Homework set 8 - due 11/22/22

Math 5047

1. (Exercise 1, do Carmo, Chapter 6, page 139.) Let M_1 and M_2 be Riemannian manifolds, and consider the product $M_1 \times M_2$, with the product metric. Let ∇^1 be the Riemannian connection on M_1 and let ∇^2 be the Riemannian connection on M_2.

   (a) Show that the Riemannian connection ∇ of $M_1 \times M_2$ is given by

   $$\nabla_{Y_1+Y_2}(X_1 + X_2) = \nabla^1_{Y_1} X_1 + \nabla^2_{Y_2} X_2,$$

   where $X_1, Y_1 \in \mathfrak{X}(M_1)$, $X_2, Y_2 \in \mathfrak{X}(M_2)$.

   (b) For every $p \in M_1$, the set

   $$(M_2)_p = \{(p,q) \in M_1 \times M_2 : q \in M_2\}$$

   is a submanifold of $M_1 \times M_2$, naturally diffeomorphic to M_2. Prove that $(M_2)_p$ is a totally geodesic submanifold of $M_1 \times M_2$.

   (c) Let $\sigma(x, y) \subseteq T_{(p,q)}(M_1 \times M_2)$ be a plane such that $x \in T_pM_1$ and $y \in T_qM_2$. Show that $K(\sigma) = 0$.

2. (Exercise 2, do Carmo, Chapter 6, page 139.) Show that $\mathbf{x} : \mathbb{R}^2 \to \mathbb{R}^4$ given by

   $$\mathbf{x}(\theta, \varphi) = \frac{1}{\sqrt{2}}(\cos\theta, \sin\theta, \cos\varphi, \sin\varphi), \quad (\theta, \varphi) \in \mathbb{R}^2$$

   is an immersion of $\mathbb{R}^2$ into the unit sphere $S^3(1) \subseteq \mathbb{R}^4$, whose image $\mathbf{x}\left(\mathbb{R}^2\right)$ is a torus $\mathbb{T}^2$ with sectional curvature zero in the induced metric.

3. (Exercise 5, do Carmo, Chapter 6, page 139.) Prove that the sectional curvature of the Riemannian manifold $S^2 \times S^2$ with the product metric, where S^2 is the unit sphere in $\mathbb{R}^3$, is non-negative. Find a totally geodesic, flat torus, $\mathbb{T}^2$, embedded in $S^2 \times S^2$. (You may use the result of any of the other exercises from this chapter of the textbook that seem relevant.)

4. (Exercise 6, do Carmo, Chapter 6, page 140.) Let G be a Lie group with a bi-invariant metric. Let H be a Lie group and let $h : H \to G$ be an immersion that is also a homomorphism of groups (that is, H is a Lie subgroup of G). Show that h is a totally geodesic immersion.

5. (Exercise 11, do Carmo, Chapter 6, page 141.) Let $f : \overline{M}^{n+1} \to \mathbb{R}$ be a differentiable function. Define the *Hessian*, Hess f of F at $p \in \overline{M}$ as the linear operator

   $$\text{Hess } f : T_p\overline{M} \to T_p\overline{M}, \quad (\text{Hess } f)Y = \overline{\nabla}_Y \text{grad } f, \quad Y \in T_p\overline{M},$$

   where $\overline{\nabla}$ is the Riemannian connection of $\overline{M}$. Let a be a regular value of f and let $M^n \subseteq \overline{M}^{n+1}$ be the hypersurface in $\overline{M}$ defined by $M = \left\{p \in \overline{M} : f(p) = 1\right\}$. Prove that:

(a) The Laplacian $\overline{\Delta} f$ is given by

$$\overline{\Delta} f = \text{trace Hess } f.$$

(For the definition of the Laplacian, see Exercise 9, Chapter 3, page 83.)

(b) If $X, Y \in \mathfrak{X}\left(\overline{M}\right)$, then

$$\left\langle (\text{Hess } f) Y, X \right\rangle = \left\langle Y, (\text{Hess } f) X \right\rangle.$$

Conclude that Hess f is self-adjoint, hence determines a symmetric bilinear form on $T_p\overline{M}$, $p \in \overline{M}$, given by $(\text{Hess } f)(X, Y) = \left\langle (\text{Hess } f) X, Y \right\rangle$, $X, Y \in T_p\overline{M}$.

(c) The mean curvature H of $M \subseteq \overline{M} \subseteq M$ is given by

$$H = -\frac{1}{n} \text{div}\left(\frac{\text{grad } f}{|\text{grad } f|}\right).$$

(See page 142 for hints on this part of the problem.)

(d) Observe that every embedded hypersurface $M^n \subseteq \overline{M}^{n+1}$ is locally the inverse image of a regular value. Conclude from the last item of this problem that the mean curvature H of such a hypersurface is given by

$$H = -\frac{1}{n} \text{div } N,$$

where N is an appropriate local extension of the unit normal field on $M^n \subseteq \overline{M}^{n+1}$.