Open Billiards and Ergodic Theory

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1 Flashing light into a mirrored cave

Imagine a cave with a smooth, mirror-reflecting inner surface. Equipped with a laser pointer you flash a beam of light into the cave and wait for it to reemerge. Although not at all an essential point, we make the mathematically convenient assumption that the cave is two dimensional. See figure 1. Here are several questions you might like to ask:

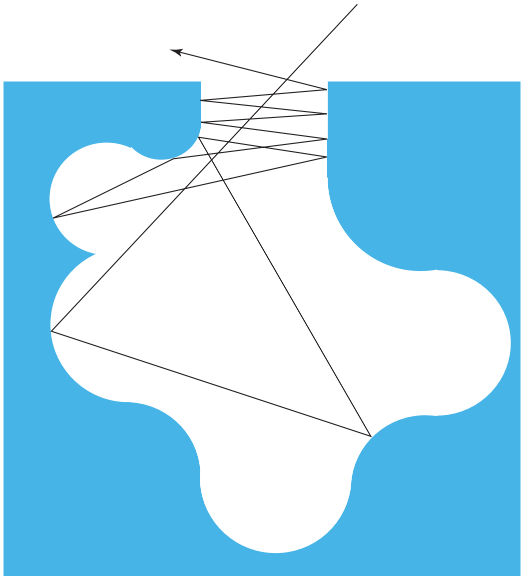

Figure 1: A light beam inside a mirrored cave.

- How long, on average, does it take for a light beam to reemerge?
- How many times, on average, does a light beam bounce around inside?

- What is the average duration of free flight between bounces?
- What is the likely angle at which the light ray will exit the cave?
- How do these quantities depend on the interior shape of the cave?

There are a few points to clarify before we can ask these questions in a more meaningful way. Here are some issues to deal with:

- What do we mean by "on average?"
- How do we know that light eventually reemerges from the cave?
- If light can get trapped inside, how likely is this to happen?
- What exactly should we mean by "likely?"
- Do we have any right to expect simple answers to these questions?

2 Trapped trajectories

Clearly, if "too many" trajectories remain inside forever, there is no sense in even asking about the time spent in the cave. (It would be infinite, a somewhat uninteresting answer since it does not reveal much about the interior shape of the cave.) Therefore, a reasonable place to begin our investigation is to ask: Can trajectories become trapped inside?

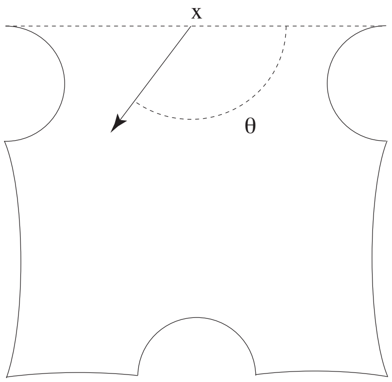

Figure 2: Can there be trapped trajectories for this example?

To study this possibility, we look at the example of figure 2. Notice that the cave is not really smooth. We will allow it to be piecewise smooth, with possibly a finite number of corners.

Before going further, ask yourself the question: Is it possible in this specific example for a light ray emitted from the outside of the dashed line to enter the cave and never again reemerge across the dashed line? If you think this is a possibility, how many initial conditions would lead to such trapped trajectories? Finitely many? Uncountably many?

I would like to convince you that there are uncountably many such trajectories. To be more precise, let us label the trajectory by the coordinates (x, θ) of the initial condition, where x designates the position along the open side (the dashed line) and θ the angle as shown in the figure. Let us assume that x lies in the interval $[0, 1]$, so the initial condition is a point on the rectangle $[0, 1] \times [0, \pi]$. Let $\mathcal{T}$ denote the subset of the rectangle corresponding to trapped rays. My claim is that $\mathcal{T}$ has the same cardinality as the rectangle itself. I won't offer you an exact proof of this fact, but I hope the argument will be sufficiently convincing that, with some work, you could produce your own proof based on it. (I convinced myself that this is true without going into all the details of a proof. I hope you will find my optimism justified after you see the outline provided below.)

