Homework set 10

Math 5047 – Renato Feres

This is a reading assignment only. Solutions are provided. Please let me know if you see typos/mistakes. These notes elaborate on some material needed for Chern's proof of the Gauss-Bonnet theorem in even dimensions, as given in Walter Poor's book, Section 3.59.

1. **Induced connection on a pullback bundle.** (See section 22.10, page 210.) Let \( \pi : E \to M \) be a smooth vector bundle of rank \( r \) over the manifold \( M \). Let \( f : N \to M \) be a smooth map and consider the pullback vector bundle \( f^*E \). (This is defined in Section 20.4, page 177.) Given a connection \( \nabla \) on \( E \), we wish to define a connection \( \nabla \) on \( f^*E \) that satisfies the following properties:

   (a) If \( e \) is a smooth section of \( E \) and \( f^*e \) denotes the pullback section \((f^*e)(q) := (q,e(f(q)))\) then

   \[ \nabla_u (f^*e) = \left(q, \nabla_{df_q u} e\right) \]

   for any \( u \in T_q N \). (Here \( df_q u = (f_*)_q u : T_q N \to T_{f(q)} M \) is the standard tangent map.)

   (b) The fundamental properties for a connection. (Definition 10.1, page 72.)

Show the following:

(a) The above properties uniquely specify a connection on \( f^*E \).

(b) If \( e_1, \ldots, e_r \) is a local frame defined on an open set \( U \subseteq M \) and \( \omega_{ij} \) are the connection 1-forms for \( \nabla \) relative to this frame, then the pull-back \( \tilde{\omega}_{ij} := f^*\omega_{ij} \) are the connection 1-forms for \( \nabla \) relative to the local frame \( f^*e_1, \ldots, f^*e_r \) on \( f^{-1}(U) \).

(c) With the same notation as in the previous item, let \( \Omega_{ij} \) be the curvature 2-forms for \( \nabla \) relative to the local frame \( e_i \). Then \( \tilde{\Omega}_{ij} = f^*\Omega_{ij} \) are the curvature 2-form for \( \nabla \) relative to the pullback frame.

(d) Let \( \overline{R} \) denote the curvature tensor of \( \nabla \) and \( R \) the curvature tensor of \( \nabla \). Show that

\[ \overline{R}_q(u,v)\tilde{\xi} = (q,R_{f(q)}(df_q u, df_q v)\xi) \]

for \( u, v \in T_q N \) and \( \tilde{\xi} = (q,\xi) \in (f^*E)_q \).

Further comments: We may consider the vector bundle \( \Lambda^k(T^*M) \otimes E \) over \( M \), whose sections are differential \( k \)-forms with coefficients in \( E \). For such a form \( \mu \) we define its pullback under \( f \) as

\[ (f^*\mu)_q(v_1, \ldots, v_k) = (q,\mu_{f(q)}(df_q v_1, \ldots, df_q v_k)). \]

If \( \mu = \alpha \otimes \xi \) is the tensor product of an ordinary differential form on \( M \) and a section of \( E \), then the definition just given amounts to \( f^*(\alpha \otimes \xi) = (f^*\alpha) \otimes (f^*\xi) \), where \( f^*\alpha \) is the ordinary pullback of differential forms. If
we think of the curvature tensor $R$ as a section of $\Lambda^2(T^*M) \otimes \text{End}(E)$, then under this definition $\tilde{R} = f^* R$ and 
$\tilde{R}_q(v_1, v_2) \tilde{e}_q = (f^* R)_q(v_1, v_2)(f^* \xi)_q$. Also note that the definition of $\nabla$ implies $\nabla(f^* \xi) = f^*(\nabla \xi)$.

**Solution.**

(a) Let $\xi$ be a section of $f^* E$ and $e_1, \ldots, e_r$ a local frame of $E$ on the open set $U \subset M$. On $f^{-1}(U)$, we may express $\xi$ in terms of the pullback frame $\tilde{e}_i := f^* e_i; \xi = \sum_i h_i \tilde{e}_i$. Then a connection $\tilde{\nabla}$ having the desired properties must satisfy:

$$
\tilde{\nabla} u \xi = \sum_i \left( (u h_i) \tilde{e}_i(q) + h_i(q) \tilde{\nabla} u \tilde{e}_i \right) = \sum_i \left( (u h_i) \tilde{e}_i + h_i(q) \left( q, \tilde{\nabla}_u h_i \right) \right)
$$

for all $u \in T_q N$ and $q \in f^{-1}(U)$. This implies uniqueness. We need to check that this is well-defined (i.e., independent of the local frame) and is indeed a connection.

To verify that $\tilde{\nabla}$ is well-defined, let $\xi_1, \ldots, \xi_r$ be another choice of local frame on $U$. Let the change of frame matrix-valued function be denoted $A = (a_{ij}) : U \to GL(r, \mathbb{R})$, so that $\xi_j = \sum_i a_{ij} e_j$. The change of frame matrix-valued function on $f^{-1}(M)$ for the pullback frames is then $\tilde{A} := A \circ f = (a_{ij} \circ f)$.

