Homework set 7 - due 11/13/22

Math 5047

1. (do Carmo's text, Chapter 5, Exercise 1.) Let M be a Riemannian manifold with sectional curvature identically zero. Show that, for every $p \in M$, the mapping $\exp_p : B_\epsilon(0) \subseteq T_pM \to B_\epsilon(p)$ is an isometry, where $B_\epsilon(p)$ is a normal ball at p.

2. (do Carmo's text, Chapter 5, Exercise 2.) Let M be a Riemannian manifold, $\gamma : [0,1] \to M$ a geodesic, and J a Jacobi field along γ. Prove that there exists a parametrized surface $f(t,s)$, where $f(t,0) = \gamma(t)$ and the curves $t \to f(t,s)$ are geodesics, such that $J(t) = \frac{\partial f}{\partial s}(t,0)$. (See the textbook, on page 119, for the hints given for this exercise.)

3. (do Carmo's text, Chapter 5, Exercise 3.) Let M be a Riamnnian manifold with non-positive sectional curvature. Prove that, for all p, the conjugate locus $C(p)$ is empty. (See the textbook, on top of page 120, for the hints given for this exercise.)

4. (do Carmo's text, Chapter 5, Exercise 5.) *Jacobi fields and conjugate points on locally symmetric spaces* (Cf. Exercise 6 of Chapter 4.) Let $\gamma : [0,\infty) \to M$ be a geodesic in a locally symmetric space M and let $v = \gamma'(0)$ be its velocity at $p = \gamma(0)$. Define a linear transformation $K_v : T_pM \to T_pM$ by

$$K_v(x) = R(x,v)v, \quad x \in T_pM.$$

(This is a little different than the definition in the text, and in agreement with the definition for sectional curvature used in class. You may keep it as stated in the text if you prefer.)

   (a) Prove that K_v is self-adjoint.

   (b) Choose an orthonormal basis $\{e_1,\dots,e_n\}$ of T_pM that diagonalizes K_v; that is,

$$K_v(e_i) = \lambda_i e_i, \quad i = 1,\dots,n.$$

   Extend the e_i to fields along γ by parallel transport. Show that, for all t,

$$K_{\gamma'(t)}(e_i(t)) = \lambda_i e_i(t),$$

   where λ_i does not depend on t. (*Hint*: Use Exercise 6(a), of Chapter 4.)

   (c) Let $J(t) = \sum_i x_i(t)e_i(t)$ be a Jacobi field along γ. Show that the Jacobi equation is equivalent the the system

$$\frac{d^2x_i}{dt^2} + \lambda_i x_i = 0, \quad i = 1,\dots,n.$$

   (d) Show that the conjugate points of p along γ are given by $\gamma\left(\pi k/\sqrt{\lambda_i}\right)$, where k is a positive integer and λ_i is a positive eigenvalue of K_v.

5. (do Carmo's text, Chapter 3, Exercise 5.) Let M be a Riemannian manifold and $X \in \mathfrak{X}(M)$. Let $p \in M$ and let $U \subseteq M$ be a neighborhood of p. Let $\varphi : (-\epsilon, \epsilon) \times U \to M$ be a differentiable mapping such that for any $q \in U$ the curve $t \to \varphi(t, q)$ is a trajectory of X passing through q at $t = 0$. (U and φ are given by the fundamental theorem for ordinary differential equations, Cf. Theorem 2.2.) X is called a *Killing field* (or and *infinitesimal isometry*) if, for each $t_0 \in (-\epsilon, \epsilon)$, the mapping $\varphi(t_0, \cdot) : U \subseteq M \to M$ is an isometry. Prove the following:

   (a) A vector field v on $\mathbb{R}^n$ may be seen as a map $v : \mathbb{R}^n \to \mathbb{R}^n$; we say that the field is linear if v is a linear map. A linear field on $\mathbb{R}^n$, defined by a matrix A, is a Killing field if and only if A is anti-symmetric.

   (b) Let X be a Killing field on M, $p \in M$, and let U be a normal neighborhood of p on M. Assume that p is a unique point of U that satisfies $X(p) = 0$. Then, in U, X is tangent to the geodesic spheres centered at p.

   (c) Let X be a differentiable vector field on M and let $f : M \to N$ be an isometry. Let Y be a vector field on N defined by $Y(f(p)) = df_p(X_p(p))$, $p \in M$. (This is the *push-forward* of X under f.) Then Y is a Killing field if and only if X is also a Killing vector field.

   (d) X is Killing if and only if
   $$\langle \nabla_Y X, Z \rangle + \langle \nabla_Z X, Y \rangle = 0$$
   for all $Y, Z \in \mathfrak{X}(M)$. This equation is called the *Killing equation*.

   (e) Let X be a Killing field on M with $X(q) \neq 0$, $q \in M$. Then there exists a system of coordinates $(x_1, \dots, x_n)$ in a neighborhood of q, so that the coefficients g_{ij} of the metric in this system of coordinates do not depend on x_n.

   (Look at the textbook, on page 82, for the hints given for part (d) of this exercise.)

6. (do Carmo's text, Chapter 3, Exercise 6.) Let X be a Killing field (Cf. Exercise 5) on a connected Riemannian manifold M. Assume that there exists a point $q \in M$ such that $X(q) = 0$ and $\nabla_Y X(q) = 0$, for all $Y(q) \in T_q M$. Prove that $X \equiv 0$.

7. (do Carmo's text, Chapter 5, Exercise 8.) Let $\gamma : [0, a] \to M$ be a geodesic and let X be a Killing field on M.

   (a) Show that the restriction $X(\gamma(s))$ of X to $\gamma(s)$ is a Jacobi field along γ.

   (b) Use item (a) to show that (Cf. Exercise 6 of Chapter 3) if M is connected and there exists $p \in M$ with $X(p) = 0$ and $\nabla_Y X(p) = 0$, for all $Y(p) \in T_p M$, then $X = 0$ on M.