ERGODIC THEORY AND DYNAMICS OF G-SPACES (with special emphasis on rigidity phenomena)

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Contents

CHAPTER 1

Introduction

1. Dynamics of group actions in mathematics and applications

The theory of dynamical systems deals with properties of groups or semi-groups of transformations that are asymptotic in character, that is, that become apparent as "one goes to infinity in the group." Since its beginnings in classical and statistical mechanics, the theory has been dedicated primarily to actions of the additive groups $\mathbb{R}$ and $\mathbb{Z}$. These groups are typically interpreted as parametrizing time - the corresponding actions usually represent the time evolution of a system, respectively as a flow or by the iteration of an invertible transformation.

Somewhat more recently, dynamical methods and ideas began to be applied to actions of more general groups. To cite a few examples, the work of G. Mackey shows an important role played by ergodic theory in the study of unitary representations of general (second countable locally compact) groups. (See [**Ma2**]; see also [**Va1**] for the relevance of Mackey's work to the foundations of quantum theory.) In probability theory, actions of $\mathbb{Z}^n$ arise in the context of the statistical physics of ferromagnetic materials and the Lenz-Ising spin lattices. (See [**Rue**] and references therein.) A certain action of $SL(2,\mathbb{R})$ on Teichmuller space plays a central role in the study of rational billiards (see survey [**S-MT**]), whereas the dynamics of homogeneous actions of unipotent and other subgroups of certain semisimple Lie groups has been applied with great success to problems in number theory. (See survey [**S-KSS**].) Also, the study of the geometry and topology of manifolds of nonpositive curvature, as well as the study of linear representations of lattices in semisimple Lie groups propelled the development of the ergodic theory of noncompact semisimple Lie groups and their lattice subgroups, a topic that will be explored in some detail in the present survey. Seminal work in this direction was done by Mostow, Margulis, and Furstenberg. See [**Mos, Mar2, Z1**].

According to the general scheme presented in [**S-HK**] we will consider actions of a group G on a set X that leave invariant some structure on X such as, say, a topology, a finite or infinite measure or a measure class, a smooth manifold structure, or a symplectic form. The various branches of dynamics correspond to putting the focus on each of these different structures, respectively: ergodic theory, topological, smooth, symplectic, holomorphic, algebraic (homogeneous and affine) dynamics and so on.

Of course, these are not fully independent: for example, differentiable structure produces topology and symplectic systems preserve a smooth volume form and hence a canonical (Liouville) measure; similarly, homogeneous systems may preserve a Haar measure.

2. Properties of groups relevant to dynamics

a. Amenable vs. non-amenable groups. In the context of the general theory of dynamical systems, the acting groups should not be too big: locally compact second countable is a standard assumption. Following [**S-HK**] we will call actions of $\mathbb{Z}$, $\mathbb{Z}_+$, $\mathbb{R}$ or $\mathbb{R}_+$ *cyclic dynamical systems*. Most of the surveys in this volume deal either exclusively or primarily with such systems. Certain general facts about cyclic dynamical systems that are discussed in this volume, particularly in [**S-HK**], either extend directly to, or have counterparts for, actions of locally compact second countable groups. Some topics that do not have a ready extension to this general class of groups as, for example, entropy theory and ergodic theorems, can be extended to certain classes of groups that share essential properties with the above "small" ones, such as amenability. (See Section 2.5 and [**S-HK**].)

Surveys [**S-B, S-LS, S-KSS**] pay considerable attention to noncyclic dynamical systems although in the first two cases actions of amenable groups and semigroups are primarily considered.

In what follows we will restrict the discussion to the case of group, as opposed to semigroup, actions. There are various technical complications that arise in passing from invertible to non-invertible actions and in general the non-amenable non-invertible case has not been given sufficient attention as yet to warrant inclusion into a general survey like this one.

The primary focus of the survey is on those aspects of the ergodic theory and differentiable dynamics of group actions that are most distinct from the theory for $\mathbb{R}$ and $\mathbb{Z}$. Since ergodic theory for actions of general amenable groups share with $\mathbb{R}$ and $\mathbb{Z}$ many key properties, this survey will be concerned in large part with actions of non-amenable groups. Orbit equivalence provides a particularly compelling example: on the one hand, all finite measure–preserving ergodic actions of discrete amenable groups are orbit equivalent [**CFW**]; on the other hand, for certain groups, which are both "sufficiently large" and "rigid", orbit equivalence essentially implies isomorphism (cf. Sections 6.2c and 6.2d).

b. Complexity of the group structure. An underlying theme that runs through this survey is the influence that the greater complexity of the acting group itself has on the dynamical properties of the actions. That complexity presents itself in several ways. Amenability, for example, is associated with low complexity in the sense of moderate (volume) growth in the group, or in the limited "ways" or "directions" along which one can go out to infinity within the group. This can be expressed more precisely by the various notions of group boundary. (See [**Kai**] for a survey on the various probabilistic and algebraic notions of group boundaries.) The very existence of invariant probability measures for an action is an issue connected with properties of these boundaries, as will be seen later. (See Chapter 4, for a brief discussion of H. Furstenberg's work on boundary actions, and Chapter 6, on the work by A. Nevo and R. Zimmer on the structure of actions of higher rank semisimple Lie groups without invariant measures.)

Yet another indication of complexity in the group structure is seen in the way subgroups are "interlocked" within the group. For example, for a simple Lie group, dynamical properties of the restriction of the action to a maximal abelian $\mathbb{R}$-diagonalizable subgroup (an $\mathbb{R}$-split Cartan subgroup) goes a long way to determine properties of the action of the whole group. (The Howe-Moore ergodicity theorem illustrates this point well. See Section 3.3 of Chapter 2.) This is specially true when the $\mathbb{R}$-split Cartan subgroup has rank (as an abelian group) two or greater, as will be seen in many of the rigidity phenomena for "higher-rank" actions discussed later in this survey.

The theory of unitary group representations is an effective tool for relating the structure of a group with the dynamics and ergodic theory of its actions. Kazhdan's property T (Section 3.2) is a prominent example of a property defined by means of the unitary representations of the group that has considerable (although not fully understood as yet) ergodic theoretic implications. Property T is a very useful ingredient in the analysis of actions of more special classes of groups, such as semisimple Lie groups of real-rank greater than one.

Amenability can also be formulated in terms of the unitary representations of the group, and it is, in fact, a natural opposite to property T. For example, the only amenable groups with property T are compact groups, which are trivial from the point of view of asymptotic behavior.

Connected semisimple Lie groups of real-rank greater than one possess all the attributes of complexity described above: a complicated web of Cartan subgroups, responsible for a rich algebraic structure of the boundary at infinity (Bruhat-Tits buildings), Kazhdan's property T, and in general a theory of irreducible unitary representations that has very different properties compared to amenable groups. Among the non-amenable groups, higher real-rank semisimple Lie groups and their lattices are the ones for which both measurable and smooth dynamics are best understood. As a result, such groups will have a prominent place in our discussion. (See Chapter 6.)

3. Rigidity phenomena

a. The various types of rigidity. The term "rigidity" (of actions of non-compact groups) will be applied to a number of situations.

First, there is rigidity of orbit structure, which refers to the situation in which a limited collection of models, usually of an algebraic nature, describes the orbit structure of a particular class of group actions up to the appropriate notion of isomorphism for the category under consideration, such as measurable, topological, or differentiable. This includes

(i) global rigidity, when the class of actions under study is defined by global conditions of a measure-theoretic (e.g. ergodicity), topological (e.g. topology of the G-space or a homotopy type of the action), or dynamical nature (e.g. some form of hyperbolicity), and

(ii) local rigidity, when small perturbations of a given action in various natural topologies are considered.

There is also rigidity of standard constructions performed over a given action, such as time change, group extensions, induced actions and so on. These constructions are closely related to cocycles over group actions, and their corresponding rigidity properties are often expressed in cohomological terms. (See Section 2.3 and [**S-HK**], Sections 1.3, 2.2, 3.4.)

The term rigidity will also be used to describe situations in which a weaker structure determines a stronger one within a given class of actions. For example, two actions of a given group that are *a priori* only measurably isomorphic might turn out to be differentiably conjugate, given that both belong to the class of, say, volume preserving smooth actions. In this case the measurable structure of the class of actions would be said to be rigid.

b. Rigidity in actions of $\mathbb{Z}$ and $\mathbb{R}$. Some of the types of rigidity phenomena mentioned above appear already for cyclic dynamical systems. Here is an assortment of examples drawn from elliptic, parabolic, and hyperbolic dynamics:

(1) The discrete spectrum theorem [**S-HK**, Theorem 3.6.3] implies that among minimal translations on compact abelian groups, measurable isomorphism is equivalent to topological and, in the case of tori, differentiable conjugacy; furthermore, the measurable isomorphism class is determined by the spectral data;
(2) every vector–valued smooth cocycle over a Diophantine translation on a torus is cohomologous to a constant cocycle [**S-HK**, Proposition 7.3.2];
(3) the invariant curve theorem for twist maps and more general KAM type results assert local differentiable rigidity of the orbit structure of a non–degenerate completely integrable system on a large invariant subset in the phase space (the union of Diophantine tori);
(4) any two minimal unipotent affine maps of a torus which are measurably isomorphic are in fact affinely conjugate; measurable isomorphism class is not determined by the spectral data [**Ab**];
(5) for horocycle flows on surfaces of constant negative curvature, as well as their time changes and other actions of unipotent groups on homogeneous spaces, a measurable isomorphism has to be essentially smooth and, in the homogeneous case, algebraic [**S-KSS**];
(6) Anosov diffeomorphisms, expanding maps and restrictions of diffeomorphisms to locally maximal hyperbolic sets are structurally stable (C^0 rigidity of the orbit structure)[**S-H**];
(7) any Anosov diffeomorphism of a torus, or, more generally, an infranilmanifold, is topologically conjugate to an (algebraic) automorphism [**S-H**];
(8) any symplectic Anosov diffeomorphism with smooth (C^∞) stable and unstable foliations is differentiably conjugate to an automorphism of an infranilmanifold [**BL2**];
(9) contact Anosov flows with smooth stable and unstable foliations are differentiably conjugate to algebraic models given by geodesic flows on compact locally symmetric spaces of noncompact type [**BFL**].

For cyclic systems, C^1 local rigidity never takes place [**S-H**], and it appears extremely unlikely that there are C^r locally rigid systems for any $2 \le r \le \infty$. As will be seen in Chapter 6 on the other hand, local differentiable rigidity is quite common for actions of semisimple Lie groups and their lattices.

c. Rigidity for higher real–rank semisimple Lie groups and lattices. Rigidity phenomena of various kinds are prevalent and relatively well understood for actions of higher real–rank semisimple Lie groups and for lattices in such groups. For such actions, rigidity appears in the measurable, topological, and differentiable categories. Some of the main results will be presented later in this survey.

A central result in the ergodic theory of actions of higher real–rank semisimple Lie groups and their lattices is Zimmer's *Cocycle Superrigidity Theorem.* The theorem is an extension to G-spaces of the celebrated work of G. Margulis on the rigidity of linear representations of lattices in higher rank semisimple groups. Zimmer's theorem asserts that any measurable cocycle over a finite measure preserving action of one of these higher–rank groups, taking values in another Lie group H, is cohomologous to a constant cocycle, modulo a cocycle into a compact subgroup of H. The theorem has numerous applications, in the general context of measurable ergodic theory as well as for smooth actions. A detailed sketch of the proof is provided in Chapter 6.

Topological superrigidity is a close counterpart of the cocycle superrigidity with assumptions about recurrence replacing existence of a finite invariant measure.

Local differentiable rigidity has been established for most standard classes of algebraic volume–preserving actions of higher–rank semisimple Lie groups and lattices.

There are also a number of global classification results, as well as conjectures pointing to a detailed description of such actions. [**MQ, GS2**]

d. Rigidity for actions of other groups. Outside the domain of higher–rank semisimple Lie groups and their lattices rigidity phenomena are not as well understood and are probably less prevalent.

First, there are some rank-one simple Lie groups (e.g. $Sp(n, 1)$) which, together with their lattices, possess property T and complicated combinatorial structure. For actions of such groups versions of the cocycle superrigidity have been established. However, implications beyond the measurable category have not been fully explored.

It is quite remarkable that certain classes of actions of higher rank *abelian* groups also display rigidity behavior in a prominent way. This is the case for actions that are "sufficiently large," in the sense, for example, that the group contains many elements that act with some form of hyperbolicity. The underlying cause of rigidity in such cases is the interplay of the hyperbolic dynamics of these various elements. As it turns out, there are often only a few ways in which such an interplay can be arranged. Results in this direction include local and, to a lesser extent global, differentiable rigidity of the orbit structure, cocycle rigidity for regular (e.g. smooth or Hölder) cocycles, and situations in which measurable orbit structure determines the smooth one, not unlike items 1, 4 and 5 in the previous subsection. However, since abelian groups are amenable, there are no universal rigidity results like the cocycle superrigidity at a measurable or continuous level.

In a subsequent volume of this series a separate survey will be dedicated to rigidity phenomena for actions of amenable, primarily abelian groups, and related mostly with hyperbolic behavior.

4. Rigid geometric structures

Many of the main classes of smooth dynamical systems are defined as groups of automorphisms of particular geometric structures such as, for example, a smooth volume form, a symplectic form, a complex structure, or a contact structure, having the common feature that their total (pseudo) groups of (local) automorphisms are infinite–dimensional. There is another class of geometric structures for which these (pseudo)groups are finite–dimensional, that is, Lie (pseudo) groups. Following Gromov [**Gro**] (see Chapter 5) such geometric structures are called *rigid*. The homogeneous structures which form the basis of algebraic dynamics are of that type, as well as such classical structures as Riemannian and pseudo–Riemannian metrics, affine connections, conformal Riemannian and pseudo–Riemannian metrics in dimension greater than two, and projective structures.

Rigidity behavior is also seen in actions that preserve such Gromov-rigid geometric structures. (In many results in this direction the acting group is not specified in advance, as will be seen in Chapter 5.) It is natural to expect that homomorphisms of a given group G into the automorphism group of a Gromov-rigid geometric structure are rather restricted. One can ask, for example, which groups allow homomorphisms of this kind, or when the actions defined by these homomorphisms possess dynamical properties such as hyperbolicity. While these questions have not been seriously discussed in full generality there are results covering special cases, some of which will be discussed later in the survey.

Also note that items 6 and 7 in Section 1.3b above are examples of rigidity of dynamical systems preserving a rigid geometric structure. While the symplectic or contact structures as such are not rigid, smoothness of stable and unstable foliations provides additional data that allow to define an invariant affine connection (given in fact by a pseudo–Riemannian metric), which is rigid.

5. Preliminaries on Lie groups and lattices

The survey will by necessity make use of a variety of basic facts about Lie groups, Lie algebras, algebraic groups and lattice groups that are not standard lore in the theory of cyclic dynamical systems. In this section we provide basic definitions and background, although it is likely that many readers will find the information insufficient. This can be augmented with the material in Chapter 1 of [**S-KSS**], which contains a concise overview of the most basic facts about Lie groups and homogeneous spaces. There are many further sources that could be used alongside the present text; we mention, for example, [**OV2**] for the general theory of Lie groups and algebraic groups and [**Bor**] and [**Rag**], for general information about lattices in semisimple Lie groups.

a. Semisimple Lie groups and algebras.

1. *Reductive and semisimple Lie groups and Lie algebras.* We only consider here linear Lie groups, that is, (real) subgroups of $GL(n,\mathbb{C})$. This assumption simplifies some of the definitions given below. The more standard definitions can be found in any text on Lie groups and Lie algebras, such as [**Va2**], for example.

Let A^* denote the complex conjugate transpose of A. The *Cartan involution* of $GL(n,\mathbb{C})$ is the homomorphism $\Theta : A \mapsto (A^*)^{-1}$. The Cartan involution of $\mathfrak{gl}(n,\mathbb{C})$ (the Lie algebra of $GL(n,\mathbb{C})$) is the Lie algebra isomorphism induced from Θ, and is given by $\theta : X \mapsto -X^*$.

Let G be a connected Lie subgroup of $GL(n,\mathbb{C})$. We say that G is a *reductive* group if it is conjugate to a subgroup that is stable under the Cartan involution Θ. In other words, G is reductive if there is $g \in GL(n,\mathbb{C})$ such that gGg^{-1} is mapped into itself by Θ. A Lie algebra $\mathfrak{g} \subset \mathfrak{gl}(n,\mathbb{C})$ is reductive if it is conjugate by an element in $GL(n,\mathbb{C})$ to a θ-stable subalgebra. In particular, G is reductive if and only if $\mathfrak{g}$ is.

We recall that the center of a group G is the subgroup

$$Z(G) = \{a \in G \mid ag = ga \text{ for all } g \in G\}.$$

The center is clearly a normal subgroup of G.

A subalgebra $\mathfrak{n} \subset \mathfrak{g}$ is an *ideal* if $[X,Y] \in \mathfrak{n}$ for all $X \in \mathfrak{n}$ and all $Y \in \mathfrak{g}$. The center of a Lie algebra $\mathfrak{g}$ is the subalgebra

$$\mathfrak{z}(\mathfrak{g}) = \{X \in \mathfrak{g} \mid [X,Y] = 0 \text{ for all } Y \in \mathfrak{g}\}$$

and it is an ideal of $\mathfrak{g}$.

