Homework set 1 - due 09/09/22

Math 5047

1. Read Chapter 0 of do Carmo's text.
2. **Chapter 0, Exercise 2.** Prove that the tangent bundle of a differentiable manifold M is orientable (even though M itself may not be).
3. **Chapter 0, Exercise 5.** Let $F : \mathbb{R}^3 \to \mathbb{R}^4$ be given by

   $$F(x,y,z) = \left(x^2 - y^2, xy, xz, yz\right), \quad (x,y,z) = p \in \mathbb{R}^3.$$

   Let $S^2 \subseteq \mathbb{R}^3$ be the unit sphere with the origin $0 \in \mathbb{R}^3$. Observe that the restriction $\varphi = F|_{S^2}$ is such that $\varphi(p) = \varphi(-p)$, and consider the mapping from projective space $\widetilde{\varphi} : P^2(\mathbb{R}) \to \mathbb{R}^4$ given by

   $$\widetilde{\varphi}([p]) = \varphi(p), \quad [p] = \{p, -p\} = \text{ equivalence class of } p.$$

   Prove that

   (a) $\widetilde{\varphi}$ is an immersion.

   (b) $\widetilde{\varphi}$ is injective. Together with (a) and the compactness of $P^2(\mathbb{R})$, this implies that $\widetilde{\varphi}$ is an embedding.
4. **Chapter 0, Exercise 7.** Let $C = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 = 1\}$ be a right circular cylinder, and let $A : C \to C$ be the symmetry with respect to the origin $0 \in \mathbb{R}^3$, that is, $A(x,y,z) = (-x,-y,-z)$. Let M be the quotient space of C with respect to the equivalence relation $p \sim A(p)$, and let $\pi : C \to M$ be the projection $\pi(p) = \{p, A(p)\}$.

   (a) Show that it is possible to give M a differentiable structure such that π is a local diffeomorphism.

   (b) Prove that M is non-orientable.
5. **Chapter 0, Exercise 8.** Let M_1 and M_2 be differentiable manifolds. Let $\varphi : M_1 \to M_2$ be a local diffeomorphism. Prove that if M_2 is orientable, then M_1 is orientable.
6. Read the statements of Exercises 9 and 12 in Chapter 0.