A section $\xi$ of $f^* E$ can then be expressed in both frames at any $q \in f^{-1}(U)$ as:

$$
\tilde{\xi}(q) = \sum_j g_j(q) \tilde{\xi}_j(q) = \sum_i g_i(q) a_{ij}(q) \tilde{e}_i(q).
$$

We observe that

$$
\tilde{\nabla} u \sum_j g_j \tilde{\xi}_j = \tilde{\nabla} u \sum_i g_i a_{ij} \tilde{e}_i
$$

$$
= \sum_i \left( (u g_i) a_{ij}(q) \tilde{e}_i(q) + g_j(q) a_{ij}(q) \tilde{\nabla} u \tilde{e}_i \right)
$$

$$
= \sum_i \left( u g_i \tilde{a}_{ij}(q) \tilde{e}_i(q) + g_j(q) \left( q, \tilde{\nabla}_u a_{ij} \right) \tilde{e}_i \right)
$$

$$
= \sum_j \left( (u g_j) \tilde{\xi}_j(q) + g_j(q) \tilde{\nabla} u \tilde{\xi}_j \right).
$$

Thus we have

$$
\sum_i (u h_i) \tilde{e}_i(q) + h_i(q) \tilde{\nabla} u \tilde{e}_i = \sum_j \left( (u g_j) \tilde{\xi}_j(q) + g_j(q) \tilde{\nabla} u \tilde{\xi}_j \right).
$$

Therefore the definition does not depend on the choice of local pullback frame.

It is clear from the expression for $\tilde{\nabla}$ given in terms of a pullback frame that $\tilde{\nabla}_u \xi$ is $\mathcal{T}$-linear in $u$ is $\mathbb{R}$-linear in $\xi$. The Leibniz property also holds:

$$
\tilde{\nabla}_u (g \xi) = \sum_i \left( (u h_i) g \tilde{e}_i(q) + h_i(q) g(q) \tilde{\nabla} u \tilde{e}_i \right)
$$

$$
= \sum_i \left( u g h_i(q) \tilde{e}_i(q) + g(q) \sum_i (u h_i) \tilde{e}_i(q) + h_i(q) \tilde{\nabla} u \tilde{e}_i \right)
$$

$$
= (u g) \tilde{\xi}(q) + g(q) \tilde{\nabla} u \tilde{\xi}.
$$

(b) Let $\tilde{\omega}_{ij}$ be the connection 1-forms for $\tilde{\nabla}$ relative to the pullback frame $f^* e_i$. Then

$$
\sum_i \tilde{\omega}_{ij}(u) \tilde{e}_i(q) = \tilde{\nabla}_u \tilde{\xi}_j = \left( q, \tilde{\nabla}_u h_i(e) \right) = \left( q, \sum_i \omega_{ij}(df_q u) e_i(f(q)) \right) = \sum_i (f^* \omega_{ij})(u) \tilde{e}_i(q).
$$

We conclude that $\tilde{\omega}_{ij} = f^* \omega_{ij}$. 

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(c) Using the second structural equation, the curvature forms for \( \bar{\nabla} \) are

\[
d\bar{\omega}_{ij} + \sum_k \bar{\omega}_{ik} \wedge \bar{\omega}_{kj} = d(f^* \omega_{ij}) + \sum_k f^* \omega_{ik} \wedge f^* \omega_{kj} = f^*(d\omega_{ij} + \sum_k \omega_{ik} \wedge \omega_{kj}) = f^* \Omega_{ij} := \bar{\Omega}_{ij}.
\]

(d) It is enough to check the relation for \( \bar{\xi} = \bar{\xi}_j \). In this case

\[
\bar{R}_q(u, v)\bar{\xi}_j(q) = \sum_i \bar{\Omega}_{ij}(u, v)\bar{\xi}_i(q) = \left(q, \sum_i \Omega_{ij}(df_q u, df_q v)e_i(f(q))\right) = (q, R_{f(q)}(df_q u, df_q v)e_j(f(q))).
\]

2. More on vector bundle-valued differential forms. Let \( \pi : E \to M \) be a smooth vector bundle over the manifold \( M \). Recall that differential \( k \)-forms on \( M \) with coefficients in \( E \) are defined as sections of the vector bundle \( \Lambda^k(T^* M) \otimes E \). The space of such sections will be denoted \( \Gamma(\Lambda^k(T^* M) \otimes E) \). In this definition, \( E \) may be replaced with other vector bundles obtained from \( E \). We are especially interested in the bundle \( \Lambda^*(E^*) \), that is, the exterior algebra bundle of alternating forms on \( E \). This is the vector bundle over \( M \) whose fiber over \( p \in M \) is the exterior algebra \( \Lambda^*(E^*_p) = \bigoplus_k \Lambda^k(E^*_p) \), where \( E^*_p \) is the dual vector space to \( E_p \). An element \( \mu \in \Gamma(\Lambda^k(T^* M) \otimes \Lambda^*(E^*)) \) will be called a \( (k, \ell) \)-differential form. Note that at any \( p \in M \) and given \( v_1, \ldots, v_k \in T_p M \),

\[
\mu_p(v_1, \ldots, v_k) \in \Lambda^\ell(E^*_p).
\]

Thus it makes sense to write \( \mu_p(v_1, \ldots, v_k)(u_1, \ldots, u_\ell) \in \mathbb{R} \) (or \( \mathbb{C} \) if \( E \) is complex), where the \( u_i \) are elements of \( E_p \), and the resulting number is anti-symmetric separately in the \( v_i \) and in the \( u_j \) arguments.

For example, if \( E \) is Riemannian, with metric \( \langle \cdot, \cdot \rangle \) and \( \nabla \) is a metric connection on \( E \), then

\[
\Omega_p(v_1, v_2)(u_1, u_2) := \langle R(v_1, v_2)u_1, u_2 \rangle_p
\]

defines a section \( \Omega \in \Gamma(\Lambda^2(T^* M) \otimes \Lambda^2(E^*)) \), where \( R \) is the curvature tensor of \( \nabla \). If \( \mu \) is a differential \( (r, j) \)-form on \( M \) and \( \nu \) is a differential \( (s, k) \)-form, we define their wedge product as the \( (r + s, j + k) \)-form \( \mu \wedge \nu \) as follows:

\[
(\mu \wedge \nu)(v_1, \ldots, v_{r+s}) := \frac{1}{r!s!} \sum_{\sigma \in S_{r+s}} \text{sign}(\sigma) \mu(v_{\sigma(1)}, \ldots, v_{\sigma(r)}) \wedge \nu(v_{\sigma(r+1)}, \ldots, v_{\sigma(r+s)}).
\]

Notice the need for the wedge product symbol on the right-hand side of this identity. This is the wedge product on the exterior algebra on the dual of \( E \).