A Lie algebra $\mathfrak{g} \subset \mathfrak{gl}(n,\mathbb{C})$ is *semisimple* if it is reductive and has trivial center. $G \subset GL(n,\mathbb{C})$ is a *semisimple Lie group* if its Lie algebra is semisimple.

A Lie algebra is said to be *simple* if its only ideals are $\{0\}$ and itself. A Lie group will be called *simple* if its Lie algebra is simple. When dealing with algebraic groups (see 1.5c), we will use a somewhat more strict definition, namely, the (algebraic k-) group will be called *k-simple* (where k is some number field) if every proper normal k-subgroup is trivial. If every proper normal k-subgroup is finite, we say that the group is *almost k-simple.*

2. *Real rank.* The (complex) rank of a semisimple Lie algebra is the maximal dimension of an abelian subalgebra which is diagonalizable over $\mathbb{C}$. The complex rank of a semisimple Lie group is the rank of its Lie algebra. In the standard notations for classical and exceptional simple Lie groups and algebras (like A_n or E_6) the index is equal to the complex rank. Since θ is an involution, i.e. $\theta^2 = \mathrm{id}$, its only eigenvalues are 1 and -1. We define subspaces $\mathfrak{k}$ and $\mathfrak{p}$ of the θ-stable Lie algebra $\mathfrak{g}$ as follows:

$$\begin{aligned} \mathfrak{k} &:= \{X \in \mathfrak{g} \mid \theta(X) = X\} \\ \mathfrak{p} &:= \{X \in \mathfrak{g} \mid \theta(X) = -X\}. \end{aligned}$$

Since θ is a Lie algebra automorphism, $\mathfrak{k}$ is a Lie subalgebra. Let K be the subgroup of G fixed by Θ. Its Lie algebra is clearly $\mathfrak{k}$.

As a vector space, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Introduce in $\mathfrak{g}$ an inner product by

$$\langle X,Y \rangle = -\mathrm{Re}(\mathrm{Tr}(\mathrm{ad}(X) \circ \mathrm{ad}(\theta Y)))$$

where $\mathrm{ad}(X)$ is the linear map on $\mathfrak{g}$ defined by $\mathrm{ad}(X)Z = [X,Z]$. For each $X \in \mathfrak{p}$, the operator $\mathrm{ad}(X)$ on $\mathfrak{g}$ is self-adjoint with respect to the given inner product. Therefore $\mathrm{ad}(X)$ is diagonalizable with real eigenvalues. Let $\mathfrak{a}$ be a maximal abelian algebra in $\mathfrak{p}$. More precisely, $\mathfrak{a}$ is abelian and is not properly contained in a subspace of $\mathfrak{p}$ consisting of commuting elements. The operators $\mathrm{ad}(X)$, $X \in \mathfrak{a}$, commute since

$$0 = \mathrm{ad}([X,Y]) = \mathrm{ad}(X) \circ \mathrm{ad}(Y) - \mathrm{ad}(Y) \circ \mathrm{ad}(X)$$

for $X, Y \in \mathfrak{a}$. Therefore, it is possible to find a basis for $\mathfrak{g}$ which simultaneously diagonalizes all the operators $\mathrm{ad}(X)$, $X \in \mathfrak{a}$. The subalgebra $\mathfrak{a}$ will be called an *$\mathbb{R}$-split Cartan subalgebra* of $\mathfrak{g}$. It is a maximal abelian $\mathbb{R}$-diagonalizable subalgebra of $\mathfrak{g}$. The dimension of $\mathfrak{a}$ is called the *real rank* of $\mathfrak{g}$. This definition seems to depend on the choice of $\mathfrak{a}$ in $\mathfrak{p}$. It turns out, however, that any two such subalgebras are conjugate by an element of K, so that their dimensions are the same. Again the real rank of a semisimple Lie group is defined as the real rank of its Lie algebra.

It is thus obvious that the real rank does not exceed the complex rank. Compact semisimple Lie algebras are exactly those of real rank 0. At the opposite end lie *split* Lie algebras for which the real rank is equal to the complex rank.

It is not hard to show, for example, that the real rank of $SL(n,\mathbb{R})$ is $n-1$ and hence $SL(n,\mathbb{R})$ is split.

3. *Root space decomposition.* Let $\mathfrak{a}$ be an $\mathbb{R}$-split Cartan subalgebra contained in $\mathfrak{p}$. Denote by $\mathfrak{a}^*$ the space of real linear functionals on $\mathfrak{a}$. We now define

$$\mathfrak{g}_\lambda := \{X \in \mathfrak{g} \mid [H, X] = \lambda(H)X, \text{ for all } H \in \mathfrak{a}\}.$$

If $\lambda \in \mathfrak{a}^*$ is nonzero and $\mathfrak{g}_\lambda$ is nonzero, we say that λ is a *root* of $(\mathfrak{a}, \mathfrak{g})$ (or a *restricted root* of $\mathfrak{g}$), with associated root space $\mathfrak{g}_\lambda$. The set of all such roots is denoted $\Phi(\mathfrak{a}, \mathfrak{g})$. We denote by $\mathfrak{g}_0$ the centralizer of $\mathfrak{a}$ in $\mathfrak{g}$, i.e., $\mathfrak{g}_0$ is the subspace of all X in $\mathfrak{g}$ such that $[X, H] = 0$ for all $H \in \mathfrak{a}$. Therefore, we have the direct sum decomposition of vector spaces:

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\lambda \in \Phi(\mathfrak{a}, \mathfrak{g})} \mathfrak{g}_\lambda$$

called the *restricted root space decomposition* of $\mathfrak{g}$.

As an example, we consider the root spaces for $SL(n, \mathbb{F})$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Let $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{F})$. In this case $\mathfrak{k}$ is the subalgebra of skew-Hermitian $n \times n$ matrices (skew-symmetric if $\mathbb{F} = \mathbb{R}$) and $\mathfrak{p}$ is the subspace consisting of Hermitian $n \times n$ matrices of trace 0. Denote by $\mathfrak{a}$ the abelian algebra consisting of real diagonal matrices of trace 0. Then $\mathfrak{a} \subset \mathfrak{p}$, and by a simple computation we see that the subalgebra consisting of all matrices in $\mathfrak{p}$ that commute with each element of $\mathfrak{a}$ is $\mathfrak{a}$ itself. Therefore, $\mathfrak{a}$ is an $\mathbb{R}$-split Cartan subalgebra of $\mathfrak{sl}(n, \mathbb{F})$ and the real rank of $\mathfrak{sl}(n, \mathbb{F})$ is $n - 1$.

For each i, $1 \leq i \leq n$, define $f_i \in \mathfrak{a}^*$ by $f_i(\text{diag}[a_1, \ldots, a_n]) = a_i$ and set $\alpha_{ij} := f_j - f_i$. Define $\mathfrak{g}_{ij} := \mathbb{F}E_{ij}$, where E_{ij} is the matrix with 1 at the ij-entry and 0 at the other positions. Note that $\dim \mathfrak{g}_{ij} = 1, 2$ for $\mathbb{R}, \mathbb{C}$, respectively. One can easily check that

$$\mathfrak{sl}(n, \mathbb{F}) = \mathfrak{g}_0 \oplus \bigoplus_{i \neq j} \mathfrak{g}_{ij}$$

where $\mathfrak{g}_0$ is the subalgebra of all diagonal matrices of trace 0. One has $\mathfrak{g}_0 = \mathfrak{m} \oplus \mathfrak{a}$, where $\mathfrak{m}$ is trivial if $\mathbb{F} = \mathbb{R}$, and it is the subalgebra of all diagonal matrices with imaginary entries if $\mathbb{F} = \mathbb{C}$.

4. *The Iwasawa decomposition.* Let G be a semisimple Lie group, $\mathfrak{g}$ its Lie algebra, and let $\mathfrak{a}$ and $\mathfrak{k}$ be as before. Let X be an element of $\mathfrak{a}$ in the complement of the union of subspaces $\ker(\lambda)$, for all $\lambda \in \Phi(\mathfrak{a}, \mathfrak{g})$. Let $\mathfrak{n}$ be the direct sum of root spaces $\mathfrak{g}_\lambda$, for all $\lambda \in \Phi(\mathfrak{a}, \mathfrak{g})$ such that $\lambda(X) > 0$. Then $\mathfrak{n}$ is a nilpotent Lie algebra and it is not difficult to show that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ (a vector space direct sum). We will refer to this decomposition as the *Iwasawa decomposition* of $\mathfrak{g}$.

THEOREM 1.5.1 (Iwasawa decomposition). *Let G be a semisimple Lie group and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ be the Iwasawa decomposition of the Lie algebra of G. Let A, N be connected subgroups of G with Lie algebras $\mathfrak{a}, \mathfrak{n}$, respectively, and K the maximal compact subgroup of G with Lie algebra $\mathfrak{k}$. Then the multiplication map*

$$K \times A \times N \to G, \quad (k, a, n) \mapsto kan$$

is a diffeomorphism onto G. The groups A and N are simply connected.

For $SL(n, \mathbb{R})$ the Iwasawa decomposition is a direct consequence of the existence and uniqueness of the Gram-Schmidt orthogonalization process. In this case we can take K to be the group $SO(n)$ of orthogonal matrices of determinant 1, whereas A is the subgroup of diagonal matrices with positive diagonal coefficients and determinant 1 and N is the subgroup of upper triangular matrices with diagonal coefficients 1.

b. Lattices in Semisimple Lie groups. Further details about the subject of this section can be found, for example, in [**OV1**].

1. *Definition of lattices.* Let G be a locally compact topological group and μ a right-invariant Haar measure on G. If Γ is a discrete subgroup of G, then μ projects to a measure, $\bar{\mu}$, on the quotient G/Γ.

Γ is said to be a *lattice* in G if $\bar{\mu}(G/\Gamma) < \infty$. If this is the case, it is not difficult to show that μ is also left-invariant (i.e., G is a unimodular group) and $\bar{\mu}$ is a G-invariant measure on G/Γ. If G/Γ is compact, Γ is called a *uniform* (or *cocompact*) lattice.

Since left translations, $L_g : h \mapsto gh$, $g, h \in G$, commute with right translations, $(L_g)_*\mu$ is also a right invariant Haar measure, hence there is a constant $\lambda(g)$ such that $(L_g)_*\mu = \lambda(g)\mu$. G is said to be a *unimodular group* if $\lambda(g) = 1$ for all $g \in G$, so that a right-invariant measure is also left invariant.

The simplest example of a (uniform) lattice is $\mathbb{Z}^n \subset \mathbb{R}^n$. Another example is $SL(n,\mathbb{Z}) \subset SL(n,\mathbb{R})$. The latter is not uniform.

2. *Commensurability.* Two subgroups $\Gamma, \Gamma' \subset G$ are said to be *commensurable* if $\Gamma \cap \Gamma'$ is a subgroup of finite index in both Γ and Γ'. If this is the case we write $\Gamma \sim \Gamma'$. Commensurability is an equivalence relation on the set of subgroups of G. If $\Gamma \sim \Gamma'$ and Γ is a (uniform) lattice, then Γ is also a (uniform) lattice.

For example, suppose that Γ, Γ' are lattices in $\mathbb{R}^n$. Then $\Gamma \sim \Gamma'$ if an only if their $\mathbb{Q}$-span coincide.

As another example, $\Gamma(p) := \{\gamma \in SL(n,\mathbb{Z}) | \gamma \equiv I (\mathrm{mod}\ p)\} \sim SL(n,\mathbb{Z})$ for any prime p; these lattices are the so-called *principal congruence subgroups* of $SL(n,\mathbb{Z})$.

3. *Irreducible lattices.* A group G is said to be an *almost direct product* of normal subgroups $G_1, \cdots, G_k \subset G$ if the map $(g_1, \cdots, g_k) \mapsto g_1 \ldots g_k$ is a surjective homomorphism with finite kernel from $G_1 \times \cdots \times G_k$ to G.

Note that if G is an almost direct product of G_1, G_2 and $\Gamma_1 \subset G_1$, $\Gamma_2 \subset G_2$ are lattices, then the image of $\Gamma_1 \times \Gamma_2$ in G under the quotient map $G_1 \times G_2 \to G$ is a lattice in G. If Γ is not commensurable with any such subgroup of G we say that Γ is an *irreducible lattice* of G.

Clearly, any lattice in a simple Lie group is irreducible, while any lattice in $\mathbb{R}^m$, $m \geq 2$, is reducible.

c. Algebraic linear groups and arithmetic lattices. A subgroup $G \subset GL(n, K)$, where K is a field, is said to be an *algebraic matrix group* if it is the set of zeros of an ideal of polynomial functions with coefficients from K. If k is a subfield of K, then G is said to be a *k-group*, or to be *defined over k*, if the ideal defining G is generated by polynomials having coefficients in k. If F is another field such that $k \subset F \subset K$, we write $G(F) = G \cap GL(n, F)$.

More generally, let V be an n-dimensional vector space over a field K. A subgroup $G \subset GL(V)$ (of invertible linear transformations of V) is *algebraic* if for some choice of basis of V, G is identified with an algebraic matrix subgroup of $GL(n, K)$. In this case, G is said to be defined over a subfield k if there exists a k-linear subspace $V_k \subset V$ that spans V over K and a basis of V_k such that the matrix representation of G with respect to this basis is defined over k.

Let V be a real vector space and G a $\mathbb{Q}$-algebraic subgroup of $GL(V)$. If L is a lattice in $V_\mathbb{Q}$ we write $G_L := \{g \in G : gL = L\}$. Then G_L is a subgroup of $G(\mathbb{Q})$. Furthermore, given two lattices L_1, L_2 of $V_\mathbb{Q}$, the groups G_{L_1} and G_{L_2} are commensurable.

A subgroup Γ of a $\mathbb{Q}$-group G is said to be an *arithmetic subgroup* if Γ is commensurable with G_L for a lattice L of $V_{\mathbb{Q}}$.

For example, if $G \subset GL_n(\mathbb{R})$ is an algebraic matrix group defined over $\mathbb{Q}$, then $G(\mathbb{Z}) = G \cap GL(n, \mathbb{Z})$ is an arithmetic subgroup of G.

The following theorem is due to Borel and Harish-Chandra.

THEOREM 1.5.2 ([**BH-C**]). *Let $G \subset GL(n, \mathbb{R})$ be a $\mathbb{Q}$-algebraic semisimple group. Then $G(\mathbb{Z})$ is a lattice in G.*

If $G \subset GL(n, \mathbb{C})$ is more generally a k-group, where $k \subset \mathbb{C}$ is a field of algebraic numbers, then $G(k)$ still can be regarded as the set of $\mathbb{Q}$-points of an algebraic matrix group defined over $\mathbb{Q}$ by using the *field restriction functor* [**W**]. In essence, this amounts to constructing a $\mathbb{Q}$-group $R_{k/\mathbb{Q}}(G)$ such that $G(k)$ is in a natural way isomorphic to the group of $\mathbb{Q}$-points of $R_{k/\mathbb{Q}}(G)$. In matrix form, $R_{k/\mathbb{Q}}(G)$ is obtained as follows. Choose a basis $\alpha = \{\alpha_1, \cdots, \alpha_m\}$ for the field k over $\mathbb{Q}$, and denote by s_λ the matrix of the linear transformation $a \mapsto a\lambda$, $a \in k$, in the basis α. For each $g = (a_{ij}) \in G(k)$, define the matrix $\bar{g}$ by replacing each entry a_{ij} of g by the matrix $s_{a_{ij}}$. Now define a group $\bar{G}$ to be the algebraic subgroup of $GL(mn, \mathbb{C})$ generated by the matrices $\bar{g}$, $g \in G(k)$. Then $\bar{G}$ is a matrix realization of $R_{k/\mathbb{Q}}(G)$.

Finally, we give a definition of arithmetic lattice for a semisimple Lie group G (without an à priori given $\mathbb{Q}$-structure). A subgroup $\Gamma \subset G$ is an arithmetic subgroup if there is an algebraic $\mathbb{Q}$-group H, and a surjective homomorphism $\varphi : \tilde{H}^0 \to G^0$ such that φ has compact kernel and Γ is commensurable with $\varphi(\pi^{-1}(H^0(\mathbb{Z})))$, where $\pi : \tilde{H}^0 \to H^0$ is the universal covering map.

By [**Bor2**] (see also [**Rag**]) every connected semisimple Lie group G contains arithmetic lattices, both cocompact and non-cocompact. Furthermore, if G is a simple Lie group of real rank 2 or greater, all lattices are arithmetic. More generally, one has the following fundamental theorem due to Margulis.

THEOREM 1.5.3 (Margulis' arithmeticity theorem [**Mar1**]). *Suppose that G is a connected semisimple Lie group without compact factors. If the real rank of G is at least 2, then every irreducible lattice in G is arithmetic.*

At the other extreme lies the group $SL(2, \mathbb{R})$. While there are continuous multiparametric families of mutually nonconjugate lattices in $SL(2, \mathbb{R})$, both uniform and nonuniform, only countably many of those lattices are arithmetic.

CHAPTER 2

Basic ergodic theory

1. Measurable G-actions

The study of group actions preserving a measure or a measure class is an important subject in its own right. It is also important because for differentiable or algebraic actions the underlying measure space structure determined by the Borel sets and the measure obtained by integrating the volume form often contain essential dynamical information. In particular, ergodic theory plays a crucial role in establishing various kinds of rigidity properties. Thus, it makes sense to start with a discussion of some basic notions of ergodic theory, along the lines of [**S-HK**, Chapter 3], but now in the broader context of general group actions.