Fix an angle θ, not too close to 0 or π, but otherwise arbitrary, and focus attention on the set

$$\mathcal{T}_\theta = \{x \in [0, 1] : (x, \theta) \in \mathcal{T}\}.$$

Proposition 2.1 *$\mathcal{T}_\theta$ contains a Cantor set, for all θ not too close to 0 or π. In particular, $\mathcal{T}$ is uncountable.*

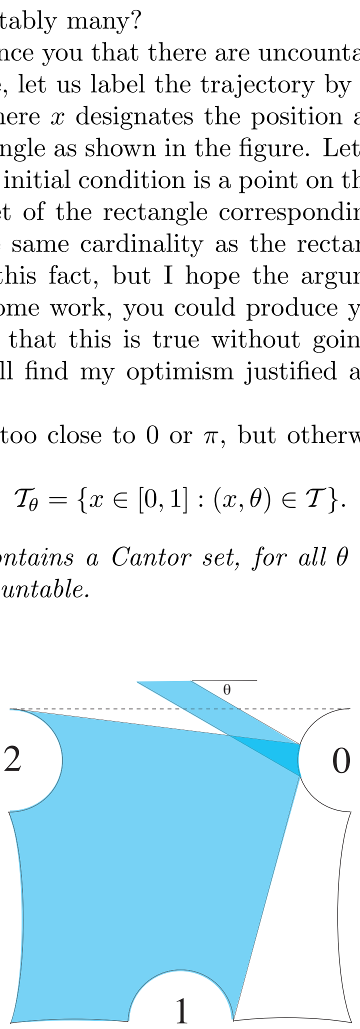

Figure 3: First stage in the construction of a Cantor set in $\mathcal{T}_\theta$.

Figure 3 shows the first stage in the construction of a Cantor set in $\mathcal{T}_\theta$. We label the three bumps inside the cave 0, 1 and 2, and shine a thick beam of light on bump 0. Light rays disperse and illuminate the two other bumps. Let J_0 denote the interval in $[0, 1]$ of initial positions of this thick beam.

Let I_{01} and I_{02} be disjoint closed intervals in J_0 such that a ray with initial position in I_{0i} and initial direction θ illuminates bump i after bouncing on 0, for $i = 1, 2$. (See the left-hand side of the figure 4.) Denote their union by $J_1 = I_{01} \cup I_{02}$.

In I_{01} there are disjoint closed subsets I_{010} and I_{012}, where I_{01i} consisting of initial positions of rays that first illuminate 0, then 1, then i, for $i = 0, 2$. Similarly, consider in I_{02} disjoint closed subsets I_{021} and I_{020}. The right-hand side of figure 4 shows the sets I_{021} and I_{020}. Define $J_2 = I_{010} \cup I_{012} \cup I_{021} \cup I_{020}$.

Continuing in this way we obtain for each positive integer n a family of 2^n closed disjoint intervals $I_{a_1 \dots a_n}$ where $a_i \in \{0, 1, 2\}$ and $a_i \neq a_{i-1}$ for each i from 2 to n. The key property characterizing $I_{a_1 \dots a_n}$ is that, if x belong to this interval, then (x, θ) is the initial condition of a light ray that first hits bump 0, then bumps a_1, a_2, …, a_n, after which point the trajectory is not specified. We denote the union of all these 2^n intervals by J_n.

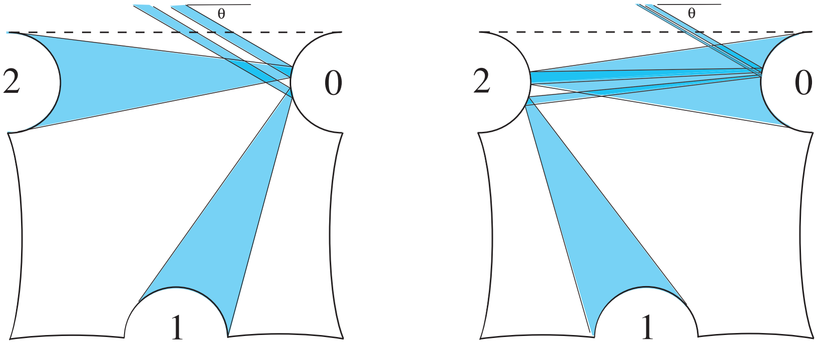

Figure 4: The second and part of the third stage in the construction of the Cantor set inside of $\mathcal{T}_\theta$.