(a) Show that if \( \mu \) is a \( (r, i) \)-form and \( \nu \) is an \( (s, j) \)-form, then

\[
\mu \wedge \nu = (-1)^{r+s+i+j} \nu \wedge \mu.
\]

(b) Show that if \( \mu = \alpha \otimes \xi \) and \( \nu = \beta \otimes \eta \), then \( \mu \wedge \nu = (\alpha \wedge \beta) \otimes (\xi \wedge \eta) \).

(c) Suppose \( E \) is an oriented Riemannian vector bundle over \( M \) of rank \( 2m \). (The even parity of the rank is not required in this part of the exercise, but will be needed later.) Let \( e_1, \ldots, e_{2m} \) be a local positive (relative to the given orientation) orthonormal frame for \( E \) over an open set \( U \subset M \). Let \( e_1, \ldots, e_{2m} \) be the dual frame, so that \( e_i = \langle e_i, \cdot \rangle \). Then \( \omega = e_1 \wedge \cdots \wedge e_{2m} \) is the Riemannian volume form on \( E \).

i. Argue that \( \omega \) is globally defined.
ii. Show that $\omega$ is parallel: $\nabla \omega = 0$.

(d) We define the Chern-Euler form $\chi \in \Gamma \left( \Lambda^{2m}(T^*M) \otimes \Lambda^{2m}(E^*) \right)$ as

$$\chi = \frac{(-1)^m}{m!(2\pi)^m} \Omega^m,$$

where $\Omega$ is the curvature form defined above. (Note: this is a frame-independent version of the curvature matrix previously defined.) I am writing the exterior power $\Omega \wedge \cdots \wedge \Omega$ as $\Omega^m$.

Show that

$$\chi = \frac{(-1)^m}{(2\pi)^m} \text{Pf}([\Omega]) \otimes \omega.$$

Here $[\Omega]$ is meant to designate the curvature matrix relative to a choice of positive orthonormal frame, whose entries are:

$$\Omega_{ij}(v_1, v_2) = \Omega(v_1, v_2)(e_i, e_j).$$

In other words,

$$\Omega(v_1, v_2) = \sum_{i<j} \Omega_{ij}(v_1, v_2) e_i \wedge e_j.$$

(Suggestion: use the definition of the Pfaffian given in Exercise 1 of Homework set 9.)

Solution.

(a) Let $\eta$ be the permutation

$$\eta(1) = r + 1, \eta(2) = r + 2, \ldots, \eta(s) = r + 2, \eta(1 + s) = 1, \eta(2 + s) = 2, \ldots, \eta(r + s) = r,$$

which has sign $\text{sign}(\eta) = (-1)^r$. First note that

$$(\mu \wedge \nu)(v_1, \ldots, v_{r+s}) := \frac{1}{r!s!} \sum_{\sigma \in \Sigma_{r+s}} \text{sign}(\sigma) \mu(v_{\sigma(1)}, \ldots, v_{\sigma(r)}) \wedge \nu(v_{\sigma(r+1)}, \ldots, v_{\sigma(r+s)})$$

$$= (-1)^{ij} \frac{1}{r!s!} \sum_{\sigma \in \Sigma_{r+s}} \text{sign}(\sigma) \nu(v_{\sigma(r+1)}, \ldots, v_{\sigma(r+s)}) \wedge \mu(v_{\sigma(1)}, \ldots, v_{\sigma(r)})$$

$$= (-1)^{ij} \frac{1}{r!s!} \sum_{\sigma \in \Sigma_{r+s}} \text{sign}(\sigma) \nu(v_{\sigma(1)}, \ldots, v_{\sigma(r)}) \wedge \mu(v_{\sigma(r+1)}, \ldots, v_{\sigma(r+s)})$$

$$= (-1)^{ij+s} \frac{1}{r!s!} \sum_{\sigma \in \Sigma_{r+s}} \text{sign}(\sigma) \nu(v_{\sigma(1)}, \ldots, v_{\sigma(s)}) \wedge \mu(v_{\sigma(r+1)}, \ldots, v_{\sigma(r+s)})$$

$$= (-1)^{ij+s} (\nu \wedge \mu)(v_1, \ldots, v_{r+s}).$$

(b) We may assume that $\mu$ is an $(r, i)$-form and $\nu$ is an $(s, j)$-form. Then

$$(\mu \wedge \nu)(v_1, \ldots, v_{r+s}) = \frac{1}{r!s!} \sum_{\sigma \in \Sigma_{r+s}} \text{sign}(\sigma) \mu(v_{\sigma(1)}, \ldots, v_{\sigma(r)}) \wedge \nu(v_{\sigma(r+1)}, \ldots, v_{\sigma(r+s)})$$

$$= \frac{1}{r!s!} \sum_{\sigma \in \Sigma_{r+s}} \text{sign}(\sigma) (\alpha(v_{\sigma(1)}, \ldots, v_{\sigma(r)}) \xi \wedge (\beta(v_{\sigma(r+1)}, \ldots, v_{\sigma(r+s)}) \eta$$

$$= \frac{1}{r!s!} \sum_{\sigma \in \Sigma_{r+s}} \text{sign}(\sigma) \alpha(v_{\sigma(1)}, \ldots, v_{\sigma(r)}) \beta(v_{\sigma(r+1)}, \ldots, v_{\sigma(r+s)}) \xi \wedge \eta$$

$$= \xi \wedge \eta$$

$$= (\alpha \wedge \beta)(v_1, \ldots, v_{r+s}) \xi \wedge \eta$$

$$= \left( (\alpha \wedge \beta) \otimes (\xi \wedge \eta) \right) (v_1, \ldots, v_{r+s}).$$
We conclude that $\mu \wedge \nu = (\alpha \wedge \beta) \otimes (\xi \wedge \eta)$.