Throughout the rest of this survey G will denote a locally compact second countable topological group. A *measurable action* of G on a Lebesgue measure space $(X, \mathcal{A}, \mu)$ is defined by a measurable map $\Phi : G \times X \to X$, where $G \times X$ has the product measurable structure and Φ has the usual properties of a group action:

(1) $\Phi(e, x) = x$ for all $x \in X$
(2) $\Phi(g_1, \Phi(g_2, x)) = \Phi(g_1 g_2, x)$ for all $g_1, g_2 \in G$ and $x \in X$.

The measure μ is said to be *G-invariant*, or simply invariant, if

$$(\Phi_g)_* \mu = \mu$$

for each $g \in G$. (Recall that a measurable map $T : X \to Y$ between measurable spaces takes a measure μ on X to a measure $T_*\mu$ on Y defined by $(T_*\mu)(A) = \mu(T^{-1}A)$, for each measurable set $A \subset Y$). It is said to be *quasi-invariant* if, for each $g \in G$, $(\Phi_g)_*\mu$ and μ have the same sets of measure zero. A measurable space equipped with a measurable action of G and a quasi-invariant measure will be referred to as a *G-space*.

Ergodic theory deals exclusively with actions having at least quasi-invariant measures. Actions that preserve a finite measure on one hand, and actions with quasi-invariant, but no invariant measures on the other, comprise broad areas in ergodic theory that have somewhat separate identities.

It is also possible to define a measure preserving measurable action of G as a homomorphism of G into the group of automorphisms of the measure space $(X, \mathcal{A}, \mu)$ provided with the weak topology (which coincides with the strong topology for the corresponding unitary operators in $L^2(X, \mu)$.) One issue that immediately arises is whether the map $g \mapsto \Phi(g, \cdot)$ from G into the group of automorphisms of X really makes sense. It is possible to show that a measurable action with an invariant measure μ can indeed be defined as a homomorphism $\tilde{\Phi}$ from G into the group of automorphisms of the Lebesgue measure space $(X, \mathcal{A}, \mu)$ (modulo the equivalence relation that identifies two maps if they coincide on a *conull* set, i.e., on the complement of a set of measure 0), with the further assumption that for each $g \in G$,

$\tilde{\Phi}(g)$ leaves μ invariant and for all $A, B \in \mathcal{A}$, the map $g \mapsto \mu\left(\tilde{\Phi}(g)(A) \cap B\right)$ is continuous. (See [**S-HK**].) More precisely, beginning with Φ, the induced $\tilde{\Phi}$ can be shown to have the continuity property just mentioned, and any such $\tilde{\Phi}$ admits a unique (modulo 0) measurable realization Φ. [**Ma1, Ver**].

In what follows, we shall often write $\Phi_g(x)$, or simply gx, instead of $\Phi(g, x)$.

2. Ergodicity and recurrence

We discuss here some of the notions in ergodic theory that carry on to general G-spaces without much change.

a. Invariance mod 0 **and ergodicity.** A measurable map $F : X \to Y$ between G-spaces is called a *G-map* if

$$F(gx) = gF(x)$$

for all $g \in G$ and $x \in X$. If the G-action on $F(X)$ is trivial, we say that F is *G-invariant.* Given a quasi-invariant measure μ on X, we say that a measurable map F between G-spaces X and Y is a *G-map relative to* μ if for each $g \in G$

$$\mu\left(\{x \in X : F(gx) \neq gF(x)\}\right) = 0.$$

For emphasis, we call a G-map (without reference to any measure) a *strict* G-map. Similarly, we talk about a G-invariant function relative to μ. A measurable set is G-invariant (relative to μ) if its characteristic function is G-invariant (relative to μ.)

It is natural to ask whether a G-map relative to μ agrees μ-a.e. with some strict G-map. If G is countable, the positive answer easily follows since a countable union of sets of measure zero also has measure zero. An essentially straightforward application of the Fubini theorem allows to extend this to general groups. We provide a proof of this fact (Proposition 2.2.1) since it contains a general idea that is very useful when dealing with actions of continuous groups. (Cf. [**Z1**, B.5].) Whenever reference is made to a measure class on G, it is assumed to be that class defined by a Haar measure. In particular, open subsets of G have positive measure.

A *standard Borel G-space* X is a measurable space isomorphic to a Borel subset of a complete separable metric space such that the action of G on X is Borel measurable.

PROPOSITION 2.2.1. *Let X be a standard Borel G-space with quasi-invariant measure μ, and Y a standard Borel G-space. Let $F : X \to Y$ be a G-map relative to μ. Then there is a G-invariant conull Borel subset $X_0 \subset X$ and a Borel (strict) G-map $\tilde{F} : X_0 \to Y$ such that $\tilde{F}$ is μ-almost everywhere equal to F.*

PROOF. Define

$$X_0 := \{x \in X \mid g \mapsto g^{-1}F(gx) \text{ is a.e. constant in } G\}.$$

We claim that X_0 is conull. Namely, by definition of a G-map relative to μ, the measurable set $\mathcal{A} := \{(x, g) \in X \times G \mid g^{-1}F(gx) = F(x)\}$ has the property that for each $g \in G$, $\{x \in X \mid (x, g) \in \mathcal{A}\}$ is conull. Let ν be a Haar measure on G. Then, by Fubini's theorem, $\mathcal{A}$ is conull with respect to the product measure $\mu \times \nu$. By Fubini's theorem again, we conclude that for μ-a.e. $x \in X$, the set $\{g \in G \mid (x, g) \in \mathcal{A}\}$ is ν-conull. The claim that X_0 is conull now follows.

For each $x \in X_0$, we can define $\tilde{F}(x) \in Y$ by

$$\tilde{F}(x) := g^{-1}F(gx),$$

for a.e. $g \in G$. Then $\tilde{F}$ is a Borel map since we can write

$$\tilde{F}(x) = \int_G g^{-1}F(gx)d\nu(g),$$

where ν is any probability measure in the class of Haar measure. It also follows from the definitions that $\tilde{F}$ and F agree almost everywhere on X_0. From the expression

$$g^{-1}F(ghx) = h[(gh)^{-1}F(ghx)]$$

we deduce that $hx \in X_0$ and $\tilde{F}(hx) = h\tilde{F}(x)$ for each $x \in X_0$ and each $h \in G$. □

We recall that a measurable G-space X, with a quasi-invariant measure μ, is said to be *ergodic* if every G-invariant measurable set is either null (i.e, it has zero measure) or conull. We also say that μ is an *ergodic measure* for the G-space X. Therefore, the action is ergodic if X cannot be decomposed as the disjoint union of two G-invariant measurable subsets, both with positive measure.

The proof of the following elementary propositions are essentially the same as for cyclic dynamical systems. The first proposition shows one way in which ergodicity relates to topological properties of the action.

Proposition 2.2.2. *Suppose that X is a second countable topological space and that a continuous action of a locally compact second countable topological group G is defined on X. Suppose that the action is ergodic relative to a quasi-invariant measure μ that is positive on open sets. Then, for almost every $x \in X$, the orbit $\{gx : g \in G\}$ is dense in X.*

Proof. Let $\{V_1, V_2, \dots\}$ be a countable basis for the topology of X, consisting of non-empty sets. The image of V_i under a $g \in G$ will be denoted gV_i. Then, for each i, the set $\bigcup_{g\in G} gV_i$ is a G-invariant set of positive measure, hence of full measure, due to ergodicity. Therefore, the countable intersection $\bigcap_i \bigcup_{g\in G} gV_i$ also has full measure. But this intersection is precisely the set of points of X with a dense orbit. In fact, for each x in the intersection and each i, there is $g \in G$ such that $g^{-1}x \in V_i$. □

Proposition 2.2.3. *Suppose that X is an ergodic G-space, Y is a standard Borel space, and $f : X \to Y$ is a G-invariant (relative to a quasi-invariant measure) measurable function. Then f is almost everywhere constant.*

Proof. By Proposition 2.2.1, it can be assumed that f is strictly G-invariant. If f were not almost everywhere constant, it would be possible to find disjoint measurable sets $A, B \subset Y$ such that $f^{-1}(A)$ and $f^{-1}(B)$ have positive measure. But these are G-invariant sets, since f is G-invariant, and disjoint, which contradicts ergodicity. □

b. Ergodic decomposition. A nonergodic G-invariant measure can be disintegrated into its ergodic components. For actions preserving a finite measure, this ergodic decomposition is discussed in [**S-HK**, Section 4.2b]. The arguments outlined there use an isomorphic representation of any measurable action as a continuous action on a compact metrizable space, and the Choquet theory for representing elements in compact convex sets as integrals over the set of extremal points. Here we state a form of the Ergodic Decomposition Theorem proved by Greschönig and Schmidt which works in the general situation of quasi-invariant measures.

THEOREM 2.2.4 (Ergodic decomposition [**GrSch**]). *Suppose that* $(X, \mathcal{A})$ *is a standard Borel* G*-space, where* G *is a locally compact second countable group. Let* μ *be a probability measure on* $\mathcal{A}$ *which is quasi-invariant under the* G*-action. Then there exist: a standard Borel space* $(Y, \mathcal{B})$*, a probability measure* ν *on* $\mathcal{B}$ *and a family* $\{p_y : y \in Y\}$ *of probability measures on* $(X, \mathcal{A})$ *with the following properties:*

(1) *For every* $B \in \mathcal{A}$*, the map* $y \mapsto p_y(B)$ *is Borel on* Y *and*

$$\mu(B) = \int_Y p_y(B) d\nu(y);$$

(2) *For every* $y \in Y$*,* p_y *is quasi-invariant and ergodic;*

(3) *If* $y, y' \in Y$ *and* $y \neq y'$*, then* p_y *and* $p_{y'}$ *are mutually singular.*

In the proof given in [**GrSch**] one first supposes that the acting group is countable. Then, for a general locally compact second countable G, it is shown that the ergodic decomposition associated to a countable dense subgroup of G is also an ergodic decomposition for G itself. Here we restrict the discussion to countable groups only.

For countable groups, the previous theorem can be derived from the next one. Before stating it we need a few definitions. Let $\mathcal{M}(X, \mathcal{A})_1$ be the space of probability measures on $(X, \mathcal{A})$ and denote by $\mathcal{S}$ the smallest σ-algebra of subsets of $\mathcal{M}(X, \mathcal{A})_1$ for which the maps $\mu \mapsto \mu(B)$ from $\mathcal{M}(X, \mathcal{A})_1$ to $\mathbb{R}$ are measurable for every $B \in \mathcal{A}$. Then $(\mathcal{M}(X, \mathcal{A})_1, \mathcal{S})$ is a standard Borel space.

Associated to the quasi-invariant measure μ is (the logarithm of) its *Radon-Nikodym cocycle* (see Section 3 of the present chapter for a systematic discussion of cocycles over group actions), which is a Borel map $\rho : G \times X \to \mathbb{R}$ such that for each $g \in G$ and μ-a.e. $x \in X$, the value at x of the Radon-Nikodym derivative of $g_*^{-1}\mu$ with respect to μ is $e^{\rho(g,x)}$. By modifying ρ on a set of measure zero, it can be assumed that the *cocycle identity*

$$\rho(gh, x) = \rho(g, hx) + \rho(h, x), \text{ for every } g, h \in G \text{ and all } x \in X \tag{2.1}$$

holds and that

$$\rho(g, x) = 0 \text{ for all } g \in G \text{ and } \mu\text{-a.e.} x \in M \text{ such that } gx = x. \tag{2.2}$$

Given a Borel map ρ satisfying the above property 2.2, a probability measure μ on $(X, \mathcal{A})$ will be called ρ*-admissible* if $e^{\rho(g,x)}$ coincides with the Radon-Nikodym derivative of all g and μ-a.e. x. Let $\mathcal{M}_\rho^G$ denote the set of ρ-admissible probability measure on $(X, \mathcal{A})$. We note that $\mathcal{M}_\rho^G \in \mathcal{S}$. If non-empty, $\mathcal{M}_\rho^G$ is a convex set. Let $\mathcal{E}_\rho^G \subset \mathcal{M}_\rho^G$ be the set of extremal points, which is precisely the set of ergodic elements in $\mathcal{M}_\rho^G$. Any two distinct elements of $\mathcal{E}_\rho^G$ are mutually singular.

Let $\mathcal{A}^G$ be the sub-σ-algebra of $\mathcal{A}$ consisting of $B \in \mathcal{A}$ such that $g(B) = B$ for all $g \in G$. Given a $\nu \in \mathcal{M}(X, \mathcal{A})_1$, denote by $E_\nu(f|\mathcal{A}^G)$ the conditional expectation of a Borel map $f : X \to \mathbb{R}$ with respect to $\mathcal{A}^G$ and ν. We recall that the conditional expectation can be defined by the following property: if $f \in L^2(X, \mathcal{A}, \nu)$ then $E_\nu(f|\mathcal{A}^G)$ is the orthogonal projection of f to $L^2(X, \mathcal{A}^G, \nu)$.

We can now state the following more detailed description of the ergodic decomposition theorem.

THEOREM 2.2.5 ([**GrSch**]). *Consider a Borel action of a countable group G on a Borel space $(X, \mathcal{A})$ and let $\rho : G \times X \to \mathbb{R}$ be a Borel map such that $\rho(g, x) = 0$ for every $(g, x) \in G \times X$ such that $gx = x$ and $\mathcal{M}_\rho^G$ is non-empty.*

(1) *There exists a Borel map $p : x \mapsto p_x$ from X to $\mathcal{E}_\rho^G$ with the following properties:*
   (a) *$p_x = p_{gx}$ for every $x \in X$ and $g \in G$;*
   (b) *$\int_X f dp_x = E(f|\mathcal{A}^G)(x)$ for every $\nu \in \mathcal{M}_\rho^G$ and every nonnegative Borel map $f : X \to \mathbb{R}$. In particular, $\int_C p_x(B) d\nu(x) = \nu(B \cap C)$ for every $B \in \mathcal{A}$ and $C \in \mathcal{A}^G$.*

(2) *If $p' : x \mapsto p'_x$ is another Borel map satisfying (1), then the set of $x \in X$ such that $p_x \neq p'_x$ has measure zero for all $\nu \in \mathcal{M}_\rho^G$.*

(3) *Let $\mathcal{T}$ be the smallest σ-algebra of $\mathcal{A}^G$ such that the map $x \mapsto p_x$ in (1) is $\mathcal{T}$-measurable. Then $\mathcal{T}$ is countably generated, it coincides with $\mathcal{A}^G$ up to measure zero for every $\nu \in \mathcal{M}_\rho^G$ and the following holds. For each $y \in X$ denote by $[y]_\mathcal{T}$ the atom of y in $\mathcal{T}$, that is, the intersection of all $C \in \mathcal{T}$ that contains y. Then $p_x([y]_\mathcal{T}) = 1$ if $x \in [y]_\mathcal{T}$ and $p_x([y]_\mathcal{T}) = 0$ if $x \notin [y]_\mathcal{T}$, for all $x, y \in X$.*

The measure ν in the Ergodic Decomposition Theorem associated to a quasi-invariant measure μ on X can be obtained from the previous theorem (where ρ is the cocycle associated to μ) by defining $\nu := p_* \mu$.

c. Poincaré recurrence. Recurrence is another dynamical concept first defined for $G = \mathbb{R}$ or $\mathbb{Z}$ that can be readily adapted to more general groups. A measurable action with quasi-invariant measure μ will be called *recurrent* if for every measurable set $A \subset X$ of positive measure and for almost every $x \in A$ the set $\{g \in G \mid gx \in A\}$ is not relatively compact, i.e., its closure is not compact.

It should be remarked that, in general, recurrence of an action of a group G does not imply recurrence for its restriction to a subgroup $H \subset G$. (This can even happen for $G = \mathbb{Z}$ and $H = 2\mathbb{Z}$.) For actions preserving a finite measure, however, Poincaré's recurrence theorem still holds (in the form described below), so that in this case recurrence is inherited by restrictions of an action to a closed noncompact subgroup.

THEOREM 2.2.6. (Poincaré recurrence; cf. [**S-HK**, Theorem 3.4.1].) *Suppose that G is a locally compact, second countable, non-compact group and let X be a measurable G-space with a finite invariant measure μ. Then the G-action on X is recurrent.*

PROOF. Let A be a measurable subset of X and consider the set B of all $x \in A$ for which there exists a compact $K \subset G$ such that $gx \in A^c$ (the complement of A) for all g in K^c. We would like to show that B has measure 0. Since G is a countable union of compact sets, B is a countable union of sets of the form $B_K = \{x \in A \mid gx \in A^c \text{ for all } g \in K^c\}$ where K is a fixed compact subset of G. Let L be a countable dense subset of G. Then

$$C := \{x \in A \mid gx \in A^c \text{ for all } g \in L \cap K^c\}$$

is a measurable set that contains B_K. If we show that it has measure 0 the theorem will follow. From a countable dense subset of G, which we may assume contains the inverse of each of its elements, we form a countable dense subgroup L consisting of all finite products of elements in that set. We assume that L in the definition of C

is this subgroup. Arguing by induction, it is possible to find a sequence $g_1, g_2, \dots$ in L such that $g_l g_m^{-1} \in L \cap K^c$ for all l, m, $l \neq m$. In fact, starting with an arbitrary $g_1 \in L$, choose g_2 such that $g_2 g_1^{-1} \in L \cap K^c$. Having chosen $g_1, \dots, g_{l-1}$, let g_l be any element of the intersection of the $(L \cap K^c)g_i$, $i = 1, \dots, l-1$. Since $g_i \in L$, $Lg_i = L$ and the intersection is nonempty. It follows from the choice of the g_i that the sets $g_i^{-1}C$ are all disjoint for different i and have the same measure. Therefore $\mu(C) = 0$, as μ is a finite measure. □

Recurrence along a given "direction" in the group may be defined in a number of ways. If a direction is defined by a choice of closed noncompact subgroup $H \subset G$, then the previous theorem immediately implies that the action is recurrent in every direction. One may also define direction in the following "transverse" sense. Let $H \subset G$ be a closed subgroup and let $\pi : G \to G/H$ be the natural projection. Then an action Φ of G is called *recurrent transversally to* H if for every measurable $A \subset X$ and almost all $x \in A$, $\{p = \pi(g) \in G/H \mid gx \in A\}$ is not relatively compact in G/H.