The Cantor set we seek is given by

$$\mathcal{C} = \bigcap_{n=1}^{\infty} J_n.$$

For each $x \in \mathcal{C}$, the light ray with initial condition (x, θ) is indeed in $\mathcal{T}_\theta$ since it will forever bounce from one bump to another. You should try to convince yourself that given an infinite sequence $a_1, a_2, \dots$, then $\bigcap_{n=1}^{\infty} I_{a_1 \dots a_n}$ consists of exactly one point. (This intersection is non-empty by compactness!)

3 What is the probability of trapping?

We have seen that there is a large set of initial conditions for trajectories that go into the cave and never again reemerge from it. But if we were to pick such an initial condition at random, what is the probability that we would pick one that actually leads to trapping? The answer is given by the next proposition.

Proposition 3.1 *The probability of trapping is exactly* 0.

But I'm getting a bit ahead of myself. How do we define the probability of an event such as trapping in the first place? This requires introducing the phase space for this cave dynamical system and a probability measure on that space. From this point on I will use billiards-related terminology and make no further reference to specular speleology (the science of mirrored caves!) since the former is already the well established metaphor for this subject.

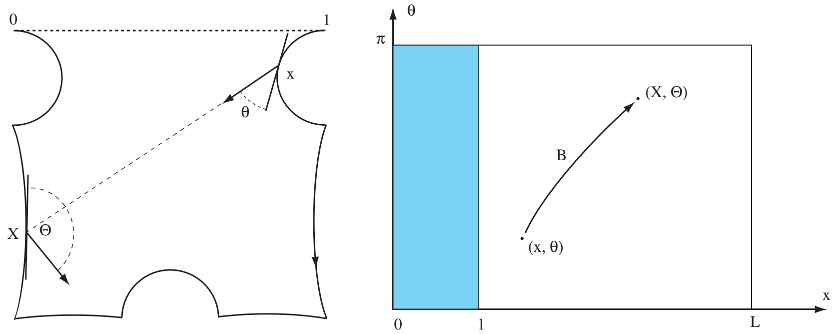

Figure 5: Billiard trajectories are described by the sequence of collisions with the table boundary. The rectangle on the right-hand side is the coordinate space describing the possible collisions. Angles represent direction of the billiard particle immediately after collision with the wall. The billiard map is represented by B. The shaded aread is E.

Suppose that the entire inner perimeter of the billiard table, including the length of the open side (the dashed line in figure 5), is L. We take the length of the open side to be 1 for simplicity, so that $L > 1$. The contour of the table (its inner wall plus open side) is a piecewise smooth curve of length L, which we orient clockwise so as to have the open side directed in the ordinary way as a subinterval of the real line. A trajectory of the system is completely specified by giving only the sequence of coordinates of collisions between the billiard particle and the boundary of the billiard table. The direction is represented by the angle $\theta \in [0, \pi]$ immediately after collision, as shown in figure 5, and the position is given by $x \in [0, L]$. If $x \in [0, 1]$, it is understood that the billiard particle lies on the open side, in which case θ is the angle of reflection after bouncing off an imaginary flat wall placed between 0 and 1.