(c) Let $\omega$ be the Riemannian volume form on $E$, as defined locally above.

i. If $\tilde{e}_1, \ldots, \tilde{e}_{2m}$ is another local, positive orthonormal frame on some neighborhood $\tilde{U}$ that overlaps with $U$, then on their intersection the change of frame matrix $A$ lies in $SO(2m)$. It follows that

$$e_1 \wedge \cdots \wedge e_{2m} = \det(A) \tilde{e}_1 \wedge \cdots \wedge \tilde{e}_{2m} = \tilde{e}_1 \wedge \cdots \wedge \tilde{e}_{2m}.$$ 

Thus $\omega$ does not depend on the choice of the positive orthonormal frame. Therefore it defines a global form.

ii. Let $v \in T_pM$ for $p \in M$ and $\gamma(t)$ a smooth curve satisfying $\gamma(0) = p$ and $\gamma'(0) = v$. Let $e_1(t), \ldots, e_{2m}(t)$ be parallel sections of $E$ along $\gamma$, so that $\frac{\nabla e_i(t)}{dt}(0) = 0$. Then

$$\nabla_v \omega = \frac{\nabla}{dt}\bigg|_{t=0} (e_1(t) \wedge \cdots \wedge e_{2m}(t)) = \sum_{i=1}^{2m} e_i(0) \wedge \cdots \wedge e_{2m}(t).$$

But if the $e_i(t)$ are parallel along $\gamma$, then so are the $e_i(t)$. This follows from the fact that the connection is metric. Therefore $\nabla_v \omega = 0$.

(d) As noted, we can write $\Omega = \frac{1}{2} \sum_{ij} \Omega_{ij} \otimes e_i \wedge e_j$. Using the definition of the Pfaffian from Homework 9 (1), we readily obtain

$$\Omega^m = m! \text{Pf}(\Omega) \otimes e_1 \wedge \cdots \wedge e_{2m} = m! \text{Pf}(\Omega) \otimes \omega.$$ 

The expression for the Chern-Euler form in terms of the Pfaffian follows from this identity.

\phi

3. **More on the exterior covariant derivative.** Different from Homework 9, I’ll define here the exterior covariant derivative for a vector bundle $E$ as the map

$$d^\nabla : \Gamma \left( \Lambda^k(T^*M) \otimes E \right) \rightarrow \Gamma \left( \Lambda^{k+1}(T^*M) \otimes E \right)$$

such that, at a point $p \in M$,

$$d^\nabla \psi (v_0, \ldots, v_k) = \sum_i (-1)^i \nabla_{v_i} \psi (v_0, \ldots, \hat{v}_i, \ldots, v_k) + \sum_{i<j} \psi ([v_i, v_j], v_0, \ldots, \hat{v}_i, \ldots, \hat{v}_j, \ldots, v_k)$$

for $\psi \in \Gamma \left( \Lambda^k(T^*M) \otimes E \right)$, $v_0, \ldots, v_k \in T_pM$, and $V_0, \ldots, V_k \in \mathfrak{X}(M)$ such that $V_i(p) = v_i$. It follows from this definition that if $\xi$ is a section of $E$, hence an element of $\Gamma \left( \Lambda^0(T^*M) \otimes E \right) = \Gamma(E)$, then $d^\nabla \xi = \nabla \xi$; and for separable sections $\omega \otimes \xi \in \Gamma \left( \Lambda^k(T^*M) \otimes E \right)$,

$$d^\nabla (\omega \otimes \xi) = d\omega \otimes \xi + (-1)^k \omega \wedge d^\nabla \xi.$$ 

As above, we are especially interested here in the case where the vector bundle in question is the exterior algebra $\Lambda^*(E^*)$ over a given vector bundle $E$. Keep in mind that if $\xi, \eta$ are sections of $\Lambda^1(E^*)$ and $\Lambda^1(E^*)$, respectively, and $\nu \in T_pM$, then

$$\nabla_\nu (\xi \wedge \eta) = (\nabla_\nu \xi) \wedge \eta + \xi \wedge (\nabla_\nu \eta).$$

(The definition in Section 22.5, Proposition 22.7, page 206, in the textbook extends to tensors over a general vector bundle $E$, not only $TM$. See also the initial remarks in Exercise 6, of Homework 8.) Note that, different from expressions involving the exterior derivative, there is no sign in this Leibniz rule for the covariant derivative.
(a) Show that the definition of \( d^\nabla \) given by Equation (1) implies the identity (2).

(b) If \( \mu \) is an \((r, i)\)-form and \( \nu \) is \((s, j)\)-form, show that

\[
d^\nabla (\mu \wedge \nu) = (d^\nabla \mu) \wedge \nu + (-1)^i \mu \wedge d^\nabla \nu.
\]

(c) If \( f : N \to M \) is a smooth map and \( E \to M \) is a smooth vector bundle over \( M \), let \( \nabla^\ast \) be the pullback connection on \( f^\ast E \). Show that \( f^\ast \circ d^\nabla = d^\nabla \circ f^\ast \).

