Homework set 6 - due 10/23/22

Math 5047

1. (do Carmo's text, page 103, exercise 1.) Let G be a Lie group with a bi-invariant metric $\langle\cdot,\cdot\rangle$. Let $X, Y, Z \in \mathfrak{X}(G)$ be unit length left-invariant vector fields on G.

   (a) Show that $\nabla_X Y = \frac{1}{2}[X, Y]$.
   Hint: Use the symmetry of the connection and the fact that $\nabla_X X = 0$ (cf. Exercise 3 of Chapter 3, HW 4.).

   (b) Conclude from the first item that $R(X, Y)Z = \frac{1}{4}[[X, Y], Z]$. (Depending on the definition we adopt for R, we have the negative of this vector.)

   (c) Prove that if X and Y are orthonormal, the sectional curvature $K(\sigma)$ of G with respect to the plane σ generated by X and Y is given by
   $$K(\sigma) = \frac{1}{4}\|[X, Y]\|^2.$$
   (Regardless of the definition adopted for the curvature tensor R, we do want the definition of K to be such that this identity holds with a non-negative quantity on the right-hand side.) Therefore the sectional curvature $K(\sigma)$ of a Lie group with bi-invariant is non-negative and is zero if and only if σ is generated by vector fields X, Y which commute, that is, such that $[X, Y] = 0$.

2. (do Carmo's text, page 105, exercise 4.) Let M be a Riemannian manifold with the following property: given any two points $p, q \in M$, the parallel transport from p to q does not depend on the curve that points p to q. Prove that the curvature of M is identically zero, that is, $R(X, Y)Z = 0$ for all $X, Y, Z \in \mathfrak{X}(M)$. (The long hint given on page 105 for this exercise is essentially the proof! It leaves little further to do.)

3. (do Carmo's text, page 105, exercise 6, Locally Symmetric Spaces.) Let M be a Riemannian manifold. M is a *locally symmetric space* if $\nabla R = 0$, where R is the curvature tensor M. (The geometric significance of this condition will be given in Exercise 14 of Chapter 8.)

   (a) Let M be a locally symmetric space and let $\gamma : [0, \ell) \to M$ be a geodesic of M. Let X, Y, Z be parallel vector fields along γ. Prove that $R(X, Y)Z$ is a parallel vector field along γ.

   (b) Prove that if M is locally symmetric, connected, and has dimension two, then M has constant (sectional) curvature.

   (c) Prove that if M has constant (sectional) curvature, then M is a locally symmetric space.

4. (do Carmo's text, page 106, exercise 8, Schur's Theorem.) Let M^n be a connected Riemannian manifold with $n \geq 3$. Suppose that M is *isotropic*, that is, for each $p \in M$ the sectional curvature $K(p, \sigma)$ does not depend on $\sigma \subseteq T_pM$. Prove that M has constant sectional curvature, that is, $K(p, \sigma)$ also does not depend on p.