Homework set 9 - due 12/11/22

Math 5047

1. (do Carmo, Chapter 8, Exercise 2, page 180.) Show that if M^k is a (connected,) closed, totally geodesic submanifold of H^n, $k \le n$, then M^k is isometric to H^k. Determine all the totally geodesic submanifolds of H^n.

2. (do Carmo, Chapter 8, Exercise 4, page 181.) Identify $\mathbb{R}^4$ with $\mathbb{C}^2$ by letting (x_1, x_2, x_3, x_4) correspond to $(x_1 + ix_2, x_3 + ix_4)$. Let

$$S^3 = \left\{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\right\},$$

and let $h : S^3 \to S^3$ be given by

$$h(z_1, z_2) = \left(e^{\frac{2\pi i}{q}} z_1, e^{\frac{2\pi i r}{q}} z_2\right), \quad (z_1, z_2) \in S^3,$$

where q and r are relatively prime integers, $q > 2$.

   (a) Show that $G = \left\{\mathrm{id}, h, \cdots, h^{q-1}\right\}$ is a group of isometries of the sphere S^3, with the usual metric, which operates in a totally discontinuous manner. The manifold S^3/G is called a *lens space*.

   (b) Consider S^3/G with the metric induced by the projection $p : S^3 \to S^3/G$. Show that all the geodesics of S^3/G are closed but can have different lengths.

3. (do Carmo, Chapter 8, Exercise 8, page 185. *Riemannian submersions.*) A differentiable mapping $f : \overline{M}^{n+k} \to M^n$ is called *submersion* if f is surjective, and for all $\overline{p} \in \overline{M}$, the differential $df_{\overline{p}} : T_{\overline{p}}\overline{M} \to T_{f(\overline{p})}M$ has rank n. In this case, for all $p \in M$, the *fiber* $f^{-1}(p) = F_p$ is a submanifold of $\overline{M}$ and a tangent vector of $\overline{M}$, tangent to some F_p, $p \in M$, is called a *vertical vector* of the submersion. If, in addition, $\overline{M}$ and M have Riemannian metrics, the submersion f is said to be *Riemannian* if $df_p : T_p\overline{M} \to T_{f(p)}M$ preserves lengths of vectors orthogonal to F_p, for all $p \in \overline{M}$.

   (a) If $M_1 \times M_2$ is the Riemannian product, then the natural projections $\pi_i : M_1 \times M_2 \to M_i$, $i = 1, 2$, are Riemannian submersions.

   (b) If the tangent bundle TM is given the Riemannian metric as in Exercise 2 of Chapter 3, then the projection $\pi : TM \to M$ is a Riemannian submersion.

4. (do Carmo, Chapter 8, Exercise 9, page 186. *Connection of a Riemannian submersion.*) Let $f : \overline{M} \to M$ be a Riemannian submersion. A vector $\overline{x} \in T_{\overline{p}}\overline{M}$ is *horizontal* if it is orthogonal to the fiber. The tangent space $T_{\overline{p}}\overline{M}$ then admits a decomposition $T_{\overline{p}}\overline{M} = \left(T_{\overline{p}}\overline{M}\right)^h \oplus \left(T_{\overline{p}}\overline{M}\right)^v$, where $\left(T_{\overline{p}}\overline{M}\right)^h$ and $\left(T_{\overline{p}}\overline{M}\right)^v$ denote the subspaces of horizontal and vertical vectors, respectively. If $X \in \mathfrak{X}(M)$, the *horizontal lift* $\overline{X}$ of X is the horizontal field defined by $df_{\overline{p}}\left(\overline{X}(\overline{p})\right) = X(f(p))$.

   (a) Show that $\overline{X}$ is differentiable.

   (b) Let ∇ and $\overline{\nabla}$ be the Riemannian connections of M and $\overline{M}$, respectively. Show that

$$\overline{\nabla}_{\overline{X}}\overline{Y} = \overline{\nabla_X Y} + \frac{1}{2}\left[\overline{X}, \overline{Y}\right]^v, \quad X, Y \in \mathfrak{X}(M),$$

where Z^v is the vertical component of Z.

(c) $\left[\overline{X},\overline{Y}\right]^v(\overline{p})$ depends only on $\overline{X}(\overline{p})$ and $\overline{Y}(\overline{p})$.

See hints on page 186 of the textbook.

5. (do Carmo, Chapter 8, Exercises 10 and 11. **These won't be collected.** They are needed for the next exercise.)

6. (do Carmo, Chapter 8, Exercise 12, page 188. *Curvature of the complex projective space.*) Define a Riemannian metric on $\mathbb{C}^{n+1}\setminus\{0\}$ in the following way: If $Z\in\mathbb{C}^{n+1}\setminus\{0\}$ and $V,W\in T_Z(\mathbb{C}^{n+1}\setminus\{0\})$,

$$\langle V,W\rangle_Z = \frac{\mathrm{Real}(V,W)}{(Z,Z)}.$$

Observe that the metric $\langle\cdot,\cdot\rangle$ restricted to $S^{2n+1}\subseteq\mathbb{C}^{n+1}\setminus\{0\}$ coincides with the metric induced from $\mathbb{R}^{2n+2}$.

(a) Show that, for all $0\le\theta\le 2\pi$, $e^{i\theta}: S^{2n+1}\to S^{2n+1}$ is an isometry, and that, therefore, it is possible to define a Riemannian metric on $P^n(\mathbb{C})$ in such a way that the submersion f is Riemannian.

(b) Show that, in this metric, the sectional curvature of $P^n(\mathbb{C})$ is given by

$$K(\sigma) = 1+3\cos^2\varphi,$$

where σ is generated by the orthonormal pair X,Y, $\cos\varphi = \left\langle\overline{X}, i\overline{Y}\right\rangle$, and $\overline{X},\overline{Y}$ are the horizontal lifts of X and Y, respectively. In particular, $1\le K(\sigma)\le 4$.

See hints on page 189.