We write $V = [0, L] \times [0, \pi]$ and call this set the *state space* (or phase space) of the billiard system. Successive collisions are described by a map

$$B : (x, \theta) \to (X, \Theta)$$

on V, as shown in figure 5. B is not generally continuous, even if the boundary contour is smooth, but away from a small subset of V, B is perfectly nice. The set of initial billiard states will be written as $E = [0, 1] \times [0, \pi] \subset V$. A trajectory of the billiard system is thus represented by a sequence

$$\xi \in E,\ B(\xi), B^2(\xi), B^3(\xi), \cdots \in V.$$

The trapping set $\mathcal{T}$ introduced earlier can now be described as the subset of states $\xi \in E$ such that $B^n(\xi)$ does not belong to E for all n. A natural way to assign a probability measure to a subset of V or E is to make this measure proportional to the area of the set. If $f(x, \theta)$ is any continuous function on V taking positive values on the interior of the rectangle, then assigning a probability measure to a subset $A \subset V$ by

$$P(A) = \int_A f(x, \theta) dx d\theta$$

will be just as reasonable since the notion of zero probability does not depend on the choice of f. For reasons that are explained shortly we adopt the function

$$f(x, \theta) = \frac{1}{2L} \sin \theta.$$

The constant in front of $\sin \theta$ is needed so that $P(V) = 1$. If we are measuring the probability of an event conditioned on it being in E, then we naturally take $f(x, \theta) = (1/2) \sin \theta$ so that $P(E) = 1$.

These preliminary remarks in place, the proposition now amounts to:

$$P(\mathcal{T}) = \frac{1}{2} \int_{\mathcal{T}} \sin \theta \ dx d\theta = 0.$$

The reason for choosing this particular function f is revealed in the following fundamental property of billiard maps:

Theorem 3.2 *The probability measure P given by $dP = (1/2) \sin \theta dx d\theta$ is invariant under the billiard map $B : V \to V$.*

The meaning of the theorem is as follows. Suppose that A is a subset of V and let $B^{-1}(A)$ be the set of states that in the next collision fall into A. Then

$$P(B^{-1}(A)) = P(A)$$

for an arbitrary event A. (If you know a little measure theory you are aware of the fact that some work is needed to specify the subsets A that qualify as legitimate probability events, i.e., for which a value $P(A)$ can be assigned. These are the so called *measurable subsets*. We will not concern ourselves with such details here.)

The proof of Theorem 3.2 is elementary but a little involved. I will simply ask you to trust me here or look at some book on dynamical systems. From the point of view of classical mechanics the theorem says that the billiard system is a conservative mechanical system. The invariance of the measure P under the billiard map B is a special case of a very general theorem in mechanics that goes by the name of *Liouville theorem*. Liouville's theorem says that the flow describing the evolution of the system in its phase space V, for a conservative mechanical system, is an incompressible flow with respect to the so-called *Liouville measure*. For a billiard system we do not have a flow, but a discrete-time system. The time evolution in the present case is obtained by iterating (taking powers of) the transformation B. We can say that B is "incompressible" in the sense that it can distort the shapes of sets in the reactangle V, but not change their "volumes" as defined by P.

We are now ready to prove proposition 3.1. It is, in fact, an immediate corollary of the simple but fundamental fact from ergodic theory given next, known as Poincaré recurrence. To apply the recurrence theorem to the case at hand, we close down the open side with a flat wall and imagine trajectories originating at that side as having just bounced off it from the inside of the table. By this way we can assign a history trajectory to each billiard state in E.

Theorem 3.3 (Poincaré recurrence) *Let $B : V \to V$ be a measure preserving transformation of a probability space V. Let P be the invariant probability measure on V and let E be a subset of V such that $P(E) > 0$. Then the trajectories starting from E that never return to E form a set of zero probability.*

