Homework set 4 - due 10/02/22

Math 5047

1. Read do Carmo's text, Chapter 3. (I believe up to page 66 should be enough for this assignment.)

2. (do Carmo, Exercise 1, page 77: Geodesics of a surface of revolution.) Denote by (u, v) the Cartesian coordinates of $\mathbb{R}^2$. Define the function $\varphi : U \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$\varphi(u, v) = (f(v)\cos u, f(v)\sin u, g(v))$$

where $U = (u_0, u_1) \times (v_0, v_1)$, f and g are differentiable functions with $f'(v)^2 + g'(v)^2 \neq 0$ and $f(v) \neq 0$.

   (a) Show that φ is an immersion. (The image $\varphi(U)$ is the surface generated by the rotation of the curve $(f(v), g(v))$ around the axis $0z$ and is called a *surface of revolution* S. The image by φ of the curves $u =$ constant and $v =$ constant are called *meridians* and *parallels*, respectively, of S.)

   (b) Show that the induced metric in the coordinates (u, v) is given by

$$g_{11} = f^2, \quad g_{12} = 0, \quad g_{22} = f'^2 + g'^2.$$

   (c) Show that local equations of a geodesic γ are

$$\frac{d^2u}{dt^2} + \frac{2ff'}{f^2}\frac{du}{dt}\frac{dv}{dt} = 0$$
$$\frac{d^2v}{dt^2} - \frac{ff'}{f'^2 + g'^2}\left(\frac{du}{dt}\right)^2 + \frac{f'f'' + g'g''}{f'^2 + g'^2}\left(\frac{dv}{dt}\right)^2 = 0.$$

   (d) Obtain the following geometric meaning of the equations above: the second equation is, except for meridians and parallels, equivalent to the fact that the "energy" $|\gamma'(t)|^2$ of a geodesic is constant along γ; the first equation signifies that if $\beta(t)$ is the oriented angle, $\beta(t) < \pi$, of γ with a parallel P intersecting γ at $\gamma(t)$, then

$$r\cos\beta = \text{ constant},$$

   where r is the radius of the parallel P. (The equation above is called *Clairaut's relation.*)

   (e) (You don't have to do this part. If you are curious, see do Carmo's *Differential Geometry of Curves and Surfaces,* Dover Publications, 2016, page 262.) Use Clairaut's relation to show that a geodesic of the paraboloid

$$\left(f(v) = v, g(v) = v^2, v > 0, -\epsilon < u < 2\pi + \epsilon\right)$$

   which is not a meridian, intersects itself an infinite number of times. (See Figure 6 on page 79.)

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3. (do Carmo, Exercise 2, page 78.) It is possible to introduce a Riemannian metric in the tangent bundle TM of a Riemannian manifold M in the following manner. Let $(p,v) \in TM$ and V, W be tangent vectors in TM at (p,v). Choose curves in TM

$$\alpha : t \mapsto (p(t), v(t)), \quad \beta : s \mapsto (q(s), w(s)),$$

with $p(0) = q(0) = p$, $v(0) = w(0) = v$, and $V = \alpha'(0)$, $W = \beta'(0)$. Define an inner product on TM by

$$\langle V, W\rangle_{(p,v)} = \langle d\pi(V), d\pi(W)\rangle_p + \left\langle \frac{Dv}{dt}(0), \frac{Dw}{dt}(0)\right\rangle_p,$$

where $d\pi$ is the differential of $\pi : TM \to M$.

(a) Prove that this inner product is well-defined and introduces a Riemannian metric on TM.

(b) A vector at $(p,v) \in TM$ that is orthogonal (for the metric above) to the fiber $\pi^{-1}(p) \cong T_pM$ is called a *horizontal vector*. A curve $t \mapsto (p(t), v(t))$ in TM is *horizontal* if its tangent vector is horizontal for all t. Prove that the curve

$$t \mapsto (p(t), v(t))$$

is horizontal if and only if the vector field $v(t)$ is parallel along $p(t)$ in M.

(c) Prove that the geodesic field is a horizontal vector field (i.e., it is horizontal at every point).