(d) Let \( \Omega \) be the curvature \((2, 2)\)-form defined in the previous exercise for a given metric connection \( \nabla \) on the Riemannian vector bundle \( E \). Show that \( d^\nabla \Omega = 0 \). This is an expression of the Second Bianchi Identity. Also show that \( d^\nabla R = 0 \). The latter identity does not depend on the connection being metric and on \( E \) being Riemannian.

(e) Recalling from the previous exercise the relation between \( \Omega^m \) and the Pfaffian, show that if \( E \) has rank \( 2m \), then \( \text{Pf}(\Omega) \) is a closed \( 2m \)-form (of the standard kind) on \( M \).

**Solution.**

(a) Identity (2) can be seen as follows. Let \( X_0 \),

\[
d^\nabla (\omega \otimes \xi) (v_0, \ldots, v_k) = \sum_{i} (-1)^{i} \nabla_{\pi_i} (\omega (V_0, \ldots, \hat{V_i}, \ldots, V_k) \xi) + \sum_{i < j} \psi ([V_i, V_j], v_0, \ldots, \hat{v_i}, \hat{v_j}, \ldots, v_k) \xi
\]

\[
= \left( \sum_{i} (-1)^{i} v_i \omega (V_0, \ldots, \hat{V_i}, \ldots, V_k) + \sum_{i < j} \psi ([V_i, V_j], v_0, \ldots, \hat{v_i}, \hat{v_j}, \ldots, v_k) \right) \xi
\]

\[
+ \sum_{i} (-1)^{i} \omega (v_0, \ldots, \hat{v_i}, \ldots, v_k) \nabla_{\pi_i} \xi
\]

\[
= d\omega (v_0, \ldots, v_k) \xi + \sum_{i} (-1)^{i} \omega (v_0, \ldots, \hat{v_i}, \ldots, v_k) \nabla_{\pi_i} \xi.
\]

It is helpful here to use the shuffle form of the wedge product (Proposition 19.15, page 172 of the textbook). Let \( \pi_i \) be the shuffle permutation given in cycle form as \( \pi_i = (i \ i + 1 \ \ldots \ k) \). Note that \( \text{sign}(\pi_i) = (-1)^{k+i} \) and

\[
(v_{\pi_i(0)}, \ldots, v_{\pi_i(k)}) = (v_0, \ldots, \hat{v_i}, \ldots, v_k, v_i).
\]

It follows that

\[
(\omega \wedge d^\nabla \xi) (v_0, \ldots, v_k) = \sum_{i} (-1)^{k+i} \omega (v_0, \ldots, \hat{v_i}, \ldots, v_k) \nabla_{\pi_i} \xi.
\]

Therefore \( d^\nabla (\omega \otimes \xi) = d\omega \otimes \xi + (-1)^{k} \omega \wedge d^\nabla \xi \).

(b) We may assume that \( \mu = \alpha \otimes \xi \) and \( \nu = \beta \otimes \eta \). According to the above Exercise 2b, \( \mu \wedge \nu = (\alpha \wedge \beta) \otimes (\xi \wedge \eta) \). Now using Equation (2):

\[
d^\nabla (\mu \wedge \nu) = d(\alpha \wedge \beta) \otimes (\xi \wedge \eta) + (-1)^{r+s}(\alpha \wedge \beta) \wedge \nabla (\xi \wedge \eta)
\]

\[
= [d\alpha \wedge \beta + (-1)^{r} \alpha \wedge d\beta] \otimes (\xi \wedge \eta) + (-1)^{r+s}(\alpha \wedge \beta) \wedge (\nabla \xi \wedge \eta + \xi \wedge \nabla \eta)
\]

\[
= (d\alpha \otimes \xi) \wedge (\beta \otimes \eta) + (-1)^{r} (\alpha \otimes \xi) \wedge (d\beta \otimes \eta) + (-1)^{r+s} [(-1)^{r} (\alpha \wedge \nabla \xi) \wedge (\beta \otimes \eta) + (\alpha \otimes \xi) \wedge (\beta \wedge \nabla \eta)]
\]

\[
= (d\alpha \otimes \xi + (-1)^{r} \alpha \wedge \nabla \xi) \wedge (\beta \otimes \eta) + (-1)^{r} (\alpha \otimes \xi) \wedge [(d\beta \otimes \eta + (-1)^{r} \beta \wedge \nabla \eta)]
\]

\[
= (d^\nabla \mu) \wedge \nu + (-1)^{r} \mu \wedge d^\nabla \nu.
\]

(c) We make use here of the remarks in Exercise 1. We may assume that the forms in question are separable,
that is, \( \mu = \alpha \otimes \zeta \), where \( \alpha \) is an ordinary \( k \)-form on \( M \). Then
\[
d^V f^*(\alpha \otimes \zeta) = d^V [(f^* \alpha) \otimes (f^* \zeta)] = d(f^* \alpha) \otimes (f^* \zeta) + (-1)^k (f^* \alpha) \wedge \nabla(f^* \zeta).
\]

We know that \( df^* \alpha = df^* \alpha \), and from the comments of Exercise 1 we have
\[
(f^* \alpha) \wedge \nabla(f^* \zeta) = (f^* \alpha) \wedge (f^* (\nabla \zeta)) = f^* (\alpha \wedge \nabla \zeta).
\]

Therefore,
\[
d^V f^*(\alpha \otimes \zeta) = (f^* df \alpha) \otimes (f^* \zeta) + (-1)^k f^* (\alpha \wedge \nabla \zeta) = f^*(d \alpha \otimes \zeta).
\]

(d) Let \( V_0, V_1, V_2 \) be vector fields on \( M \) and \( \zeta_0, \zeta_1 \) sections on \( E \). We wish to evaluate \( (d^V \Omega)(V_0, V_1, V_2)(\zeta_0, \zeta_1) \).