PROPOSITION 2.2.7. *If the action of G preserves a finite invariant measure and $H \subset G$ is such that* $\mathrm{vol}(G/H) = \infty$*, then the action is recurrent transversally to H.*

The proof is essentially the same as for Theorem 2.2.6.

d. Tame actions. Although the most interesting group actions, from the viewpoint of dynamics, exhibit some form of nontrivial recurrence, there are actions that play an important subsidiary role in the general theory for which such behavior is absent, namely the proper and tame actions.

Suppose that X is a complete second countable metrizable G-space, G is a locally compact second countable group and let the orbit space X/G have the quotient topology. The G-action on X is *proper* if for each $x, y \in X$ there exist neighborhoods U of x and V of y such that $\{g \in G \mid V \cap gU \neq \emptyset\}$ is relatively compact in G. Clearly, if G is compact the action is proper. It follows from the definition of proper action that X/G is a Hausdorff space. In this case, each orbit Gx is closed in X. Therefore a free proper action does not have recurrent orbits.

A somewhat more complicated but still dynamically simple situation corresponds to the case in which the σ-algebra $\mathcal{B}$ of Borel sets in X/G, i.e. the σ-algebra generated by the open sets in the quotient topology of X/G, is countably separated. If this happens, the G-action will be called *tame.* In particular, a proper action is tame. The importance of tame actions in the ergodic theory context is based on the following property: any ergodic quasi-invariant measure for a tame G-action is supported on a single orbit.

The orbit Gx of a topological G-space X is said to be *locally closed* if it is open in its closure $\overline{Gx} \subset X$.

Tame actions will arise several times later in this survey (explicitly or not) and it will always occur in the following context. Suppose that $f : X \to V$ is a G-map, where X is an ergodic G-space with invariant probability measure μ, and V is a tame G-space. Since the measure $f_*\mu$ is invariant with respect to a tame action it follows that f takes values almost surely on a single orbit $G \cdot v$ which, as a result, has a G-invariant probability measure $f_*\mu$.

The next theorem gives a useful characterization of tame actions.

THEOREM 2.2.8 ([**Eff, Gl**]). *Suppose that Φ is a continuous action of a locally compact second countable group G on a complete second countable metrizable space X. Then the following are equivalent:*

(1) *All orbits are locally closed.*
(2) *The action is tame.*
(3) *For every $x \in X$, the natural map $G/G_x \to Gx$ is a homeomorphism, where Gx has the relative topology as a subset of X.*

SKETCH OF PROOF. The implication $2 \Rightarrow 1$ is the hardest to prove and will not be discussed here. We refer the reader to [**Z1**, 2.1.14] for a proof. Note that in [**Z1**] tame actions are called *smooth*. We begin with the assertion $1 \Rightarrow 2$. Since the topology of X has a countable basis and the projection $\pi : X \to G\backslash X$ is open, the topology of $G\backslash X$ also has a countable basis. To prove that the Borel measurable structure is countably separating it suffices to show that $G\backslash X$ is a T_0-space, i.e., that we can separate any two points by an open set that contains only one of the points. Let $x, y \in X$. If $\pi(x)$ and $\pi(x)$ are not separated by an open set, $Gy \subset \overline{Gx}$ and $Gx \subset \overline{Gy}$. Therefore Gy is dense in $\overline{Gx}$. But by assumption Gx is open in its closure, so $Gy \cap Gx \neq \emptyset$. This implies that $\pi(x) = \pi(y)$.

We now show that 3 and 1 are equivalent. We may assume without loss of generality that Gx is dense in X. If this is not the case, simply let X denote the closure of that orbit. We begin with $3 \Rightarrow 1$ and assume that $G/G_x \to Gx$ is a homeomorphism. Then Gx with the subspace topology satisfies the Baire category theorem, because G/G_x satisfies it. (G is locally compact, hence a Baire space. It follows that the quotient is also a Baire space.) Now, G is σ-compact, being second countable and locally compact. Therefore, by Baire's theorem, some compact set $A \subset Gx$ contains a nonempty open set, i.e., for some nonempty open set $U \subset X$, $U \subset U \cap \overline{Gx} \subset A$. Thus $Gx = GU$, which is open.

For the converse, suppose that Gx is open in X. Note that $G/G_x \to Gx$ is continuous, so it suffices to prove that it is also open. We call $U \subset G$ a *symmetric set* if $g \in U$ implies $g^{-1} \in U$. Any open neighborhood V of e contains a symmetric neighborhood: $V \cap V^{-1}$. We claim that it suffices to show that for any compact symmetric set $U \subset G$ whose interior is an open neighborhood of $e \in G$, Ux contains a nonempty open set. Namely, let N be any neighborhood of e and choose U compact symmetric with $U^2 \subset N$. If Ux contains a neighborhood of some ux, $u \in U$, then $u^{-1}Ux$ contains a neighborhood of x, and hence so does Nx. Therefore $G/G_x \to Gx$ is an open map. To show that Ux contains an open set, choose a countable dense set $\{g_i\} \subset G$. Then $Gx = \bigcup_i g_i Ux$, a union of compact sets, so by the Baire category theorem, we have that one $g_i Ux$ contains an open set, and hence so does Ux. □

e. Algebraic actions are tame. A general class of actions having the equivalent properties of the previous theorem consists of algebraic actions of algebraic groups on varieties. (The action map $\Phi : G \times X \to X$ defines an *algebraic action* if Φ is a morphism of varieties, that is, if in local affine coordinates Φ is a polynomial map.) General definitions regarding algebraic groups and actions can be found in [**Z1**].)

By a theorem of Rosenlicht, (See [**Ros, Gro**]) the orbit space of an algebraic action is close to being itself a variety. Rosenlicht's theorem implies that given a real algebraic action of a real algebraic group G on a real algebraic variety V, then

V decomposes into G-invariant sets

$$V = M_0 \cup M_1 \cup \cdots \cup M_l$$

such that, for each $0 \leq i \leq s$, the union $M_i \cup \cdots \cup M_l$ is Zariski closed in V and contains M_i as a Zariski-open subset. Moreover, for each i, M_i/G has the structure of a smooth variety defined over $\mathbb{R}$. G-orbits in M_l are closed sets. It follows that each orbit of a real algebraic action Φ is locally closed and is an embedded submanifold of V. It will shown in Section 5.3 how tame (mostly algebraic) actions arise in the context of ergodic actions.

As an example, the Borel density theorem (see Theorem 2.2.11 and Theorem 2.2.12 of the next subsection) implies that if G is a noncompact connected simple Lie group (say, a subgroup of $GL(n,\mathbb{R})$ for some n) and H is an Zariski closed subgroup (that is, H is a subgroup having the property that any polynomial in the matrix coefficients of $GL(n,\mathbb{R})$ that vanishes identically on H also vanishes identically on G) such that G/H has a G-invariant probability measure, then $H = G$. If, for example, H is the isotropy subgroup of a point $x \in V$, where the G-action on V is algebraic then $H = G_x$ is Zariski closed. Therefore the following proposition holds.

Proposition 2.2.9. *Suppose that $f : X \to V$ is a G-map where X is an ergodic G-space with invariant probability measure, G is a noncompact connected simple Lie group and V is an algebraic G-space. Then f is constant almost everywhere and its value is a fixed point of G in V.*

f. The Borel density theorem. The Borel density theorem describes a fundamental restriction on measure preserving algebraic actions. We first present it in the following form.

Theorem 2.2.10. *Let H be a Zariski connected non-compact, real algebraic group which is generated by its one parameter algebraic subgroups, and $L \subset H$ an algebraic subgroup. Suppose that there is an H-invariant probability measure on H/L. Then $H = L$.*

A simple proof is given in [**A'C-Bu**], where the present theorem corresponds to Théorème 6.2.

Since any semisimple connected Lie group without compact factors satisfies the assumptions of the theorem we obtain as a corollary the most commonly used form of the Borel Density Theorem.

Theorem 2.2.11 (Borel Density Theorem). *Let G be a semisimple connected Lie group without compact factors, and $L \subset H$ an algebraic subgroup. Suppose that there is an H-invariant probability measure on H/L. Then $H = L$.*

A well known corollary is that any lattice in G is a Zariski dense subgroup.

Combining the Borel Density Theorem and Proposition 2.4.2 one concludes the following fact about actions of semisimple Lie groups.

Corollary 2.2.12. *Suppose that G is a semisimple real algebraic group without compact factors. Let $f : X \to V$ be a measurable G-equivariant map between an ergodic G-space X with finite invariant measure and a smooth variety V with an algebraic action of G. Then f is constant almost everywhere.*

g. Actions on the spaces of measures. If H is a real algebraic group and Q a real algebraic subgroup such that H/Q is compact, then the natural action of H on the space $\mathcal{M}(H/Q)_1$ of Borel probability measures on H/Q provided with the weak-* topology behaves in some respects like algebraic actions on varieties, as the next theorem indicates. The theorem plays an important role in the proof of the Cocycle Superrigidity Theorem. (Chapter 6.)

THEOREM 2.2.13 ([**Z1**]). *Let H be a real algebraic group and Q a real algebraic subgroup of H such that $V = H/G$ is compact. Consider the induced action of H on the space $\mathcal{M}(H/Q)_1$. Then this action is tame and the isotropy group H_μ of each $\mu \in \mathcal{M}(H/Q)_1$ is a real algebraic subgroup of H. If H is moreover a simple Lie group with finite center and H_μ is not compact, then H_μ has dimension strictly less than the dimension of H.*

The theorem depends on a key lemma, due to H. Furstenberg, that describes the limit behavior of probability measures $g_m\mu$, where μ is a probability measure on projective space $P^{n-1}(\mathbb{R})$ and g_n is a sequence in $PGL(n, \mathbb{R})$. [**Fur1**]

3. Cocycles and related constructions

Many of the main concepts and constructions that apply to general group actions were introduced in [**S-HK**, Sections 1.3, 3.4]. The present section develops some of these constructions further for later reference.

a. Cocycles and orbit equivalence. Let $(X, \mathcal{A}, \mu)$ be a G-space and $(Y, \mathcal{B}, \nu)$ a G'-space, where μ and ν are quasi-invariant measures. The actions are said to be *orbit equivalent* if there are conull measurable sets $X_0 \subset X$ and $Y_0 \subset Y$ and a measurable isomorphism of Lebesgue spaces $\theta : X_0 \to Y_0$, preserving the measure class, such that $x_1, x_2 \in X_0$ lie on a same G-orbit if and only if $\theta(x_1)$ and $\theta(x_2)$ lie on a same G'-orbit in Y_0.

If $G = G'$ and θ is G-equivariant, i.e. $\theta(gx) = g\theta(x)$ for all g and x, then the actions are called *conjugate*.

If the action of G' on Y is *essentially free* (i.e., almost all stabilizers are trivial) and $\theta : X \to Y$ is a Borel isomorphism that realizes the orbit equivalence then, possibly after discarding sets of measure 0 from X and Y, we can define uniquely $\alpha(g, x) \in G'$, for all $x \in X$ and $g \in G$, by the equality

$$\theta(gx) = \alpha(g, x)\theta(x).$$

It is not difficult to show that α is a *cocycle over the G-action* [**S-HK**, Section 1.3m]. We recall that a measurable function $\alpha : G \times X \to G'$ is a cocycle if for all $g_1, g_2 \in G$ and almost all $x \in X$

$$\alpha(g_1g_2, s) = \alpha(g_1, g_2s)\alpha(g_2, s).$$

The cocycle is said to be *strict* if the identity holds for all g_1, g_2, x. If X is a standard Borel G-space with quasi-invariant measure and G' is a second countable topological group, then for a given cocycle $\alpha : G \times X \to G'$ there exists a strict cocycle $\beta : G \times X \to G'$ such that for all $g \in G$ and for almost all $x \in X$, one has $\beta(g, x) = \alpha(g, x)$. The proof follows the same line as in Proposition 2.2.1.

Two cocycles $\alpha, \beta : G \times X \to G'$ are called *equivalent*, or *cohomologous*, if there exists a Borel function $\varphi : X \to G'$ such that for each $g \in G$

$$\beta(g, x) = \varphi(gx)^{-1}\alpha(g, x)\varphi(x)$$

for almost all $x \in X$. If α and β are strict cocycles and the last identity holds for all x and g we say that α and β are *strictly equivalent.*

A cocycle α is said to be a *coboundary* if it is equivalent to the trivial cocycle, which maps $G \times X$ to the identity element of G'. The quotient of cocycles modulo coboundaries is the first *cohomology space* over the G-action.

As a first example, let X be a G-space with quasi-invariant measure μ and denote by $\mathbb{R}^\times$ the multiplicative group of positive real numbers. The Radon-Nikodym derivative $J_\mu(g,x) := (dg_*^{-1}\mu/d\mu)(x)$ (of the measure $g_*^{-1}\mu$ with respect to μ) gives a map

$$J_\mu : G \times X \to \mathbb{R}^\times$$

which is easily shown to be a cocycle. J_μ is a coboundary if and only if the measure class represented by μ contains an (in general, only σ-finite) G-invariant measure. In fact, if $\varphi : X \to \mathbb{R}^\times$ is such that $J_\mu(g,x) = \varphi(gx)^{-1}\varphi(x)$, then the measure ν such that $d\nu = \varphi d\mu$ is G-invariant.

Suppose that α is the cocycle defined earlier from an orbit equivalence. Let $\pi : G \to G'$ be a continuous isomorphism and introduce the constant cocycle

$$\alpha_\pi(g,x) = \pi(g).$$

Such an α_π will be called a π-*simple cocycle*, or a *constant cocycle*.

The following is a direct consequence of the definitions.

PROPOSITION 2.3.1. *Suppose $G = G'$. Let $\pi : G \times X \to G$ be a cocycle corresponding to an orbit equivalence $\theta : X \to Y$ of essentially free G-spaces X and Y. If there is an inner automorphism $\pi : G \to G$ such that α is equivalent to α_π, then the actions of G on X and on Y are isomorphic.*

b. Cocycles and automorphisms of principal bundles. Another setting where cocycles arise is the following. Consider a principal H-bundle $\pi : P \to M$ over a space (with additional structure, e.g., a topological manifold) M. This means that π is a surjective map and that the fibers $\pi^{-1}(x)$ are the orbits of a free action of H on P. If M is a topological space we assume that the bundle is *locally trivial*, i.e., each point of M has a neighborhood U such that $\pi^{-1}(U)$ is homeomorphic to $U \times H$ via a homeomorphism that intertwines the H-action on $\pi^{-1}(U)$ with the action on $U \times H$ given by right-translations. P is a *smooth* (*continuous, analytic*, etc.) principal bundle if π and the H-action are smooth (continuous, analytic, respectively).

If M is a measure space then without loss of generality one may assume that the bundle is actually trivial, i.e., there is a measurable isomorphism between P and $M \times H$ that intertwines the H-action on P with the action on the product by right-translations.

Now let us assume that M carries both a topology and a Borel measurable structure. To each measurable section $\sigma : M \to P$, one associates a cocycle $\alpha : G \times M \to H$ defined by the relation

$$g\sigma(x) = \sigma(gx)\alpha(g,x).$$

The central example of this situation is the case of a smooth manifold M of dimension n and the principal bundle associated with its tangent bundle TM. This principal bundle is actually the $GL(n,\mathbb{R})$-bundle of frames over M. The fiber above $x \in M$ is by definition the family of all linear isomorphisms from $\mathbb{R}^n$ onto the tangent space T_xM. In this case, a measurable section of P describes an identification of T_xM with $\mathbb{R}^n$ that depends measurably on x. If G acts differentiably on M, there is a natural action on the bundle of frames that to each $g \in G$ and $\xi \in P$

in the fiber of $x \in M$ associates $\xi \circ dg_x$, where dg_x is the tangent map at x for the diffeomorphism $g : M \to M$. In this case, the cocycle describes the matrix in $GL(n, \mathbb{R})$ representing $dg_x : T_xM \to T_{gx}M$ in the given choice of frame.

Up to measurable isomorphism, the action of G on a principal H-bundle can be recovered from the cocycle relative to a measurable trivialization of P by the *twisted product* construction. Let S be a G-space with (quasi) invariant measure ν and X an H-space with (quasi) invariant measure μ. If $\alpha : G \times S \to H$ is a strict cocycle, $S \times X$ can be made into a G-space with (quasi) invariant measure $\nu \times \mu$ by defining the G-action:

$$g \cdot (s, x) := (gs, \alpha(g, s)x).$$

The cocycle identity is precisely the condition needed for this to be a group action. The resulting G-space will be denoted $S \times_\alpha X$. Twisted products are special cases of skew product actions already discussed in [**S-HK**, Section 1.3l].

