i (X, Y) \nabla_{X_i} \xi - \sum_{i,j} \theta_i \wedge \theta_j (X, Y) \nabla_{X_i} \nabla_{X_j} \xi
\]

\[ = \sum_i \left( X \theta_i (Y) - Y \theta_i (X) - \theta_i ([X, Y]) \right) \nabla_{X_i} \xi - \sum_{i,j} \theta_i (X) \theta_j (Y) \nabla_{X_i} \nabla_{X_j} \xi + \sum_i \theta_i (Y) \theta_j (X) \nabla_{X_i} \nabla_{X_j} \xi
\]

\[ = \sum_i \left( X \theta_i (Y) \right) \nabla_{X_i} \xi + \theta_i (Y) \nabla_{X_i} \nabla_{X_i} \xi - \sum_{i,j} \theta_i (X) \nabla_{Y} \nabla_{X_j} \xi + \sum_i \theta_i (Y) \nabla_{X_i} \nabla_{X_i} \xi - \nabla_{[X, Y]} \xi
\]

\[ = \sum_i \nabla_{X_i} \theta_i (Y) \nabla_{X_i} \xi - \sum_{i,j} \nabla_{Y} \theta_i (X) \nabla_{X_i} \xi - \nabla_{[X, Y]} \xi
\]

\[ = \nabla_X \nabla_Y \xi - \nabla_Y \nabla_X \xi - \nabla_{[X, Y]} \xi
\]

\[ = R(X, Y) \xi. \]