Proof. For $N \geq 0$, let $E_N = \bigcup_{n=N}^{\infty} B^{-n}E$. This is the set of initial conditions in V whose trajectories eventually hit E after N collisions. Then $F = E \cap \bigcap_{N=0}^{\infty} E_N$ is the set of initial conditions in E whose trajectories return to E infinitely often. In other words, if $\xi \in F$, there is a sequence $0 < n_1 < n_2 < \dots$ such that $B^{n_i}(\xi) \in E$ for all i. Notice that for all i we have $B^{n_i}(\xi) \in F$ since $B^{n_i}(\xi)$ lies in E and returns to E infinitely many times under the iterations $B^{n_j - n_i}$, for all j. We claim that $P(F) = P(E)$. Since $B^{-1}(E_N) = E_{N+1}$, by invariance of the measure under B we have $P(E_N) = P(E_{N+1})$ for all N, so $P(E_N) = P(E_0)$ for all N. Therefore, $P(F) = P(E \cap E_0) = P(E)$ since $E \subset E_0$. $\square$

4 Averages

We have found that billiard particles originating at the open side of the table will return to it with probability exactly equal to 1. It is now time to return to our earlier question about average return time.

At this point we will add one important dynamical assumption that relates to the shape of the billiard table. We assume that (after closing the billiard table by adding a flat lid at the interval $[0, 1]$) the billiard transformation B is *ergodic*.

The definition of ergodicity is as follows. Let A be a subset of V such that $B^{-1}A = A$ (so that B maps A to itself). We say that A is an invariant set for B. The transformation is said to be ergodic with respect to the probability measure P if $P(A)$ can only take the values 0 or 1, for any invariant (measurable) A. In other words, it is not possible to partition V into invariant subsets without one of the subsets corresponding to an impossible event (i.e., having zero probability).

It can be shown, although it is not really an easy fact, that the map B for the table of figure 2 is ergodic. More generally, tables having convex sides (these are so called *dispersing* or *Sinai* billards) are ergodic. (Having one flat side as in the present situation is also fine since we can double the table at the flat side to obtain a dispersing billiard.) Ergodicity implies, as we will see in greater detail below, that the time spent by a typical trajectory of the system in a subset of the phase space, $A \subset V$, is equal to $P(A)$, where P is the invariant probability measure with respect to which B is ergodic. This is the kind of statistical regularity that allows us to obtain the value of such quantities as the mean return time to the open side of the billiard table.

But first, let us review some notation and introduce some more. We have so far assumed that the length of the open side is 1. To be a little more general let this length be equal to e. Then $L > e$ and $E = [0, e] \times [0, \pi]$ represents the part of the phase space for the flat side. The scalar velocity of the billiard particle will be taken to be u. This value remains constant during the entire trajectory.

For each $(x, \theta) \in E$, let $S(x, \theta)$ and $N(x, \theta)$ be, respectively, the time of first return and the number of collisions before returning to the open side, counting the arrival as one collision. The entire phase space is V. For any $(x, \theta) \in V$, let $\tau(x, \theta)$ denote the time duration of free flight with initial condition (x, θ) to the point of next collision. The average values of N, S are defined by

$$\langle N\rangle_E := \frac{1}{2e}\int_0^e \int_0^\pi N(x,\theta) sin\theta \, d\theta dx;$$
$$\langle S\rangle_E := \frac{1}{2e}\int_0^e \int_0^\pi S(x,\theta) sin\theta \, d\theta dx.$$

Similarly, the average of τ (over V) is

$$\langle \tau\rangle_V := \frac{1}{2L}\int_0^L \int_0^\pi \tau(x,\theta) sin\theta \, d\theta dx,$$

where L is the total perimeter of the billiard table, including the open side.

Theorem 4.1 *Suppose that the billiard system is ergodic. Then*

1. $\langle N\rangle_E = L/e$;
2. $\langle S\rangle_E = A\pi/eu$;
3. $\langle S\rangle_E = \langle N\rangle_E \langle \tau\rangle_V$.

We will discuss the proof later. As an illustration of this result, consider the two billiard tables of figure 6.