The next proposition is immediate.

PROPOSITION 2.3.2. *Given two cocycles α, β over the G-space S, with values in H, then $S \times_\alpha X$ and $S \times_\beta X$ are isomorphic G-spaces if α and β are strictly equivalent.*

c. Suspensions and twisted products. Suppose that X is an H-space, where H is a closed subgroup of G. It is possible to define in a canonical way an action of G on a larger space as follows. Let H act on $G \times X$ by the product action $h \cdot (g, x) = (gh^{-1}, hx)$ and consider the quotient $Y = (G \times X)/H$. We denote the natural projection by $\pi : Y \to G/H$. As a measure space, Y is isomorphic to the product $G/H \times X$. If X is a manifold, Y is a fiber bundle over G/H whose fibers are diffeomorphic to X. G acts on the product $G \times X$ by $g \cdot (g', x) = (gg', x)$ and, since the actions of G and H commute, G naturally acts on Y as well. The action of G on Y is called the *suspension* of the H-action on X. It is also referred to as the G-action *induced* from the H-action on X.

By projecting to Y a probability measure on $G \times X$ in the class of the product measure, one obtains a quasi-invariant measure on Y. If ν is a G-invariant (probability) measure on G/H and μ is an H-invariant (probability) measure on X one obtains a (probability) G-invariant measure λ on Y as follows. Denote by μ_{gH} the probability measure on the fiber $\pi^{-1}(gH)$ obtained by pushing-forward μ under the map that associates to each $x \in X$ the element $(g, x)H \in Y$. Since μ is H-invariant, μ_{gH} does not depend on the choice of representative, g, in gH. We can now define

$$\lambda(A) := \int_{G/H} \mu_{gH}(A)\ d\nu(gH).$$

To relate twisted products defined in the previous subsection and suspensions, let H be a closed subgroup of G and let $S = G/H$, upon which G acts by left-translations. Choose a Borel section $\sigma : G/H \to G$ and define the cocycle $\alpha : G \times G/H \to H$ by the relation

$$g\sigma(g_1H) := \sigma(gg_1H)\alpha(g, g_1H).$$

Then given any H-space X, the twisted product $G/H \times_\alpha X$ and the suspension $Y = (G \times X)/H$ are isomorphic as G-spaces. In particular, the twisted products obtained from different sections of $G \to G/H$ are also isomorphic.

PROPOSITION 2.3.3. *The suspension G-space Y is ergodic if and only if the action of H on X is ergodic.*

PROOF. The proposition is a consequence of the following remark. Denote the natural projection by $\pi : Y \to G/H$ and define $X_{[g]}$ to be the fiber of Y above $[g] \in G/H$. Note that each $g \in G$ yields a bijection from X to $X_{[g]}$ by sending x to $[g, x]$, and that the H-invariant measure class of X induces a well-defined measure class on each $X_{[g]}$ that is independent of the choice of g.) If A is a G-invariant subset of Y, then the subsets $A_{[g]} := A \cap X_{[g]}$ satisfy: $g' A_{[g]} = A_{[g'g]}$ for all $g', g \in G$. By applying the Fubini Theorem and the fact that G acts transitively on G/H one concludes that $A_{[e]}$ (the fiber of A above the coset of the identity element of G) is null (conull) if and only if A is null (respectively, conull). (Cf. 2.2.21, [**Z1**].) □

d. The Mackey range of a cocycle. There is a useful generalization of the suspension construction involving a cocycle $\alpha : H \times X \to G$ for a locally compact group G, known as the *Mackey range* of α. Here we elaborate on a brief description given in [**S-HK**, Section 1.3n,o].

If H is a closed subgroup of G and $\alpha(h, x) = h$, the Mackey range of α will be the suspension of the H-action on X. For the general case, start with $\alpha : H \times X \to G$ and form the twisted product H-space $G \times_\alpha X$. Recall that

$$h(g, x) := (\alpha(h, x)g, hx)$$

defines an H-action on $G \times X$ if the cocycle is strict. In general we only obtain a *near action* in the following sense: for every $h_1, h_2 \in H$, $(h_1 h_2) \cdot \xi = h_1 \cdot (h_2 \cdot \xi)$ for almost every $\xi \in G \times_\alpha X$. This can, however, be shown to coincide with an action almost everywhere as in Proposition 2.2.1. (Cf. [**Z1**, Appendix B.])

As in the case of the suspension of an H-action, we try to form the quotient $(G \times_\alpha X)/H$. However, this need not be a Lebesgue space; in fact, H may even act ergodically on $G \times_\alpha X$. The appropriate replacement for the space of H-orbits is defined as follows. Let $\mathcal{I}$ be the space of H-invariant essentially bounded functions on $X \times_\alpha G$. More precisely, $f \in \mathcal{I}$ is an element of $L^\infty(X \times_\alpha G)$ such that for each $h \in H$ and almost every $u \in X \times_\alpha G$, $f(hu) = f(u)$. Then $\mathcal{I}$ is a weak-* closed subalgebra of $L^\infty(X \times G)$, and hence there is a standard measure space (Z, ν) and a measure class preserving Borel map $\varphi : G \times X \to Z$ such that $\varphi^*(L^\infty(Z)) = \mathcal{I} \subset L^\infty(X \times G)$. If $(G \times_\alpha X)/H$ is countably separated, then we can naturally identify Z and $(G \times_\alpha X)/H$. As in the suspension construction, G acts on $G \times_\alpha X$ by $g_0(g, x) = (g_0 g, x)$, and this commutes with the near action of H. Therefore, G leaves the algebra $\mathcal{I}$ invariant, and hence we may choose Z to be a G-space and φ to be a G-map.

The following are elementary properties of the Mackey range. (See [**Z1**, 4.2.24] and the references cited there.)

PROPOSITION 2.3.4. *Let Z be the Mackey range associated to a cocycle α.*

1. *If the H-action on X is ergodic, the G-action on Z is also ergodic.*
2. *Z reduces to a point when the H-action on $G \times_\alpha X$ is ergodic.*
3. *Suppose that the H-action on X is ergodic. Then the cocycle is trivial if and only if the G-action on Z is the action of G on itself by translations.*
4. *The cocycle α is equivalent to a cocycle taking values in a subgroup G_0 of G if and only if Z is the suspension of an action of G_0.*

(5) *If $X = H/H_0$ and $\alpha : H \times H/H_0 \to G$ corresponds to a homomorphism $\sigma : H_0 \to G$, then the Mackey range is the action of G on $G/\overline{\sigma(H_0)}$.*

One can also define the Mackey range of a pair (H, P), where P is a principal G-bundle and H acts on P by principal bundle automorphisms, as a G-space (Z, ν) together with a measure class preserving G-map $\varphi : P \to Z$ such that the pull-back of $L^\infty(Z)$ under φ is the algebra of H-invariant essentially bounded functions on P.

4. Reductions of principal bundle extensions

a. α-invariant functions. Let X be a G-space, Y an H-space and suppose that $\alpha : G \times X \to H$ is a cocycle over the G-action on X. A measurable function $\Psi : X \to V$ is called *α-invariant* if for each $g \in G$

$$\Psi(gx) = \alpha(g, x)\Psi(x)$$

for almost all $x \in X$.

In Section 2.3b it was shown how cocycles arise in a natural way in the context of actions on principal bundles: if G acts by automorphisms of a principal H-bundle P over a manifold X with a quasi-invariant measure on X, and $\sigma : X \to P$ is a (measurable) section of P, then the expression $g\sigma(x) = \sigma(gx)\alpha(g, x)$, for $g \in G$ and $x \in X$, defines a cocycle α over the G-space X.

We have the following elementary fact.

Lemma 2.4.1. *The following statements are equivalent, where L is a subgroup of H:*

(1) *The cocycle α obtained from a measurable section $\sigma : X \to P$ is cohomologous to a cocycle β taking values in L;*
(2) *There exists an β-invariant function $\Psi : X \to H/L$, where β is a cocycle into L cohomologous to α;*
(3) *There exists a G-invariant measurable function $\mathcal{G} : P \to H/L$;*
(4) *There exists a measurable G-invariant reduction Q of P with group L.*

A similar set of equivalences holds if one replaces 'measurable' with another regularity condition, such as 'continuous', although the cocycle α will in general also depend on the indices of some open covering of X that trivializes P.

Proposition 2.4.2 (Zimmer's cocycle reduction lemma). *Suppose that G acts ergodically on (X, μ) and let $\alpha : G \times X \to H$ be a cocycle. Let V be a space on which H acts continuously and tamely. Suppose that there is an α-invariant measurable function $\Psi : X \to V$. Then α is equivalent to a cocycle β taking values in the stabilizer subgroup H_v of a $v \in V$. Furthermore, $\Psi(x) \in H_v$ for almost all $x \in X$.*

Proof. Composing the α-invariant function Ψ with the natural projection $V \to V/H$, we obtain a G-invariant function which, by ergodicity, must be constant almost everywhere, i.e., Ψ takes values in a single orbit, for almost all $x \in X$. Since the H-action on V is tame, the orbit of any $v \in V$ is locally closed and homeomorphic to H/H_v. Therefore, Ψ can be considered as an α-invariant map into H/H_v. The conclusion follows from Lemma 2.4.1. □

If the cocycle α is equivalent to a homomorphism $\rho : G \to H$, we obtain a function $\Phi : X \to V$ that is G-equivariant, that is, for each $g \in G$

$$\Phi(gx) = \rho(g)\Phi(x)$$

for almost all x. In this case, by the cocycle reduction lemma we have a measurable equivariant map from X into the real variety H/L. If G preserves a finite measure on X, the equivariant map will then push this measure forward to a $\rho(G)$-invariant measure on H/L.

b. Invariant reductions. Some of the facts discussed in the previous subsection admit a useful formulation in the language of principal bundles. We collect here for later reference two technical results of this kind.

The first result is that a geometric structure $\mathcal{G}$ whose isometry group acts topologically transitively on M (that is, having a dense orbit) must be "essentially" an L-structure. This is due to the next proposition.

PROPOSITION 2.4.3. *Let V be a real algebraic H-space and P a principal H-bundle over a manifold M. Let $\mathcal{G} : P \to V$ be a continuous, H-equivariant map and suppose that a group G of automorphisms of P acts topologically transitively on M. Suppose moreover that $\mathcal{G}$ is G-invariant. Then there exists an open and dense G-invariant subset U of M such that $\mathcal{G}$ maps $P|_U$ onto a single H-orbit, $H \cdot v_0 \subset V$, for some $v_0 \in V$. The set $\mathcal{G}^{-1}(v_0) \subset P$ is a continuous G-invariant L-reduction of P, where $L \subset H$ is the isotropy subgroup of v_0. If $H \cdot v_0$ is a closed subset of V, then $U = M$. (If $\mathcal{G}$ has greater regularity, the resulting L-reduction will have the same regularity as $\mathcal{G}$.)*

PROOF. Suppose that $x_0 \in M$ has a dense G-orbit in M and let $\xi_0 \in P_x$ be any point in the fiber of P above x_0. Set $v_0 = \mathcal{G}(\xi_0)$ and denote by W the closure of the H-orbit of v_0 in V. Since the G-orbit of x_0 is dense in M, the $G \times H$-orbit of ξ_0 is also dense in P, and maps into $H \cdot v_0$. Therefore $\mathcal{G}$ maps P into W. By the general properties of algebraic actions $H \cdot v_0$ is open in W, so $\mathcal{G}^{-1}(H \cdot v_0)$ is an open and dense subset of P. This set is saturated by H-orbits since $\mathcal{G}$ is H-equivariant, hence it is of the form $P|_U$ for some open and dense subset $U \subset M$. Moreover, U is G-invariant since $\mathcal{G}$ is itself G-invariant. If $H \cdot v_0$ is closed in V, then $W = H \cdot v_0$, so that $U = M$. Once we know that $\mathcal{G}$ maps into a single H-orbit, the remaining claims follow. □

The same ideas also prove the measurable counterpart of the previous proposition. (This is a version of the Cocycle Reduction Lemma 2.4.2.)

PROPOSITION 2.4.4. *Let V be a real algebraic H-space and P a measurable principal H-bundle over a second-countable metrizable space M. Let $\mathcal{G} : P \to V$ be a measurable H-equivariant map and suppose that a group G of automorphisms of P acts ergodically on M with respect to a quasi-invariant measure μ, and leaves $\mathcal{G}$ invariant. Then, there exists a G-invariant measurable conull subset U of M such that $\mathcal{G}$ maps $P|_U$ into a single H-orbit $H \cdot v_0$ in V. The pre-image of v_0 under $\mathcal{G}$ defines a measurable, G-invariant L-reduction of $P|_U$.*

PROOF. H-equivariance of $\mathcal{G}$ implies that $\mathcal{G}$ induces a G-invariant measurable map $\overline{\mathcal{G}} : M \to V/H$. The H-action on V is tame, since it is an algebraic action. It follows that $\overline{G}$ is constant almost everywhere. Therefore $\mathcal{G}$ sends a G-invariant set $P|_U$, $\mu(M - U) = 0$, into a single orbit in V. □

COROLLARY 2.4.5. *Let $\mathcal{G} : F^r(M) \to V$ be a continuous geometric structure on M and suppose that $Iso(M, \mathcal{G})$ has a dense orbit in M. Then, over an open dense*

$Iso(M, \mathcal{G})$-invariant subset $U \subset M$, $\mathcal{G}$ is an L-reduction. A similar result holds in the measurable case.

c. The algebraic hull. As it turns out, the idea of a "maximal geometric structure" preserved by an action can be formalized in a certain way and gives rise to a very useful dynamical invariant called the *algebraic hull* (of a cocycle over a group action. The concept will be used later in Chapters 5 and 6).

Suppose that a group G acts by automorphisms of a principal H-bundle P and that the G-action on the base M preserves a measure class represented by a probability measure μ. We say that $Q \subset P$ is a G-invariant *measurable L-reduction* of P if Q is a measurable L-reduction of $P|_U$, for some G-invariant, μ-conull, measurable subset $U \subset M$ and the G-action on P restricts to a G-action on Q.

PROPOSITION 2.4.6 (Zimmer [**Z1**]). *Let M be a second countable metrizable G-space with a quasi-invariant probability measure μ. Suppose that the action is ergodic with respect to μ. Let H be a real algebraic group and let P be a measurable principal H-bundle on which G acts by bundle automorphisms over the G-action on M. Then:*

(1) *There exists a real algebraic subgroup $L \subset H$ and a G-invariant measurable L-reduction $Q \subset P$ such that Q is minimal; i.e., Q does not admit a measurable G-invariant L'-reduction for a proper real algebraic subgroup L' of L.*
(2) *If Q_1 and Q_2 are G-invariant reductions with groups L_1 and L_2 resp., satisfying the above minimality property, then there is an $h \in H$ such that $L' = hLh^{-1}$ and $Q_2 = Q_1 h^{-1}$.*
(3) *Any G-invariant measurable L'-reduction of P, for real algebraic L', contains a G-invariant measurable L''-reduction, where L'' is a conjugate in H of the minimal L obtained in item 1.*

PROOF. Let $Q_1 \supset Q_2 \supset \cdots$ be a nested sequence of invariant reductions with groups $L_1 \supset L_2 \supset \cdots$. The groups L_i form a descending chain of real algebraic groups. By the descending chain condition the sequence must stabilize at a finite depth, so that a minimal reduction must exist.

The uniqueness claimed in item 2 can be seen as follows. A G-invariant L_i-reduction, Q_i, yields a G-invariant H-equivariant map

$$\mathcal{G}_i : P \to H/L_i.$$

Taking the product $\mathcal{G}_1 \times \mathcal{G}_2$, we obtain a G-invariant, H-equivariant map

$$\mathcal{G} : P \to H/L_1 \times H/L_2.$$

The right-hand side is an H-space for the natural product action. Applying the measurable reduction lemma to $\mathcal{G}$, we conclude that $\mathcal{G}$ maps $P|_U$ onto a single H-orbit in $H/L_1 \times H/L_2$, where U is a conull subset of M. We denote that orbit by $H \cdot (h_1L_1, h_2L_2)$. The isotropy group of (h_1L_1, h_2L_2) is

$$L = \{h \in H \mid hh_1L_1 = h_1L_1, hh_2L_2 = h_2L_2\}$$

and we have a G-invariant measurable L-reduction Q of P. Note that $L \subset h_1L_1h_1^{-1} \cap h_2L_2h_2^{-1}$. L cannot be a proper subgroup of $h_iL_ih_i^{-1}$ since, otherwise, Qh_i would define a proper reduction of Q_i, contradicting the minimality of Q_i. Therefore, $Qh_i = Q_i$, $i = 1, 2$, proving 2. The same argument also shows 3. □

The conjugacy class of the group L obtained above is called the continuous (resp., the measurable, in 2.4.6) *algebraic hull* of the G-action on P. By an abuse of language, we sometimes call L itself the algebraic hull. If the action is not ergodic, we should regard the algebraic hull as a map from the ergodic components of a quasi-invariant measure into the conjugacy classes of algebraic subgroups of H.

The following result is proved in a similar way using a the C^r form of the reduction lemma.