From the definition of \( d^V \) we have
\[
(d^V \Omega)(V_0, V_1, V_2) = \nabla_{V_0} \Omega(V_1, V_2) - \nabla_{V_1} \Omega(V_0, V_2) + \nabla_{V_2} \Omega(V_0, V_1) - \Omega([V_0, V_1], V_2) + \Omega([V_0, V_2], V_1) - \Omega([V_1, V_2], V_0).
\]

And from the general properties of \( \nabla \) acting on tensors over \( E \),
\[
\nabla_{V_0} \Omega(V_1, V_2)(\zeta_0, \zeta_1) = V_0 \Omega(V_1, V_2)(\zeta_0, \zeta_1) - \Omega(V_1, V_2)(\nabla_{V_0} \zeta_0, \zeta_1) - \Omega(V_1, V_2)(\zeta_0, \nabla_{V_0} \zeta_1)
\]
\[
= V_0 (R(V_1, V_2) \zeta_0, \zeta_1) - \langle R(V_1, V_2) \nabla_{V_0} \zeta_0, \zeta_1 \rangle - \langle R(V_1, V_2) \zeta_0, \nabla_{V_0} \zeta_1 \rangle
\]
\[
= \langle \nabla_{V_0} R(V_1, V_2) \zeta_0 - R(V_1, V_2) \zeta_0, \zeta_1 \rangle
\]
\[
= \langle \nabla_{V_0} R(V_1, V_2) \zeta_0, \zeta_1 \rangle.
\]

Thus
\[
(d^V \Omega)(V_0, V_1, V_2)(\zeta_0, \zeta_1) = \langle \nabla_{V_0} R(V_1, V_2) \zeta_0, \zeta_1 \rangle - \langle \nabla_{V_1} R(V_0, V_2) \zeta_0, \zeta_1 \rangle + \langle \nabla_{V_2} R(V_0, V_1) \zeta_0, \zeta_1 \rangle
\]
\[
- \langle R([V_0, V_1], V_2) \zeta_0, \zeta_1 \rangle - \langle R([V_0, V_2], V_1) \zeta_0, \zeta_1 \rangle + \langle R([V_1, V_2], V_0) \zeta_0, \zeta_1 \rangle
\]
\[
= \langle (d^V R)(V_0, V_1, V_2) \zeta_0, \zeta_1 \rangle.
\]

Thus it suffices to show \( d^V R = 0 \). To simplify expanding the terms of \( (d^V R)(V_0, V_1, V_2) \xi \) we will use the following notation: \( \nabla_{V_i} = [i, R(V_i, V_j)] = (i, j) \), \( [V_i, V_j] = [i, j] \). Thus, for example, we write
\[
\nabla_{V_i} R(V_j, V_k) \xi = [i, j, k] \quad R(V_j, V_k) \nabla_{V_i} \xi = ([i, j], k) \quad R([V_i, V_j], V_k) \xi = ([i, j], k) \quad \nabla_{V_i} [V_j, V_k] \xi = ([i, j], k)
\]

and so on. We can then write
\[
(d^V R)(V_0, V_1, V_2) \xi = \xi_0(1, 2) - (1, 2) \xi_0 - \xi_1(0, 2) + (0, 2) \xi_1 + \xi_2(0, 1) - (0, 1) \xi_2 + (0, 1, 2) - (0, 2, 1) - (-1, 2, 0)
\]
\[
= \xi_0(1) - \xi_0(0, 2) + (0, 2) \xi_0 - (0, 1) \xi_2 + (0, 1, 2) - (0, 2, 1) - (-1, 2, 0)
\]
\[
= \xi_0(0, 1, 2) - (1, 2, 0)
\]
\[
= 0.
\]

The Jacobi identity was used at the last step.
4. The canonical section and canonical form on $\pi^* E$. Using the bundle map $\pi : E \to M$ we can pull back $E$ under $\pi$. Then $\pi^* E$ is a vector bundle over the manifold $E$. By the definition of pullback bundles, elements of $\pi^* E$ have the form $(e, e') \in E \times E$ such that $\pi(e) = \pi(e')$. Another vector bundle we can define over the manifold $E$ is the vertical bundle $V$. The vector fiber at each $e \in E$ is the kernel of $d\pi_e : V_e = \{\xi \in T_e E : d\pi_e \xi = 0\}$. Note that $V_e$ is linearly isomorphic to $E_{\pi(e)}$, with the isomorphism given by the map

$$g_e : e' \in E_{\pi(e)} \mapsto \left. \frac{d}{dt} \right|_{t=0} (e + te') \in V_e.$$ 

Given a connection $\nabla$ on $E$, we can define the horizontal subbundle of $TE$, denoted $H$. The fiber $H_e$ is defined as the kernel of a map $K_e : T_e E \to E_{\pi(e)}$ which we define as follows. An element $\xi \in T_e E$ can be represented by a curve $e(t) \in E$, so that $e(0) = e$ and $e'(0) = \xi$. Then $e(t)$ is a section of $E$ over the curve $\gamma(t) = \pi(e(t)) \in M$. Therefore it makes sense to define

$$K_e \xi = \left. \frac{d}{dt} \right|_{t=0} e = \left. \frac{d}{dt} \right|_{t=0} \pi(e(t)).$$

Note that $K_e \xi$ is the identity map on $E_{\pi(e)}$. Moreover, the rank of $H$ is equal to the dimension of $M$ and $d\pi_e : H_e \to T_{\pi(e)} M$ and $K_e : V_e \to E_{\pi(e)}$ are linear isomorphisms for each $e$, as can be easily checked. Also observe that $g_e \circ K_e$ is the identity on $V_e$. From these definitions we obtain a splitting of $TE$ as a direct sum of subbundles: $TE = V \oplus H$.