(d) Prove that the trajectories of the geodesic field are geodesic on TM in the metric above. *Hint:* Let $\overline{\alpha}(t) = (\alpha(t), v(t))$ be a curve in TM. Show that $\ell(\overline{\alpha}) \geq \ell(\alpha)$ and that the equality is verified if v is parallel along α. Consider a trajectory of the geodesic flow passing through (p,v) which is locally of the form $\overline{\gamma}(t) = (\gamma(t), \gamma'(t))$, where $\gamma(t)$ is a geodesic on M. Choose convex neighborhoods $W \subseteq TM$ of (p,v) and $V \subseteq M$ of p such that $\pi(W) = V$. Take two points $Q_1 = (q_1, v_1)$, $Q_2 = (q_2, v_2)$ in $\overline{\gamma} \cap W$. If $\overline{\gamma}$ is not a geodesic, there exists a curve $\overline{\alpha}$ in W passing through Q_1 and Q_2 such that $\ell(\overline{\alpha}) < \ell(\overline{\gamma}) = \ell(\gamma)$. Let $\alpha = \pi(\overline{\alpha})$; since $\ell(\alpha) \leq \ell(\overline{\alpha})$, this contradicts the fact that γ is a geodesic.

(e) A vector at $(p,v) \in TM$ is called *vertical* if it is tangent to the fiber $\pi^{-1}(p) \cong T_pM$. Show that

$$\langle W, W\rangle_{(p,v)} = \langle d\pi(W), d\pi(W)\rangle_p, \quad \text{if } W \text{ is horizontal}$$
$$\langle W, W\rangle_{(p,v)} = \langle W, W\rangle_p, \quad \text{if } W \text{ is vertical,}$$

where we are identifying the tangent space to the fiber with T_pM.

4. (do Carmo, Exercise 3, page 80.) Let G be a Lie group, $\mathfrak{g}$ its Lie algebra and let $X \in \mathfrak{g}$. (See Example 2.6, Chapter 1.) The trajectories of X determine a mapping $\varphi : (-\epsilon, \epsilon) \to G$ with $\varphi(0) = e$, $\varphi'(t) = X(\varphi(t))$.

(a) Prove that $\varphi(t)$ is defined for all $t \in \mathbb{R}$ and that $\varphi(t+s) = \varphi(t)\varphi(s)$. ($\varphi : \mathbb{R} \to G$ is then called a 1-*parameter subgroup* of G.)

(b) Prove that if G has a bi-invariant metric $\langle \cdot, \cdot \rangle$ then the geodesics of G that start from e are 1-parameter subgroups of G.

Hints:

(a) Let $\varphi(t_0) = y$, $t_0 \in (-\epsilon, \epsilon)$. Show that, from the left invariance, $t \mapsto y^{-1}\varphi(t)$, $t \in (-\epsilon, \epsilon)$, is also an integral curve of X passing through e for $t = t_0$. By uniqueness, $\varphi(t_0)^{-1}\varphi(t) = \varphi(t - t_0)$, hence φ can be extended out from t_0 in an interval of radius ϵ. This shows that $\varphi(t)$ is defined for all $t \in \mathbb{R}$. In addition $\varphi(t_0)^{-1} = \varphi(-t_0)$ and, since t_0 is arbitrary, we obtain $\varphi(t+s) = \varphi(t)\varphi(s)$.

(b) Use the relation (see Eq. (9) of Chap. 2)

$$2\langle X, \nabla_Z Y\rangle = Z\langle X, Y\rangle + Y\langle X, Z\rangle - X\langle Y, Z\rangle \\ + \langle Z, [X, Y]\rangle + \langle Y, [X, Z]\rangle - \langle X, [Y, Z]\rangle$$

and the fact that the metric is left-invariant to prove that $\langle X, \nabla_Y Y\rangle = \langle Y, [X, Y]\rangle$, where X, Y and Z are left-invariant fields. Use also the fact that the bi-invariance of the metric implies that (Equation (3) on page 40)

$$\langle [U, X], V\rangle = -\langle U, [V, X]\rangle,$$

for $X, U, V \in \mathfrak{g}$. It follows that $\nabla_Y Y = 0$ for all $Y \in \mathfrak{g}$. Thus 1-parameter subgroups are geodesics. By uniqueness, geodesics are 1-parameter subgroups.