PROPOSITION 2.4.7. *Let P be a principal H-bundle over a manifold M. Suppose that a group G acts by bundle automorphisms of P so that the action on M is topologically transitive. Then, for each $r \geq 0$, there exists a real algebraic subgroup $L \subset H$ and a G-invariant C^r L-reduction $Q \subset P|_U$, over a G-invariant dense open subset $U \subset M$, such that Q is minimal in the same sense already defined in the previous proposition. Moreover, the above properties 2 and 3 also hold here after replacing 'measurable' by 'continuous' and taking into account that all reductions are only defined over a G-invariant open and dense subset of M.*

If G is a Lie group that acts via a smooth action on a manifold M, then G also acts on each of the frame bundles $F^r(M)$. One interpretation of the above proposition is that it is possible to define a "maximal A-structure" of order r that is invariant under G and that this structure is, in a sense, unique. (A-structures are defined in Section 5.2e.)

5. Amenable groups and amenable actions

a. Amenable groups. A *mean* on a group G is a continuous linear functional m on the space of bounded continuous functions on G such that $m(f) \geq 0$ whenever f is non-negative and $m(1) = 1$. It is a *left-invariant mean* if $m(f \circ L_g) = m(f)$ for all $g \in G$ and f a continuous bounded function of G, where L_g denotes left multiplication by g.

An action of G on a convex space W is called *affine* if

$$g(\lambda w_1 + (1-\lambda)w_2) = \lambda g w_1 + (1-\lambda) g w_2$$

for all $w_1, w_2 \in W$, $g \in G$ and $0 \leq \lambda \leq 1$.

A group G is called *amenable* if one of the following equivalent conditions is satisfied:

(1) The space of continuous bounded functions on G has a left-invariant mean.
(2) Every continuous affine action of G on a compact convex subset W of a locally convex topological vector space has a fixed point.
(3) Every compact non-empty G-space X admits a G-invariant probability measure.
(4) G possesses a Følner sequence. (For the definition, see survey [**S-HK**].)

Further details, as well as the characterization of amenability by Følner sequences, can be found in [**S-HK**, Sections 1.4, 4.2]. For the equivalence of these four characterizations see [**Grl**], [**Pat**] and references cited there.

For example, let $G = \mathbb{R}$ and define $\rho(t)f$ by $(\rho(t)f)(x) = f(x-t)$. Almost invariant vectors can be constructed by taking the characteristic function of a very long symmetric interval in $\mathbb{R}$ and normalizing it so as to have L^2 norm equal to 1.

Amenable connected Lie groups are characterized as follows.

PROPOSITION 2.5.1 ([**Grl**],[**Z1**]). *Let G be a connected Lie group. Then G is amenable if and only if it has a closed solvable normal subgroup H such that G/H is compact.*

However, already for discrete finitely generated groups, and, naturally, for general locally compact groups, amenability cannot be so easily characterized. For example, it follows immediately from the Følner sequences characterization that any discrete finitely generated group is amenable if the size of the balls in the word–length metric grows subexponentially. Solvable groups are amenable. Any solvable group that does not have a nilpotent subgroup of finite index has exponential growth. Polynomial growth of balls implies that the group is essentially nilpotent, by a celebrated theorem of Gromov [**Gro**]. On the other hand, there are finitely generated groups whose growth is subexponential but faster than any polynomial rate.

b. Amenable actions. It is possible for certain actions of non-amenable groups to share key properties with actions of amenable groups. This leads to the notion of amenable group actions which is based on the fixed point characterization of amenable groups (see [**Z1**]; for the definition of amenability for general measurable groupoids see [**Ren**] and [**AnRe**]). Suppose that E is a separable Banach space, E^* the dual space and $E_1^* \subset E^*$ the unit ball. Given $A \in \text{Iso}(E)$, $A^* : E^* \to E^*$ denotes the adjoint of A. Suppose that to each $x \in X$ is associated a compact convex subset $A_x \subset E_1^*$ depending measurably on x, in the sense that the unions of all $\{x\} \times A_x$ is a Borel subset of $X \times E_1^*$. We then define the space $F(X, \{A_x\})$ of all measurable $\varphi : X \to E_1^*$ such that $\varphi(x) \in A_x$ for almost all x. $F(X, \{A_x\})$ is a compact convex subset of $L^\infty(X, E^*)$, where the latter has the weak-$*$ topology as the dual space of $L^1(X, E)$. Suppose now that $\alpha : G \times X \to \text{Iso}(E)$ is a cocycle into the group of all isometric isomorphisms of E and that the family $\{A_x\}$ is α-invariant, i.e., for each g

$$\alpha(g, x)^* A_{gx} = A_x$$

for almost every x. Then $F(X, \{A_x\})$ is G-invariant, for the G-action defined by

$$(g \cdot \varphi)(x) := \alpha(g^{-1}, x)^* \varphi(g^{-1}x).$$

The action of G on X is called *amenable* if the G-action on any $F(X, \{A_x\})$, of the form just constructed (from a cocycle α), has a fixed point. In other words, there exists $\varphi \in F(X, \{A_x\})$ such that for each g

$$\alpha(g^{-1}, x)\varphi(g^{-1}x) = \varphi(x)$$

for almost every x.

For example, suppose that $\alpha : G \times X \to H$ is a cocycle into a group H and that H acts by homeomorphisms of a compact metrizable space V. For each $x \in V$ define $A_x = \mathcal{M}(V)_1$. If the G-action on X is amenable, then there exists a measurable assignment $x \mapsto \mu_x$ of probability measures on V which is α-invariant.

The following geometric situation may help illuminate the definition of amenable action. Suppose that G acts by diffeomorphisms on a smooth compact manifold M. At each $x \in M$, consider the Grassmannian variety V_x of k-dimensional linear subspaces of the tangent space T_xM. The union of the V_x forms a smooth fiber bundle over M, which we will call the *Grassmann bundle.* A subbundle L of TM is a section of this bundle. It is G-invariant if for each $g \in G$ and each $x \in M$ we have $dg_xL(x) = L(gx)$, where dg denotes the tangent map induced by g. More generally, one may consider a "random" plane field, that is, at each $x \in M$ one is given a probability measure μ_x on V_x, depending measurably on x. We say that such a random plane field is invariant under the G-action if for each $g \in G$ and each $x \in M$ the diffeomorphism between V_x and V_{gx} induced by dg_x sends μ_x to μ_{gx}. The above definition implies that if the G-action on M is amenable, then there exists a G-invariant random plane field.

It is clear that if G is an amenable group, every G-action is amenable, but the converse is not always true. It is true, however, when the G-action is also finite-measure preserving.

PROPOSITION 2.5.2. [**Z1**, 4.3.2, 4.3.3, 4.3.4, 4.3.5] *Let X and Y be G-spaces, and suppose that the G-action on X is amenable. Let H be any closed subgroup of G. Then*

1. *If X has an invariant finite measure, then G is an amenable group.*
2. *The product G-action on $X \times Y$ is amenable.*
3. *The action of H on X is amenable.*
4. *The action of G on G/H by left translations is amenable if and only if H is amenable.*

As an example of an amenable action of a non-amenable group on a compact space, consider the homogeneous space $X = SL(n, \mathbb{R})/H$, where H is the stabilizer of a flag of subspaces

$$\{0\} \subset V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{R}^n$$

and V_k has dimension k for each k. Then H is a solvable group, hence amenable, and it follows from Property 4 of the previous proposition that the action of $SL(n, \mathbb{Z})$ on X by left translations is amenable.

It turns out that any amenable ergodic G-space X is an extension of an amenable transitive action. In other words, if X is an ergodic G-space and the G-action on X is amenable, then there exists a G-equivariant map $F : X \to G/L$, where L is a closed amenable subgroup. (See [**Z6**].)

It is easy to prove that if G is an amenable group and $\varphi : G \to H$ is a continuous surjective homomorphism, then H is also amenable. An extension of this fact to cocycles also holds: if X is an amenable ergodic G-space and $\alpha : G \times X \to H$ is a cocycle into an algebraic group H, then α is equivalent to a cocycle taking values in an amenable group.

PROPOSITION 2.5.3. [**Z1**, 4.3.10] *Suppose that X is a G-space, Y is a G'-space, μ, μ' are quasi-invariant measures on X and Y, respectively, and the actions are essentially free and orbit equivalent. Then the G-action is amenable if and only if the G'-action is amenable.*

PROOF. Due to results in [**FHM**], the proof can be reduced to the case of discrete groups. Assuming that G and G' are discrete subgroups, we can pass to a conull subset and assume that α is a strict cocycle.

Let $\theta : X \to Y$ be a Borel isomorphism that realizes the orbit equivalence and $\alpha : G \times X \to G'$ the corresponding cocycle, described in Subsection 2.3a. Suppose that the G-action is amenable. In order to show that the G'-action is also amenable, we consider a cocycle $\beta' : G' \times Y \to \mathrm{Iso}(E)$ and some $F(Y, \{A_y\})$ as in the definition of amenable actions. Define a cocycle $\beta : G \times X \to \mathrm{Iso}(E)$ by

$$\beta(g, x) = \beta'(\alpha(g, x), \theta(x)). \tag{2.1}$$

Then there exists a β-invariant $\varphi \in F(X, \{A_{\theta(x)}\})$; in other words, there exists a map $x \in X \mapsto \varphi(x) \in A_x$ such that $\beta(g^{-1}, x)\varphi(g^{-1}x) = \varphi(x)$ for almost all $x \in X$ and all $g \in G$. Define $\psi = \varphi \circ \theta^{-1}$ and note that for each $g' \in G'$ and almost all $y \in Y$ there is $g \in G$ such that $g' = \alpha(g, x)$. It follows from the above equation 2.1 that ψ is G'-invariant. Therefore, the G'-action is also amenable. □

COROLLARY 2.5.4. *Let X (respectively, Y) be an essentially free G-space (respectively, G'-space) with finite invariant measure. Suppose that the G action on X and the G' action on Y are orbit equivalent. Then G is amenable if and only if G' is amenable.*

PROOF. This is an immediate consequence of the previous proposition and Proposition 2.5.2 (1). □

c. Orbit equivalence for actions of amenable groups. A G-action on X is said to be *properly ergodic* if it is ergodic but X does not contain a conull set consisting of a single orbit.

THEOREM 2.5.5 (Connes-Feldman-Weiss [**CFW**]).

(1) *A free properly ergodic action of a discrete group is amenable if and only if it is orbit equivalent to a $\mathbb{Z}$-action.*
(2) *A free properly ergodic action of a continuous group is amenable if and only if it is orbit equivalent to an $\mathbb{R}$-action.*
(3) *Any two free properly ergodic actions of continuous amenable unimodular groups with invariant measure are orbit equivalent.*

As we will see later this fundamental theorem is in sharp contrast with the behavior of actions of semisimple groups of real-rank greater than 1. For the latter groups, it was shown by Zimmer (see Theorem 6.2.4 below) that orbit equivalence often implies isomorphism of the actions.

CHAPTER 3

Groups actions and unitary representations

1. Spectral theory

a. Unitary representations. The spectral theory for measurable group actions is based on the theory of unitary group representations. In the case of cyclic systems and, more generally, actions of abelian groups, the unitary representations are well understood: they are direct integrals of one-dimensional representations. The basics about spectral theory for $\mathbb{Z}$ actions are discussed in [**S-HK**, Sections 3.4g, 3.6].

The theory of unitary representations for general groups is much more sophisticated and is not fully understood in most cases. We present here some basic notions needed in applications to ergodic theory of group actions that will be discussed later in the chapter. For a detailed introduction to the subject see, for example, [**Kn1**].

Let G be a locally compact second countable group and H a separable (complex) Hilbert space. A *unitary representation* ρ of G in H is a strongly continuous homomorphism $\rho : G \to U(H)$ of G into the group of unitary operators of H. (Strong continuity means that for each $v \in H$ the map $g \mapsto \rho(g)v$ is continuous from G into H.) The following fact is a direct consequence of Lemmas 5.4 and 5.28 of [**Va1**].

PROPOSITION 3.1.1. *Let $\rho : G \to U(H)$ be a homomorphism of groups. If for each v and w in H the function $g \mapsto \langle \rho(g)v, w\rangle$ is measurable, then ρ is strongly continuous, hence it defines a unitary representation of G on H.*

A unitary representation ρ in H is said to be *unitarily equivalent* to another unitary representation ρ' in H' if there exists a unitary isomorphism $L : H \to H'$ of H onto H' such that

$$\rho(g)' = L\rho(g)L^{-1}$$

for each $g \in G$. The representation is called *irreducible* if the only closed G-invariant subspaces of H (under the linear action of G defined by ρ) are 0 and H.

Assume that X is a G-space with a finite G-invariant measure μ and consider $H := L^2(X, \mu)$, with inner product

$$\langle f_1, f_2\rangle := \int_X f_1(x)\overline{f_2(x)}\, d\mu(x).$$

Then $(H, \langle\cdot,\cdot\rangle)$ is a separable Hilbert space.

For each $g \in G$, denote by $\pi(g)$ the linear operator on H defined as follows. To each element in H, represented by a square-integrable function f, the (class determined by the) function $\pi(g)f$ satisfies $(\pi(g)f)(x) := f(g^{-1}x)$. As the measure μ is preserved by G, each $\pi(g)$ satisfies the identity

$$\langle \pi(g)f_1, \pi(g)f_2\rangle = \langle f_1, f_2\rangle.$$

Therefore, $\pi(g)$ belongs to the group of unitary operators $U(H)$ and defines a homomorphism $\pi : G \to U(H)$.

Two measure–preserving actions are called *spectrally isomorphic* if the corresponding unitary representations associated with them are unitarily equivalent. Obviously metrically isomorphic actions are spectrally isomorphic. Properties of measure–preserving actions which are preserved under the spectral isomorphism are usually called *spectral invariants.*

There is a canonical unitary representation associated to a G-space even if the action only leaves invariant a measure class. The representation space is the same as before, but now for each $g \in G$ and $f \in L^2(X, \mu)$ one writes

$$(\pi(g)f)(x) := f(g^{-1}x)\sqrt{\frac{d(g_*\mu)}{d\mu}(x)}.$$

The homomorphism

$$\pi : G \to U(L^2(X, \mu))$$

is continuous by Proposition 3.1.1. Thus, one has a unitary representation of G into $U(L^2(X, \mu))$.

If μ_G is a left-Haar measure on G, we define the operator $\lambda(g)$ on $L^2(G, \mu_G)$ by setting for $f \in L^2(G, \mu_G)$ and $x \in G$

$$(\lambda(g)f)(x) = f(g^{-1}x).$$

Then λ is a continuous representation of G, called the *left-regular representation.* Similarly one defines the *right-regular representation* ρ of G on $L^2(G, \mu'_G)$ (where μ'_G is a right-invariant Haar measure) by

$$(\rho(g)f)(x) = f(xg).$$

The latter will be called the *regular representation* of G and will be denoted ρ_G^{reg}.

b. The unitary dual of a group. The set of equivalence classes of unitary representations (respectively, irreducible unitary representations) of G on separable Hilbert spaces will be denoted $\tilde{G}$ (respectively, $\hat{G}$.) The set $\hat{G}$ is called the *unitary dual* of G.

1. *Topology on the unitary dual.* We define a topology on $\tilde{G}$ by specifying a family of neighborhoods of a class of unitary representations as follows. Let ρ be a unitary representation of G on a Hilbert space V. Given any compact subset $K \subset G$, a finite collection of vectors $v_1, \dots, v_n \in V$ and a number $\epsilon > 0$ we define an open neighborhood $U(K, v_1, \dots, v_n, \epsilon)$ of (the class represented by) ρ as the subset of $\tilde{G}$ consisting of all (classes of) representations having the following property: a representation ρ', with representation space V', belongs to $U(K, v_1, \dots, v_n, \epsilon)$ if there exist vectors $w_1, \dots, w_n$ such that

$$|\langle \rho(g)v_i, v_j\rangle - \langle \rho'(g)w_i, w_j\rangle| < \epsilon$$

for all $g \in K$, and all i, j between 1 and n.

If the representation $\rho' \in \tilde{G}$ belongs to the closure of the single point ρ, then ρ' is said to be *weakly contained* in ρ.

When G is a locally compact abelian group, each irreducible unitary representation of G is one-dimensional and $\hat{G}$ coincides with the group of characters of G, that is, the set of homomorphisms $\chi : G \to \mathbb{T}$ (where $\mathbb{T}$ denotes here the complex numbers of unit norm). Pointwise multiplication on G makes $\hat{G}$ also a locally compact abelian group.

2. *The unitary dual of* $SL(2,\mathbb{R})$. As an example we describe the unitary dual of the group $SL(2,\mathbb{R})$ of real unimodular matrices of order 2. The reader will find the details in [**L**] or [**Kn1**], for example. The irreducible unitary representations of $G = SL(2,\mathbb{R})$ (other than the trivial representation, $g \mapsto 1 \in U(\mathbb{C})$) decomposes into three series: two continuous series, the *principal series*, the *complementary series*, and the *discrete series*.

