Homework set 5 - due 10/16/22

Math 5047

1. (Chapter 3 of do Carmo's text , Exercise 7, page 83. *Geodesic frame.*) Let M be a Riemannian manifold of dimension n and let $p \in M$. Show that there exists a neighborhood $U \subseteq M$ of p and n vector fields $E_1, \ldots, En \in \mathfrak{X}(U)$, orthonormal at each point of U, such that, at p, $\nabla_{E_i} E_j(p) = 0$. Such a family E_i, $i = 1, \ldots, n$, of vector fields is called a (local) *geodesic frame* at p.

   Hint: Consider an orthonormal basis $\{e_1, \ldots, e_n\}$ of T_pM and define $E_i(q)$ near p by parallel transporting e_i along the radial geodesic joining p to q.

2. (Chapter 3 of do Carmo's text, Exercise 8, page 83.) Let M be a Riemannian manifold. Let $X \in \mathfrak{X}(M)$ and f a differentiable function on M. Define the *divergence* of X as the function $\operatorname{div} X$ on M such that $\operatorname{div} X(p)$ is the trace of the linear mapping $v \mapsto \nabla_v X$, and the *gradient* of f as the vector field $\operatorname{grad} f$ defined by

   $$\langle \operatorname{grad} f(p), v \rangle = df_p(v)$$

   for $p \in M, v \in T_pM$.

   (a) Let E_i, $i = 1, \ldots, n = \dim M$, be a geodesic frame at p. (See the previous exercise.) Show that

   $$\operatorname{grad} f(p) = \sum_{i=1}^{n} (E_i f)(p) E_i(p), \quad \operatorname{div} X(p) = \sum_i E_i(f_i)(p) E_i(p)$$

   where $X = \sum_i f_i E_i$.

   (b) Suppose that $M = \mathbb{R}^n$, with coordinates $(x_1, \ldots, x_n)$ and $\frac{\partial}{\partial x_i} = e_i = (0, \ldots, 1, \ldots, 0)$. Show that

   $$\operatorname{grad} f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} e_i, \quad \operatorname{div} X = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i},$$

   where $X = \sum_i f_i e_i$.

3. (Chapter 3 of do Carmo's text, Exercise 9, page 83.) Let M be a Riemannian manifold. Define an operator Δ from differentiable functions to differentiable functions on M (the *Laplacian* of M) by

   $$\Delta f := \operatorname{div} \operatorname{grad} f.$$

   (a) Let E_i be a geodesic frame at $p \in M$, $i = 1, \ldots, n = \dim M$. Prove that

   $$\Delta f(p) = \sum_i E_i(E_i(f))(p).$$

   Conclude that if $M = \mathbb{R}^n$, Δ coincides with the usual Laplacian, $\Delta f = \sum_i \frac{\partial^2 f}{\partial x_i^2}$.

(b) Show that

$$\Delta(fg) = f\Delta g + g\Delta f + 2\langle \operatorname{grad} f, \operatorname{grad} g\rangle.$$

4. (Chapter 3 of do Carmo's text, Exercise 11, page 84.) Let M be an oriented Riemannian manifold. Let ν be a differential form of degree $n = \dim M$ defined in the following way:

$$\nu(v_1,\dots,v_n)(p) = \pm\sqrt{\det(\langle v_i, v_j\rangle)} = \text{oriented volume of } \{v_1,\dots,v_n\}, \quad p \in M,$$

where $v_1,\dots,v_n \in T_pM$ are linearly independent, and the oriented volume is affected by the sign + or − depending on whether or not the basis $\{v_1,\dots,v_n\}$ belongs to the orientation of M. The form ν is called the *volume element* of M. For a vector field $X \in \mathfrak{X}(M)$ define the *interior product* $i(X)\nu$ of X with ν as the $(n-1)$-form

$$i(X)\nu(Y_2,\dots,Y_n) = \nu(X, Y_2,\dots,Y_n), \quad Y_2,\dots,Y_n \in \mathfrak{X}(M).$$

Prove that

$$d(i(X)\nu) = (\operatorname{div} X)\nu.$$

Note: do Carmo offers a long hint on page 85, which I don't reproduce here. You may take for granted the property of interior product:

$$i(X)\alpha\wedge\beta = (i(X)\alpha)\wedge\beta + (-1)^k\alpha\, i(X)\beta$$

where α is a k-form.

5. Read the statements of Exercises 12 and 14. You don't have to write down the proofs.