Homework set 3 - due 09/25/22

Math 5047

1. (do Carmo, page 56, Exercise 1.) Let M be a Riemannian manifold. Consider the mapping

$$P = P_{c,t_0,t} : T_{c(t_0)}M \to T_{c(t)}M$$

defined by: $P_{c,t_0,t}(v)$, $v \in T_{c(t_0)}M$, is the vector obtained by parallel transporting the vector v along the curve c. Show that P is an isometry and that, if M is oriented, P preserves the orientation.

2. (do Carmo, page 56, Exercise 2.) Let X and Y be differentiable vector fields on a Riemannian manifold M. Let $p \in M$ and let $c : I \to M$ be an integral curve of X through p, i.e. $c(t_0) = p$ and $\frac{dc}{dt} = X(c(t))$. Prove that the Riemannian connection of M satisfies

$$(\nabla_X Y)(p) = \frac{d}{dt}(P_{c,t_0,t})^{-1}(Y(c(t)))\bigg|_{t=t_0},$$

where $P_{c,t_0,t} : T_{c(t_0)}M \to T_{c(t)}M$ is the parallel transport along c, from t_0 to t. (This shows how the connection can be recovered from the concept of parallelism.)

3. (do Carmo, page 57, Exercise 3.) Let $f : M^n \to \overline{M}^{n+k}$ be an immersion of a differentiable manifold M into a Riemannian manifold $\overline{M}$. Assume that M has the Riemannian metric induced by f (cf. Example 2.5 of Chap. 1). Let $p \in M$ and let $U \subseteq M$ be a neighborhood of p such that $f(U) \subseteq \overline{M}$ is a submanifold of $\overline{M}$. Further, suppose that X, Y are differentiable vector fields on $f(U)$ which extend to differentiable vector fields $\overline{X}, \overline{Y}$ on an open set of $\overline{M}$. Define

$$(\nabla_X Y)(p) = \text{ tangential component of } \left(\overline{\nabla}_{\overline{X}}\overline{Y}\right)(p),$$

where $\overline{\nabla}$ is the Riemannian connection of $\overline{M}$. Prove that ∇ is the Riemannian connection of M.

4. (do Carmo, page 57, Exercise 6.) Let M be a Riemannian manifold and let p be a point of M. Consider a constant curve $f : I \to M$ given by $f(t) = p$, for all $t \in I$. Let V be a vector field along f (that is, V is a differentiable mapping of I into T_pM). Show that $\frac{DV}{dt} = \frac{dV}{dt}$, that is to say, the covariant derivative coincides with the usual derivative of $V : I \to T_pM$.

5. Read Exercise 7, page 58 of do Carmo's text. Explain to yourself how parallel transport can be obtained on the sphere as described in the hint. You don't have to turn this in.