For each $g \in G$ and $z = x + y\sqrt{-1} \in \mathbb{C}^* := \mathbb{C} - 0$ we write

$$zg := xg_{11} + yg_{21} + (xg_{12} + yg_{22})\sqrt{-1}.$$

Let $H = L^2(\mathbb{R}^2)$ denote the Hilbert space of square-integrable measurable functions $f : \mathbb{C}^* \to \mathbb{C}$, with the norm $\|f\|^2 := \int_{\mathbb{R}^2 - 0} |f(x,y)|^2 dxdy$. Then

$$(\rho(g)f)(z) := f(zg) \tag{3.1}$$

defines a unitary representation $\rho : G \to U(H)$. The *principal series* of representations is obtained by decomposing ρ into irreducible representations. This can be done explicitly as follows. First write $H = H^+ \oplus H^-$–the decomposition of H into even and odd functions (H^+ and H^-, respectively). These are ρ invariant subspaces, and each one can be further decomposed into subspaces $H_s^\pm$, $s = u\sqrt{-1}$, $u \in \mathbb{R} - 0$, where $H_s^\pm$ consists of functions $f \in H^\pm$ which are $s-1$-homogeneous; that is,

$$f(tz) = |t|^{s-1} f(z). \tag{3.2}$$

The norm on $H_s^\pm$ is defined by

$$\|f\|_s^2 := \int_{|z|=1} |f(x)|^2 d\lambda(z) \tag{3.3}$$

where λ is the Haar measure on the unit circle normalized to 1. The subrepresentations on $H_s^\pm$ can be shown to be irreducible. It can also be shown that the representations corresponding to s and $-s$ are equivalent (separately for H^+ and H^-) so that it suffices to consider $s > 0$. Other than these, all the $H_s^\pm$ define inequivalent representations.

The *complementary series* can be defined as follows. Let H_s^+ be the space of even functions satisfying the same homogeneity condition 3.2, except that now we suppose $0 < s < 1$. The representation on H_s^+ is again defined by 3.1, but the inner product on H_s^+ is now:

$$\langle f_1, f_2 \rangle := \int_{|z_1|=1} \int_{|z_2|=1} f_1(z_1)\overline{f_2(z_2)} K_s(z_1,z_2) d\lambda(z_1) d\lambda(z_2),$$

where $K_s(z_1,z_2) := |\mathrm{Im}(z_1\bar{z}_2)|^{-s-1}$. It can be shown that these are all mutually inequivalent irreducible unitary representations of G.

Finally, we describe the *discrete series*. For half of the representations in this series, the representation space consists of analytic functions on the upper half-plane, $H^+ := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$, and for the other half, the representation space is the lower half-plane, $H^- := \{z \in \mathbb{C} : \mathrm{Im}(z) < 0\}$. In both cases, the representation operators are given by

$$(\rho_n(g)f)(z) := (g_{12}z + g_{22})^{-n-1} f\left(\frac{g_{11}z + g_{21}}{g_{12}z + g_{22}}\right),$$

where n is a nonnegative integer, and the inner product, for each $n > 0$ is

$$\langle f_1, f_2\rangle_\pm := \int_{H^\pm} f_1(z)\overline{f_2(z)} y^{n-1} dx dy$$

whereas, for $n = 0$, it is

$$\langle f_1, f_2\rangle_\pm := \int_{-\infty}^{+\infty} f_1(x)\overline{f_2(x)} dx.$$

These representations are all mutually inequivalent, irreducible unitary representations.

Theorem 3.1.2 (Unitary dual of $SL(2,\mathbb{R})$). *The set of irreducible unitary representations (up to equivalence) of $SL(2,\mathbb{R})$ consists of the principal series, the complementary series, the discrete series, and the trivial representation. In the topology of the unitary dual, the representations in the discrete series are all isolated; the principal series forms a closed and open subset, with the parameter u inducing the topology within the series; the latter is true for the complementary series, except for the fact that the trivial representation is the limit in the unitary dual of the representations of the complementary series as $s \to 1$.*

Let us comment on the last assertion of the theorem. It is not difficult to show that for each compact subset $K \subset SL(2,\mathbb{R})$ and each $\epsilon > 0$, it is possible to find a unit vector, f, in the Hilbert space H_s^+ of a representation in the complementary series (for s sufficiently close to 1) such that (for ρ as in 3.1) $|\langle f, \rho(g)f\rangle - 1| \leq \epsilon$ for all $g \in K$.

c. Decomposition into irreducible representations. Let X be a measure space with measure μ and V a separable Hilbert space. Assume that for each $x \in X$ one is given a closed subspace H_x of V so that the map $x \mapsto H_x$ is measurable in the following sense: for all $v, w \in V$ the function $x \mapsto \langle P_x v, w\rangle$ is measurable, where P_x is the orthogonal projection onto H_x. The collection $H = \{H_x : x \in X\}$ may be regarded as a *Hilbert bundle* with base space X. Consider the space of square integrable measurable sections of H, that is, the space $L^2(X, H, \mu)$ of measurable functions $f : X \to V$ such that

$$\int_X \|f(x)\|^2 d\mu(x) < \infty$$

and $f(x) \in H_x$ for all $x \in X$. On this space one defines an inner product by

$$\langle f_1, f_2\rangle := \int_X \langle f_1(x), f_2(x)\rangle d\mu(x).$$

After identifying functions that agree almost everywhere, we obtain a Hilbert space, $L^2(X, H, \mu)$. This is called the *continuous sum* of the Hilbert spaces H_x and is also often written $\int_X^\oplus H_x d\mu$.

Suppose now that for each $x \in X$ one is given a unitary representation ρ_x of G on H_x. The *continuous direct sum* or *direct integral* of the ρ_x is the representation ρ of G on $L^2(G, H, \mu)$ that for each $g \in G$ and each $f \in L^2(G, H, \mu)$ satisfies

$$(\rho(g)f)(x) := \rho_x(g)f(x)$$

for almost all $x \in X$. This notion reduces to ordinary direct sum of representations when X is finite or countable, and μ is discrete.

THEOREM 3.1.3. [**Dix**] *Every unitary representation ρ of G on a separable Hilbert space can be decomposed into a direct integral of irreducible unitary representations.*

This theorem is a generalization of the *spectral theorem* for a locally compact abelian group G of unitary operators. In the latter (abelian) case all irreducible representations are one-dimensional and the unitary dual is identified with the *dual group* G^*, which consists of the continuous characters of G, where group multiplication is defined by the ordinary product of functions. More precisely, define a unitary representation ρ of the locally compact abelian group G on $L^2(G^*, H, \mu)$, for a σ-finite measure μ and some Hilbert bundle H over G^*, by

$$(\rho(g)f)(\lambda) := \lambda(g)f(\lambda)$$

for $f \in L^2G^*, H, \mu)$ and $g \in G$. Up to unitary equivalence, ρ is uniquely determined by the measure class of μ and the *multiplicity function* $\lambda \mapsto \dim H_\lambda$. We denote this representation by ρ_μ^H. The next proposition summarizes the main facts about unitary representations of abelian locally compact groups. (See, for example, [**Ma3**].)

PROPOSITION 3.1.4. *Let ρ be a unitary representation of a locally compact abelian group G on a Hilbert space V. Then there exists a σ-additive measure μ on G^* and a Hilbert bundle H over G^* such that ρ is unitarily equivalent to ρ_μ^H.*

For example, if ρ is the regular representation ρ_G^{reg} of a locally compact abelian group G on $L^2(G, m)$ (m is a Haar measure), then the decomposition of ρ into irreducible representations corresponds to the Fourier inversion formula: for $f \in L^2(G, m)$,

$$f(g) = \int_{G^*} h(\chi)\overline{\chi(g)}d\hat{m}(\chi)$$

where $\hat{m}$ is a suitably normalized Haar measure on the group of characters G^* and h is the Fourier transform of f:

$$h(\chi) = \int_G f(g)\chi(g)dm(g).$$

d. Induced representations. For general locally compact groups, an important concept is that of *induced representations*, introduced in this context and systematically developed by G. Mackey. One way to define it is as follows. Let H be a closed subgroup of G and σ a unitary representation of H in a separable Hilbert space V. Let $\alpha : G \times G/H \to U(V)$ be a cocycle over the G-action on G/H by translations, taking values in the group of unitary operators on V. The cocycle will be called a *σ-cocycle* if for all $h \in H$

$$\alpha(h, [e]) = \sigma(h),$$

where $[e]$ is the coset of the identity element in G/H. Select a quasi-invariant measure μ on G/H and let $J : G \times G/H \to \mathbb{R}^+$ denote the Radon-Nikodym cocycle, which was defined earlier as $J(g, x) = (dg_*^{-1}\mu/d\mu)(x))$. The representation of G *induced* by σ is now given by the unitary operators $\pi(g)$ on the Hilbert space of V-valued square integrable functions on G/H, $L^2(G/H, V, \mu)$, defined by

$$(\pi(g)f)(x) = J(g, g^{-1}x)^{1/2}\alpha(g, g^{-1}x)f(g^{-1}x)$$

for $x \in G/H$, $g \in G$, where α is a σ-cocycle. The induced representation, π, is often denoted $\mathrm{Ind}_H^G(\sigma)$. Up to equivalence, π does not depend on the choice of μ and α.

We note for later use the following basic properties of the induced representation. A proof of the first property is given in [**Z1**, 7.3.7]. The second one is a direct consequence of the definitions.

PROPOSITION 3.1.5. *Let H be a closed subgroup of a locally compact group G.*

(1) *If σ_1 and σ_2 are unitary representations of H such that σ_1 is weakly contained in σ_2, then $\mathrm{Ind}_H^G(\sigma_1)$ is weakly contained in $\mathrm{Ind}_H^G(\sigma_2)$.*
(2) *If ρ_H and ρ_G denote the regular representations of H and G, respectively, then $\mathrm{Ind}_H^G(\rho_H)$ is unitarily equivalent to ρ_G.*

As an example, suppose that G is a complex semisimple Lie group (let us take $SL(n, \mathbb{C})$ as an example) and $H = B$ is a Borel subgroup of G (say, the subgroup of upper triangular matrices in $SL(n, \mathbb{C})$). If χ is a character of the maximal abelian subgroup D of B (the diagonal subgroup of $SL(n, \mathbb{C})$), we can regard χ as a character of B and define $L_\chi := \mathrm{Ind}_B^G(\chi)$. It was proved by Gel'fand and Naimark that the L_χ, for $\chi \in \hat{D}$, are all irreducible and that the regular representation can be decomposed as a direct integral of the L_χ.

For certain semidirect products and nilpotent Lie groups, it can be shown that all irreducible unitary representations are induced. More precisely, it can be shown, for example, that for every irreducible unitary representation of a connected nilpotent Lie group G, there is some 1-dimensional unitary representation of some subgroup of G that induces it. For semisimple groups the inducing process does not in general yield all the irreducible representations, although the ones needed for a *Plancherel formula* can be obtained by inducing from the so-called *parabolic subgroups*. For more information on this subject, see [**Kn2**]. We only mention here the following fundamental facts about unitary representations of semidirect products, due to G. Mackey.

PROPOSITION 3.1.6 ([**Ma1**]). *Let ρ be an irreducible unitary representation of a locally compact group G on a Hilbert space V. Let R be an abelian normal subgroup of G isomorphic to $\mathbb{R}^n$. Let μ and H be, respectively, a measure and Hilbert bundle on the dual space R^* (regarded here as the dual vector space to R) such that ρ is unitarily equivalent to ρ_μ^H. Then μ is ergodic, $\dim H_\lambda$ is constant μ-a.e., and the following holds: there is a $\lambda_0 \in R^*$ and an irreducible unitary representation σ of the stabilizer G_{λ_0} of λ such that*

(1) $\rho = \mathrm{Ind}_{G_{\lambda_0}}^G(\sigma)$,
(2) $\sigma|_R = (\dim \sigma) \cdot \lambda_0$, *and*
(3) *μ is a quasi-invariant measure on the orbit $G \cdot \lambda_0$.*

e. Ergodicity and mixing. Ergodicity of measure preserving actions has a natural characterization in terms of the unitary representation of G on $L^2(X, \mu)$, and is thus a spectral invariant. Let H_0 be the (G-invariant) orthogonal complement in $L^2(X, \mu)$ to the subspace $\mathbb{C}$ of constant functions.

PROPOSITION 3.1.7. *If X is a G-space with finite invariant measure μ, and π is the unitary representation of G on the orthogonal complement H_0 of $\mathbb{C}$ in $L^2(X, \mu)$, then X is ergodic if and only if there are no nonzero $\pi(G)$-invariant vectors in H_0.*

PROOF. If the action is not ergodic, we can find a nonconstant function in $L^2(X, \mu)$ which is G-invariant (for example, the characteristic function of an invariant measurable set A such that both A and A^c have positive measures). Conversely, if we can find a nonzero element v in H_0 fixed by $\pi(G)$, then we can find a square integrable Borel function f representing v which is G-invariant. Note that if h is a square integrable function representing a G-invariant element of $L^2(X, \mu)$, then in principle h is only quasi-invariant in the sense that for each $g \in G$, $h(gx) = h(x)$ for almost all $x \in X$. If G is a countable group it is easy to modify h on a set of measure 0 so as to obtain a G-invariant function that coincides with h almost everywhere. If G is not countable, the same claim can be shown by an argument based on Fubini's theorem as in Proposition 2.2.1 □

There are other natural spectral invariants related to recurrence properties stronger than ergodicity (cf. [**S-HK**, Section 3.6]). For example, the G-action is called *weakly mixing* if there are no finite dimensional G-invariant subspaces of $L^2(X, \mu)$ other than $\mathbb{C}$.

Let H_0 be, as before, the Hilbert space defined as the orthogonal complement of $\mathbb{C}$ in $L^2(X, \mu)$. A measurable G-action is called *mixing* if the matrix coefficients of the unitary representation of G on H_0 vanish at infinity, i.e.

$$\langle \pi(g)v, w \rangle \to 0 \text{ as } g \to \infty \text{ in } G$$

for all $v, w \in H_0$. Since this cannot happen for unitary operators in finite–dimensional space, one concludes that a mixing action is also weakly mixing, justifying the terminology.

f. Discrete spectrum. A G-action is called *measurably isometric* if it is measurably conjugate to the action defined by a homomorphism $\rho : G \to Iso(M, d)$, for a compact metric space (M, d), where $Iso(M, d)$ denotes the group of isometries of (M, d). By the next theorem, due to Mackey, being measurably isometric is a property of G-spaces that can be read from the spectrum of the unitary representation of G on $L^2(X, \mu)$. Mackey's theorem generalizes the von Neumann Discrete Spectrum Theorem ([**S-HK**, Theorem 3.6.3]).

Let $\pi : G \to U(L^2(X, \mu))$ denote the representation. Then π is said to have *discrete spectrum* if $L^2(X, \mu)$ is the direct sum of finite dimensional G-invariant subspaces.

If the action is measurably isometric, then the Peter-Weyl theorem implies that it has discrete spectrum. The converse is also true:

THEOREM 3.1.8 (Mackey, [**Ma1**]). *Suppose that G is a separable locally compact group that acts ergodically on a Lebesgue space $(X, \mathcal{A}, \mu)$ with invariant measure, having discrete spectrum. Then there exists a compact group K, a homomorphism $\rho : G \to K$ onto a dense subgroup of K, and a closed subgroup L of K such that the action of G on $(X, \mathcal{A}, \mu)$ is metrically isomorphic to the action of G on K/L by translations. The invariant measure on the coset space is the image under the projection map of the Haar measure on K.*

COROLLARY 3.1.9. *A measurably isometric actions is not weakly mixing.*

2. Amenability and property T

a. Spectral characterizations of amenability. It is possible to tell whether a given group is amenable from its unitary representations, as indicated in the next theorem. In the theorem, $\infty \cdot \lambda_G$ indicates the countable direct sum of unitary representations whose factors are unitarily equivalent to the regular representation of G.

THEOREM 3.2.1. [**Grl**] *Let G be a locally compact second countable topological group. Then the following conditions are equivalent:*

(1) *G is amenable;*
(2) *The trivial 1-dimensional representation of G is weakly contained in the regular representation of G;*
(3) *Every irreducible unitary representation of G is weakly contained in the regular representation.*
(4) *The trivial 1-dimensional representation of G is weakly contained in $\infty \cdot \lambda_G$;*
(5) *Every unitary representation of G is weakly contained in $\infty \cdot \lambda_G$.*

For items 4 and 5, see [**Z1**, 7.3.6].

b. Groups with property T. Let G be a locally compact second countable topological group and ρ a unitary representation of G on a Hilbert space H. If $K \subset G$ is a compact subset and $\epsilon > 0$, a vector $\xi \in H$ is said to be (ϵ, K)-*invariant* if

$$\sup\{\|\rho(g)\xi - \xi\| : g \in K\} < \epsilon.$$

The representation is said to *almost have invariant vectors* if, for all (ϵ, K) there is a (ϵ, K)-invariant unit vector. The representation is said to have invariant vectors if there is $\xi \in H$ such that

$$\rho(g)\xi = \xi$$

for all $g \in G$.

For example, the regular representation of $\mathbb{R}$ is easily seen to almost have invariant vectors. On the other hand, there cannot be invariant vectors. In fact, if the left-regular representation of a locally compact group G has a nonzero invariant vector, then G has finite Haar measure, and it follows that G is compact.

A group G is said to have the *Kazhdan property*, or the *property T*, if any unitary representation ρ of G that almost has invariant vectors actually has a unit invariant vector. The concept was introduced by D. Kazhdan in [**Kaz**].

One has the following characterization of property T in terms of the topology introduced earlier on the space of unitary representations. (Subsection 3.1b). The *trivial representation*, ρ_0, is the representation that associates to each $g \in G$ the constant $\rho_0(g) = 1$, acting by multiplication on $\mathbb{C}$. The reader is referred to [**lH-V**] for the proof of the following facts.