The pullback bundle $\pi^* E$ admits a canonical section $s \in \Gamma(\pi^* E)$ such that $s(e) = (e, e)$. Given the isomorphism of $\pi^* E$ and $V$ that you will establish below, $s$ may be regarded as a section of $V$, in which case it is given by $s(e) = g_e e$. Further define the $(0, 1)$-form $\eta$ on $\pi^* E$ (which we may regard as a section of the dual vertical bundle $V^*$ over $E$):

$$\eta_e (g_e e') = \langle s(e), g_e e' \rangle_e = \langle e, e' \rangle_{\pi(e)}.$$ 

We are using here the pullback metric to $\pi^* E$, still denote by $\langle \cdot, \cdot \rangle$. I note that $\overline{\nabla}$ is a metric connection for this metric.

Finally, we define the connection $(1, 1)$-form $\theta = d^{\pi^*} \eta \in \Gamma(T^{*\pi} E \otimes V^*)$.

(a) Show that $\pi^* E$ and $V$ are isomorphic vector bundles over $E$.
(b) If $\overline{\nabla}$ is the pullback connection on the vector bundle $\pi^* E \equiv V \to E$ and $\xi \in T_e E$, show that $\overline{\nabla}_s \xi = g_e K_e \xi$.
(c) Let $\xi \in T_e E$ and $e' \in E_{\pi(e)}$. Show that $\theta_e (\xi) (g_e e') = \langle K_e \xi, e' \rangle_{\pi(e)}$.
(d) Show that $d^\pi \theta_e (\xi_1, \xi_2) (u) = (\pi^* \Omega)(\xi_1, \xi_2) (\tau(e), u)$ for $\xi_1, \xi_2 \in T_e E$ and $u \in V_e$.
(e) Let $E_1 \subseteq E$ denote the subbundle of $E$ whose fiber at $p \in M$ consists of the vectors of unit length in $E_p$. We call it the sphere bundle of $E$. (It is a fiber bundle over $M$ but, naturally, not a vector bundle.) If $\xi$ is tangent to $E_1$ at $e \in E_1$, show that $\theta_e (\xi) (\tau(e)) = 0$. 

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(f) Consider the $(2m - 1, 2m)$-form $\Pi_j := \eta \wedge \Theta^{j-1} \wedge \pi^* \Omega^{m-j}$. Let $\xi_1, \ldots, \xi_{2m-1}$ be tangent vectors to $(E_1)_p$ at $e \in E_1$ (thus the $\xi_i$ are vertical vectors; that is, $\xi_i \in V_e$) and let $X_0, X_1, \ldots, X_{2m-1}$ form a positive orthonormal basis of $(\pi^* E)_e$ such that $X_0 = \delta(e)$. Show that

$$\Pi_j(\xi_1, \ldots, \xi_{2m-1})(X_0, \ldots, X_{2m-1}) = 0$$

for $j \leq m$ and

$$\Pi_m(\xi_1, \ldots, \xi_{2m-1})(X_0, \ldots, X_{2m-1}) = (2m - 1)! \omega_e(\xi_1, \ldots, \xi_{2m-1})$$

where $\omega$ is the volume form on the fiber of $E_1$ at $p \in M$.

**Solution.**

(a) For each $e \in E$, both $(\pi^* E)_e$ and $V_e$ are naturally isomorphic to $E_{\pi(e)}$ under the maps $pr_2 : (\pi^* E)_e \rightarrow E_{\pi(e)}$, $pr_2(e, e') = e'$, and $\mathcal{K}_e : V_e \rightarrow E_{\pi(e)}$. Define $\mathcal{B} : \pi^* E \rightarrow V$ by

$$(e, e') \in (\pi^* E)_e \mapsto \mathcal{B}(e, e') = \mathcal{B}_e e' \in V_e.$$

It can be checked that $\mathcal{B}$ is a bundle map, a linear isomorphism on each fiber, and smooth. It has an inverse bundle map given by $\mathcal{K} : V \rightarrow \pi^* E$ define for each $e \in V_e$ by $\mathcal{K}_e = (e, \mathcal{K}_e \xi)$, also smooth.

(b) Let $\xi = e'(0) \in T_e E$, where $e(t)$ is a differentiable curve in $E$ such that $e(0) = e$ and $e'(0) = \xi$. Let $\gamma(t) = \pi(e(t))$.

Then $e(t)$ is a section of $E$ along $\gamma(t)$ and $\delta(t) = (e(t), e(t))$ is a section of $\pi^* E$ along $e(t)$. We have

$$\frac{\nabla_e}{dt} \bigg|_{t=0} = (e(0), \frac{\nabla e}{dt} \bigg|_{t=0}) = (e, \mathcal{K}_e \xi).$$

The right-most term is identified with $\mathcal{B}_e \mathcal{K}_e \xi$ by the first part of this exercise.

(c) Let $X \in \mathfrak{X}(E)$ and $U \in \Gamma(\pi^* E) = \Gamma(V)$. Then

$$\theta_e(X)(U) = \left( \frac{d \nabla}{dt} \right) \eta(X)(U) = (\nabla X \eta)_e(U) = X \eta(U) - \eta(\nabla X U) = X \langle \xi, U \rangle - \langle \xi, \nabla X U \rangle_e = \langle \nabla X \xi, U \rangle_e = \langle \mathcal{B}_e \mathcal{K}_e X, U \rangle_e.$$

If $U = \mathcal{B}_e e'$ and $X_e = \xi$, then $\theta_e(\xi)(\mathcal{B}_e e') = \langle \mathcal{K}_e \xi, e' \rangle_{\pi(e)}$.