5 Birkhoff's ergodic theorem

The key ingredient in the proof of theorem 4.1 is Birkhoff's ergodic theorem. This theorem, and the notion of ergodicity, belong to the foundations of classical

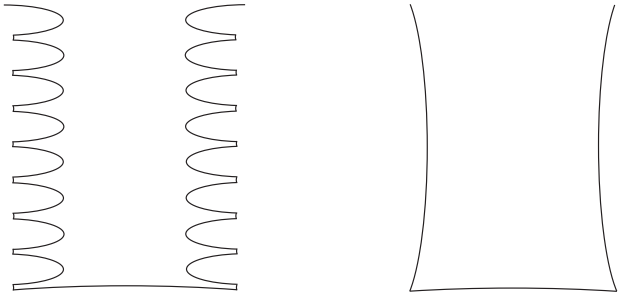

Figure 6: The average time, $\langle S\rangle_E$, is nearly the same for the two geometries shown above, although the average number of collisions, $\langle N\rangle_E$, is over three times greater for the table on the left. As the number of bumps on the boundary of the table on the left grows (assuming same height and same base), $\langle S\rangle_E$ stays nearly constant while $\langle N\rangle_E$ grows approximately linearly.

statistical physics. The overall idea behind it is that, if the system is ergodic, almost every trajectory (i.e., with probability 1) have the same statistical behavior in the following sense: averaging quantities along trajectories gives the average of those quantities over the entire phase space. The theorem is, in fact, a general statement of the law of large numbers applied to dynamical systems with invariant probability measures. The following version of the theorem (not stated in the most general form) makes this precise. We say that a function $N : V \to \mathbb{R}$ is integrable if $\int_V |N(\xi)|dP(\xi) < \infty$.

Theorem 5.1 (Birkhoff) *Let V be a probability space with probability measure P, and let $B : V \to V$ be a measure preserving transformation. Suppose that $N : V \to \mathbb{R}$ is an integrable function on V. Also assume that B is ergodic. Then, for all $\xi \in V$, except of a subset of V of zero measure, the time-average N along the orbit of ξ converges to the average of N over V; that is,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} N(B^i(\xi)) = \int_V N(\xi)dP(\xi).$$

The proof of Birkhoff's theorem can be found on any text in ergodic theory. I will only give here the sketch of a simple example to illustrate its use. I invite you to fill in the details.

The experiment of picking a number between 0 and 1 at random (with the uniform distribution) serves as a mathematical model of flipping coins an infinite number of times, since we can regard the binary expansion of $x = 0.a_1a_2a_3\ldots$, $a_i \in \{0, 1\}$, as a sequence of heads and tails (say, heads = 0 and tails = 1). For this to be a sensible mathematical definition of coin tossing, we need the average to be exactly 1/2. We can show this by applying Birkhoff's theorem, which in this case is nothing but the law of large numbers. Let $B : [0, 1) \to [0, 1)$ be the map $B(x) = 2x - [2x]$, where $[y]$ represents the integer part of y. Notice that

$B(0.a_1a_2\dots) = 0.a_2a_3\dots)$. With the help of basic Fourier series you can show that B is ergodic. (Given an invariant set A, let χ_A denote its characteristic function. Now express χ_A in Fourier series and use the fact that this is an invariant function, that is, $\chi_A \circ B = \chi_A$. From this, conclude that all but the constant term in the Fourier series is 0 so χ_A is a constant function, which must, therefore, be either 0 or 1.) Let $N : [0, 1) \to \{0, 1\}$, $N(x) = a_1$, denote the value of the first binary digit of numbers in $[0, 1)$. Notice that $\int_0^1 N(x)dx = 1/2$ since $N(x)$ is 0 on the first half of the unit interval and 1 on the second half. Then the fraction of tails in a long trial of coin tosses is

$$\begin{aligned}\lim_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n}{n} &= \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} N(B^i(x))\\ &= \int_0^1 N(x)dx\\ &= \frac{1}{2}.\end{aligned}$$

6 Proof of theorem 4.1

Let $B : V \to V$ be the billiard map that includes the top flat side. The first return to E is given by a map $T : E \to E$. The billiard flow will be denoted φ_t. Elements of V will be written $\xi = (x, \theta)$. Note that

$$S(\xi) = \tau(\xi) + \tau(B(\xi) + \cdots + \tau(B^{N(\xi)-1}(\xi)).$$

For each $\xi \in E$ and positive integer l, define

$$\begin{aligned}N^l(\xi) &:= N(\xi) + N(T(\xi)) + \cdots + N(T^l(\xi));\\ S^l(\xi) &:= S(\xi) + S(T(\xi)) + \cdots + S(T^l(\xi)).\end{aligned}$$