(d) Let $X, Y \in \mathfrak{X}(E), U \in \Gamma(V)$. Then

$$\left( \frac{d \nabla}{dt} \right) \langle X, Y \rangle(U) = \langle \nabla X \theta(Y) - \theta(Y) \nabla X U \rangle(U) = X \langle \theta(Y), U \rangle - \theta(Y) \langle X, U \rangle + \theta(X) \langle \nabla Y U \rangle - \theta(|X, Y|) \langle X, U \rangle = \langle \nabla X \theta(Y), U \rangle = \langle \mathcal{K}_e \mathcal{K}_e X, U \rangle.$$  

Writing $\xi_1 = X_e, \xi_2 = Y_e, u = U_e$, then

$$\left( \frac{d \nabla}{dt} \right) \langle \xi_1, \xi_2 \rangle(u) = \langle \mathcal{K}_e(\xi_1, \xi_2), u \rangle = \langle \pi^* \Omega \langle \xi_1, \xi_2 \rangle, \delta(e), u \rangle.$$

(e) Let $e = e'(0) \in T_e E_1$ where $e(t)$ is a smooth curve in $E_1$ representing $\xi$. Then $\langle \delta(e(t)), \delta(e(t)) \rangle = 1$ and

$$0 = 1 = \frac{1}{2} \frac{d}{dt} \bigg|_{t=0} \langle e(t), e(t) \rangle = \left( \frac{\nabla e}{dt} \bigg|_{t=0} \right) e \cdot e = \langle \mathcal{K}_e \xi, e \rangle_{\pi(e)} = \theta_e(\xi)(\delta(e)).$$
(f) If \( j < m \), the form \( \Pi_j \) contains a factor \( \pi^* \Omega \). But this term vanishes when evaluated on a vertical vector \( \xi_j \). Thus \( \Pi_j(\xi_1, \ldots, \xi_{2m-1})(X_0, \ldots, X_{2m-1}) = 0 \) if \( j < m \). Let us now look at \( \Pi_m = \eta \wedge \theta \wedge \cdots \wedge \theta = \eta \wedge \theta^{2m-1} \). Keep in mind that \( \theta(\xi_0) = 0 \) and \( \eta(e) = \chi \), and form a positive orthonormal basis. Since the latter bundle is trivial for an oriented \( E \), we can write
\[
\Pi_m(\xi_1, \ldots, \xi_{2m-1})(X_0, \ldots, X_{2m-1}) = \sum_{\sigma \in S_{2m-1}} \text{sign}(\sigma) \left[ \eta \wedge \theta(\xi_{\sigma(1)}) \wedge \cdots \wedge \theta(\xi_{\sigma(2m-1)}) \right](X_0, \ldots, X_{2m-1})
\]
\[
= (2m-1)! \left[ \eta \wedge \theta(\xi_1) \wedge \cdots \wedge \theta(\xi_{2m-1}) \right](X_0, \ldots, X_{2m-1})
\]
\[
= (2m-1)! \sum_{\sigma \in S_{2m-1}} \text{sign}(\sigma) \left[ \theta(\xi_1) \left( X_{\sigma(1)} \right) \cdots \theta(\xi_{2m-1}) \left( X_{\sigma(2m-1)} \right) \right].
\]

The vectors \( \xi \) are tangent to the fiber \((E_1)_p\) at \( e \in E_1 \) and the \( X_i \) lie in \( \pi^* E \equiv V \). So we may write \( X_i = \mathcal{G}_e e_i \) where the \( e_i \) are tangent to \( E \) at \( p \) and form a positive orthonormal basis. Since the \( X_i \) are orthogonal to \( X_1 = \theta \) for \( i \geq 1 \), the \( e_i \) are orthogonal to \( e \) for \( i \geq 1 \). This means that the \( e_1, \ldots, e_{2m-1} \) constitute a positive orthonormal basis for the tangent space to \((E_1)_p\) at \( e \). Consequently, the above alternating sum gives \( \omega_e(\xi_1, \ldots, \xi_{2m-1}) \) and \( \Pi_m(\xi_1, \ldots, \xi_{2m-1})(X_0, \ldots, X_{2m-1}) = (2m-1)! \omega_e(\xi_1, \ldots, \xi_{2m-1}) \).

5. Keeping with the notation of the previous exercise, define \( \Pi := \sum_{j=1}^{m} c_j \Pi_j \) where
\[
c_j = (-1)^{m+j+1} \frac{2^{m-j+1}(j-1)!}{(m-j)!(2j-1)!n^m}.
\]
Then \( \Pi \in \Gamma(\Lambda^{2m-1}(T^* E) \otimes \Lambda^{2m}(\pi^* E^*)) \). Show that \( d^\nabla \Pi = \pi^* \chi \) on \( E_1 \). Note that \( \Pi \) and \( \pi^* \chi \) are forms on \( E \) (or \( E_1 \)) with coefficients in the vector bundle \( \Lambda^{2m}(E^*) \). Since the latter bundle is trivial for an oriented \( E \), with the volume form \( \omega \) serving as a global section, we can write \( \Pi \) and \( \pi^* \chi \) as a product of ordinary forms on the base manifold \( E_1 \). Using the same notations \( \Pi \) and \( \pi^* \chi \) for these ordinary forms, then \( \pi^* \chi = d\Pi \), where \( d \) is now the ordinary exterior derivative.

**Solution.** This is now the calculation given in Section 3.67, pages 142 and 143 of Walter Poor’s text.