Then $N^l(\xi)$ is the total number of collisions with the table boundary during the period of l returns to the flat side, and $S^l(\xi)$ is the total time during the same period. We have $\lim_{l\to\infty} N^l(\xi)/l = \langle N\rangle_E$ and $\lim_{l\to\infty} S^l(\xi)/l = \langle S\rangle_E$. Therefore, for all $\xi \in E$ but for a set of zero probability,

$$\begin{aligned}\langle N\rangle_E^{-1} &= \lim_{l\to\infty} \frac{l}{N^l(\xi)}\\ &= \lim_{l\to\infty} \frac{1}{N^l(\xi)} \sum_{i=0}^{N^l(\xi)} \chi_E(B^i(\xi))\\ &= P(E)\\ &= \frac{\text{length of flat side}}{\text{total perimeter of table boundary}}.\end{aligned}$$

This shows (1). To obtain (3), start with

$$\sum_{k=0}^{N^l(\xi)} \tau(B^k(\xi)) = S(\xi) + S(T(\xi)) + \cdots + S(T^{l-1}(\xi))$$

and average both sides over E, using T-invariance of P. This gives:

$$\left\langle \sum_{k=0}^{N^l(\xi)} \tau(B^k(\xi)) \right\rangle_E = l\langle S\rangle_E.$$

Consequently,

$$\begin{aligned}\langle S\rangle_E &= \lim_{l\to\infty} \left\langle \left(\frac{N^l(\xi)}{l}\right) \left(\frac{1}{N^l(\xi)} \sum_{k=0}^{N^l(\xi)} \tau(B^k(\xi))\right) \right\rangle_E \\ &= \langle N\rangle_E \langle \tau\rangle_V.\end{aligned}$$

To show (2), it is convenient to first introduce the collar region U_h shown in Figure 7, where h is a small positive number.

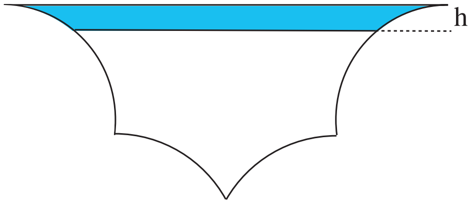

Figure 7: Define U_h as the strip of width h with the distinguished side as one of the boundary lines.

Except for a set of small measure (which goes to zero with h), the time it takes for the ray with initial condition $\xi = (x, \theta) \in E$ to traverse U_h is $\eta(\xi) = h/u \sin\theta$. An explicit integral calculation gives

$$\lim_{h\to 0} \frac{1}{h} \langle \eta\rangle_E = \pi/2u.$$

We can now conclude that, for all $\xi \in E$ except for a set of zero probability:

$$\begin{aligned}
\langle S\rangle_E \frac{\text{length of flat side}}{\text{area of billiard cell}} &= \lim_{h\to 0} \lim_{m\to\infty} \left(\frac{S^m(\xi)}{m}\right) \left(\frac{1}{hS^m(\xi)} \int_0^{S^m(\xi)} \chi_{U_h}(\varphi_t(\xi))\right) dt \\
&= \lim_{h\to 0} \lim_{l\to\infty} \frac{1}{hm} \int_0^{S^m(\xi)} \chi_{U_h}(\varphi_t(\xi)) dt \\
&= \lim_{h\to 0} \lim_{l\to\infty} \frac{1}{hm} \sum_{i=0}^{m-1} 2\eta(T^i(\xi)) \\
&= \lim_{h\to 0} \frac{2}{h} \langle \eta \rangle_E \\
&= \frac{\pi}{u}.
\end{aligned}$$

This gives the average value of S claimed in (2).

References

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