PROPOSITION 3.2.2. *Let G be a locally compact group. Then the following are equivalent:*

(1) *G has the property T;*
(2) *There exists a neighborhood V of ρ_0 in $\tilde{G}$ such that every $\rho \in V$ has nonzero invariant vectors.*
(3) *The trivial representation is an isolated point of $\hat{G}$.*
(4) *Every finite dimensional irreducible unitary representation of G is an isolated point of $\hat{G}$*

Compact groups have the property T because invariant vectors may be obtained by averaging with respect to Haar measure. Theorem 3.2.1 indicates that in a certain sense property T is opposite to amenability. In particular if an amenable group has property T, then it must be compact. This follows from the next theorem. (Cf. [**Schm, Z14**]. See also [**Z1**].)

THEOREM 3.2.3. *Let G be a group with the Kazhdan property T and X an ergodic G-space with invariant probability measure. Let $\alpha : G \times X \to H$ be a cocycle into an amenable group H. Then α is equivalent to a cocycle into a compact subgroup of H.*

Among the connected Lie groups with finite center those with property T are for the most part the semisimple groups whose simple factors have real-rank greater than 1. (See, for example, [**Mar2**].)

THEOREM 3.2.4. *Let G be a connected semisimple Lie group with finite center, each of whose factors has real rank at least 2. Then G, as well as any lattice in G, has Kazhdan's property T. Moreover, the rank 1 groups $Sp(1,n)$ $(n \geq 2)$ and $F_{4(-20)}$, as well as the semi-direct product $\mathbb{R}^n \rtimes SL(n,\mathbb{R})$ for $n \geq 3$, all have the property T.*

c. $SL(3,\mathbb{R})$ **has property T.** Theorem 3.1.2 implies that $SL(2,\mathbb{R})$ does not have Property T. Let us sketch a proof of the fact that $SL(3,\mathbb{R})$ has property T. The reader will find the details (for a general higher rank connected semisimple Lie group) in [**Z1**] or [**lH-V**], for example.

The following notation will be used: H will denote the subgroup of G consisting of matrices of the form

$$[A,b] := \begin{pmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ 0 & 0 & 1 \end{pmatrix}$$

where the block matrix A is in $SL(2,\mathbb{R})$ and $b \in \mathbb{R}^2$. Then H is isomorphic to the semidirect product, of $SL(2,\mathbb{R})$ with the subgroup R (isomorphic to $\mathbb{R}^2$) consisting of matrices of the form $[I,b]$, where I is the identity matrix.

Let ρ be a unitary representation of $G = SL(3,\mathbb{R})$. Suppose that ρ almost has invariant vectors, but (for a contradiction) ρ has no invariant vectors. By Theorem 3.3.3, $\rho|_R$ also has no invariant vectors.

We claim that $\rho|_H$ is weakly contained in $\infty \cdot \rho_H^{\mathrm{reg}}$. Before showing this fact we note that since ρ almost has invariant vectors, then the trivial representation is weakly contained in ρ, hence also in $\rho|_H$. Therefore, the claim implies that the trivial representation is weakly contained in $\infty \cdot \rho_H^{\mathrm{reg}}$. But this is in contradiction with Theorem 3.2.1 since H is not amenable.

By the general properties of direct sums of unitary representations, it will suffice to prove the claim under the assumption that $\rho|_H$ is irreducible. Then, by Proposition 3.1.5, $\rho|_H$ is unitarily equivalent to $\mathrm{Ind}_{H_0}^H(\sigma)$, where σ is an irreducible unitary representation of the stabilizer group H_0 of some $\lambda \in R^*$. If $\lambda = 0$, then by the same proposition $\rho|_R$ is trivial, which is not the case. If $\lambda \neq 0$, a simple matrix computation shows that the stabilizer subgroup of λ in H is conjugate to the Heisenberg group $\{[A,b] \in H : a_{11} = a_{22} = 1\}$. But the Heisenberg group is amenable, hence by Theorem 3.2.1 σ is weakly contained in $\infty \cdot \rho_{H_0}^{\mathrm{reg}}$. On the other hand, by Proposition 3.1.5, $\mathrm{Ind}_{H_0}^H(\sigma)$ is weakly contained in $\mathrm{Ind}_{H_0}^H(\infty \cdot \rho_{H_0}^{\mathrm{reg}}) = \infty \cdot \mathrm{Ind}_{H_0}^H(\rho_{H_0}^{\mathrm{reg}})$, and the latter is equivalent to $\infty \cdot \rho_H^{\mathrm{reg}}$. Therefore, $\rho|_H$ is weakly contained in $\infty \cdot \rho_H^{\mathrm{reg}}$ as claimed.

d. Actions with the property T. Just as amenability can be extended from groups to group actions, one can also define the notion of a group action with the property T. This was done, for discrete groups, in [**Z14**].

Let X be an ergodic G-space, where G is a discrete countable group. We write $\mathcal{G} = G \times X$ and let $\tilde{\mathcal{G}}$ denote the set of equivalence classes of cocycles $\alpha : \mathcal{G} \to U(V)$, where $U(V)$ denotes the unitary group of a separable Hilbert space V. Let $B_1(V) = \{v \in V : \|v\| \leq 1\}$ be the unit ball in V and define the space $L^\infty(X, B_1(V))$ of essentially bounded measurable functions $\varphi : X \to B_1(V)$ with $\|\varphi\|_\infty = 1$. Given $\varphi, \psi \in L^\infty(X, B_1(V))$, write $f_{\alpha,\varphi,\psi} : \mathcal{G} \to \mathbb{C}$ for

$$f_{\alpha,\varphi,\psi}(g, x) = \langle \alpha(g, x)\varphi(x), \psi(gx)\rangle.$$

A unitary cocycle $\alpha : \mathcal{G} \to U(V)$ (or, more precisely, its class in $\tilde{\mathcal{G}}$) will be called a limit of a sequence of unitary cocycles $\alpha_n : \mathcal{G} \to U(V_n)$ if given $\varphi_1, \ldots, \varphi_k \in L^\infty(X, B_1(V))$ there exist $\psi_1^n, \ldots, \psi_k^n \in L^\infty(X, B_1(V_n))$ such that for each $g \in G$ and each i, j with $1 \leq i, j \leq k$, we have that the sequence $f_{\alpha_n,\psi_i^n,\psi_j^n}$ converges in measure to $f_{\alpha,\varphi_i,\varphi_j}$. Note that if (the equivalence class of) α is contained, as a direct summand, in (the equivalence class of) α_n for all sufficiently large n, then $\alpha_n \to \alpha$.

Let I denote the identity cocycle: $I(g, x) = 1$ for all $(g, x) \in \mathcal{G}$. An ergodic G-space, X, with countable discrete group G, will be said to have the property T if whenever $\alpha_n \to I$, then I is contained in α_n for all sufficiently large n.

THEOREM 3.2.5. *Suppose that the G-space, (X, μ), and G'-space, (Y, ν), considered below are ergodic and G, G' are countable (discrete) groups.*

(1) *If the actions are essentially free and orbit equivalent, then the G-space has the property T if and only if the G'-space has the property T.*
(2) *If the G-action has a finite invariant measure and G has the property T, then the G-action also has the property T.*
(3) *If the G-action has the property T then G has the property T .*

Part (1) is a special case of [**Z14**, Theorem 2.3] and parts (2) and (3) are in [**Z14**, Proposition 2.4], although in [**Z14**, Proposition 2.4] there is the additional assumption that the action is weakly mixing. It was observed by A. Furman that one can use Theorem 8.3 of [**Furm2**], due to Bekka and Valette, to eliminate the weakly mixing assumption.

Note that if (X, μ) is an ergodic G-space with the property T, then the Radon-Nikodym cocycle for the action is equivalent to the trivial cocycle, since the only compact subgroup of the multiplicative group $\mathbb{R}^\times$ is the trivial group. Therefore, there exists a σ-finite G-invariant measure in the same class of μ.

3. Howe-Moore ergodicity theorem

The Howe-Moore ergodicity theorem is a fundamental and very useful result about the ergodic theory of noncompact semisimple groups. It implies that given an ergodic action of a noncompact simple Lie group with finite center, preserving a finite measure, every noncompact closed subgroup also acts ergodically.

Let X be a G-space with a finite invariant measure. X is called *irreducible* if every normal subgroup of G not contained in the center acts ergodically on X.

THEOREM 3.3.1 (Howe-Moore ergodicity theorem). *Let G be a semisimple Lie group with finite center and no compact simple factors, and X an irreducible G-space with finite G-invariant measure. If H is a closed noncompact subgroup of G, then H also acts ergodically on X.*

The previous theorem provides a powerful criterion of ergodicity for homogeneous actions, as the next corollary illustrates.

COROLLARY 3.3.2. *Let G be a simple noncompact Lie group with finite center and let Γ be a lattice in G. Then any closed noncompact subgroup L of G acts ergodically on G/Γ by left-translations.*

PROOF. G clearly acts ergodically on G/Γ, since the action is transitive. By the Howe-Moore theorem, the L-action must also be ergodic. □

The Howe-Moore ergodicity theorem is in fact a spectral result. In view of the characterization of ergodicity in terms of the unitary representation of G on $L^2(X,\mu)$ (Proposition 3.1.7), the theorem results from the following fact.

For any connected noncompact simple Lie group G with finite center, and unitary representation π of G with no nonzero invariant vectors, a closed subgroup L of G such that $\pi|_L$ has nonzero invariant vectors must be compact.

Observe that given a nontrivial L-invariant vector $v \in H$, the function $f(g) := \langle \pi(g)v, w\rangle$ is constant on L. Therefore, it is sufficient to prove that for all $v, w \in H$, $\langle \pi(g)v, w\rangle$ approaches 0 as $g \to \infty$ in G. This is the content of the next theorem.

THEOREM 3.3.3 ([**HM, She, Z7**]). *Let G_i be, for each i, a connected noncompact simple Lie group with finite center, and let $G = \prod_i G_i$ be a finite direct product. Let V be a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$, and $\pi : G \to \mathbf{U}(V)$ a unitary representation such that for each i, $\pi|_{G_i}$ has no invariant vectors. Then for all $u, v \in V$,*

$$\lim_{g\to\infty} \langle \pi(g)u, v\rangle = 0.$$

In other words, $\pi(g)$ tends to 0 in the weak operator topology as g tends to infinity in G.

We sketch the proof of Theorem 3.3.3 for $G = SL(2,\mathbb{R})$. We recall that a connected, semisimple Lie group G with finite center admits a KAK decomposition, where K is a maximal compact subgroup, A is abelian, and $\mathrm{Ad}(A)$ is diagonalizable over $\mathbb{R}$. For $SL(2,\mathbb{R})$, we can take $K = SO(2)$ and A the group of diagonal matrices with positive diagonal entries and determinant 1.

Since K is compact, in order to prove Theorem 3.3.3 it suffices to show that

$$\lim_{a\to\infty} \langle \pi(a)u, v\rangle = 0$$

where a goes to infinity inside A.

For any topological group G and $a \in G$, define the *stable (horospherical) group* of a, G_a^s, as the closure of

$$\{g \in G : \lim_{m\to+\infty} a^m g a^{-m} = e\}.$$

Also define the *unstable (horospherical) group* of a as $G_a^u = G_{a^{-1}}^s$.

For $SL(2,\mathbb{R})$, if we write

$$a_t := \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix},\ h_s^+ := \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix},\ h_s^- := \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix},$$

then

$$a_{-t}h_s^{\pm}a_t = h^{\pm}_{se^{\mp 2t}}$$

In this case, $h_s^{\pm}$ define 1-parameter horospherical groups for $a_t, t \geq 0$.

The next lemma contains a key idea of the proof. (This is the so called *Mautner phenomenon.*)

LEMMA 3.3.4. *Let V be a separable Hilbert space and G a locally compact, second countable group. Let $\pi : G \to \mathbf{U}(V)$ be a (strongly continuous) unitary representation of G. Let v be a vector in V that is fixed by an element $a \in G$. Then v is also fixed by all elements in G_a^s and G_a^u.*

PROOF. Suppose that $a^m g a^{-m}$ converges to the identity element and let v be any vector fixed by $\pi(a)$. Hence v is also fixed by $\pi(a)^{-1}$ and, since $\pi(a)$ is unitary, we have for all m

$$\|\pi(g)v - v\| = \|\pi(a^m g a^{-m})v - v\| \to 0.$$

Therefore, by continuity, $\pi(g)v = v$ for all $g \in G_a^s$. A similar argument applies to G_a^u. □

We follow an idea due to Ellis and Nerurkar, [**EN**]. Let V be a separable Hilbert space and $\pi : G \to \mathbf{U}(V)$ a unitary representation such that no normal subgroup of G not contained in the center leaves invariant a nonzero vector in V. Fix an element $a \in A$, $a \neq e$, and define

$$W = \{v \in V \mid \pi(a)v = v\}.$$

We claim that $W = 0$. To prove the claim, it suffices to show that W is stable under G since, if this is the case, a would be in the kernel of $g \in G \mapsto \pi(g)|_W$, so the kernel would be a noncentral normal subgroup of G fixing W pointwise, therefore $W = 0$.

We now show that W is stable under G. Note that the subgroup $\check{G}$ of G that stabilizes W is closed, hence a Lie subgroup. If $\check{\mathfrak{g}}$ denotes the Lie algebra of $\check{G}$, then it suffices to show that $\mathfrak{g} = \check{\mathfrak{g}}$.

A is diffeomorphic to its Lie algebra $\mathfrak{a}$ via the exponential map. Therefore, we can write $a = \exp X$ for some $X \in \mathfrak{a}$. It will be supposed for definiteness that the first diagonal entry of X is positive (hence the second is negative). The opposite case is treated in essentially the same way.

The Lie algebra $\mathfrak{g}$ decomposes as a direct sum

$$\mathfrak{g} = \mathfrak{u}^- \oplus \mathfrak{a} \oplus \mathfrak{u}^+$$

where $\mathfrak{u}^-$ (respectively, $\mathfrak{u}^+$) is the subalgebra spanned by a matrix whose only nonzero entry is the intersection of the first row with the second column (respectively, second row with first column). The groups $U^{\pm}$ obtained by exponentiating $\mathfrak{u}^{\pm}$ are contained in the stable and unstable subgroups of a, so that $\mathfrak{u}^{\pm}$ are contained in $\check{g}$, by the lemma, and $\mathfrak{a} \subset \check{\mathfrak{g}}$ since any element that centralizes a must stabilize W. Therefore $\mathfrak{g} = \check{\mathfrak{g}}$, as claimed. Consequently, W is stable under G.

Let now $a_m = a_{t_m} \in A, t_m \geq 0$, be a sequence tending to infinity such that $\pi(a_m)$ converges in the weak operator topology to an operator $T \in \mathcal{B}$. We want to show that $T = 0$. Write $a_m = \exp(X_m)$. We assume that the first diagonal entry of X_m is positive, hence it approaches $+\infty$ as m goes to ∞. Choose a nonzero Y

whose only nonzero entry is y_{12} and write $u^+(t) := \exp(tY), u^-(t) := \exp(-tY^T)$, where Y^T is the transpose of Y. Then, for all $t \in \mathbb{R}$, we have

$$a_m u^{\pm}(t) = u(e^{\pm\alpha(X_m)}t)a_m$$

where $\alpha(X) = x_{22} - x_{11}$.

Recall that $\pi(a_m)$ converges weakly to T and note that $\pi(u^{\pm}(e^{\alpha(X_m)}t))$ converges strongly to the identity operator as m tends to ∞. Therefore,

$$T\pi(u(t)) = T,$$

for all $t \in \mathbb{R}$. Of course we also have $T^*T\pi(u(t)) = T^*T$. These equations show that T^*T and T cannot be invertible; if they were, the kernel of π in G would contain the infinite group generated by the elements $u^+(t)$, hence it could not be contained in the finite center of G. (Recall that we are assuming that no noncentral normal subgroup of G has invariant vectors under the representation π.)

Denote $S = T^*T \in \overline{\pi(A)}\,\overline{\pi(A)} \subset \overline{\pi(A)}$. Since S is not invertible, it must belong to the boundary $\overline{\pi(A)} - \pi(A)$. Therefore, S is a weak limit for a sequence $\pi(a'_m)$, $a'_m \in A$ tending to infinity.

We now repeat for S the same argument used above for T. Namely, write $a'_m = \exp(X'_m)$, and assume that the X_m have positive first diagonal entry. Define $\alpha(X) = X_{22} - X_{11}$. Then, for all $t \in \mathbb{R}$, we have as before

$$a'_m u^+(t) = u^+(e^{\alpha(X'_m)}t)a'_m$$
$$u^-(t)a'_m = a'_m u^+(e^{\alpha(X'_m)}t)$$

for all $t \in \mathbb{R}$. Applying π and passing to the limit as $m \to \infty$, we conclude, as we did for T, that $S\pi(u^+(t)) = S$ and $\pi(u^-(t))S = S$. Note that

$$(S\pi(u^+(t)))^* = \pi(u^+(t))^*S^* = \pi(u^+(-t))S.$$

Therefore, we have $\pi(u^+(t))S = S$ and $\pi(u^-(t))S = S$ for all $t \in \mathbb{R}$.

The closure in $G = SL(2, \mathbb{R})$ of the group generated by $u^+(t)$ and $u^-(t)$, $t \in \mathbb{R}$, is G itself. By strong continuity of π, we have $\pi(a)S = S$, for $a \in A$. Note that $\pi(a)$ fixes each element in the image of S in H so that $W := \{v \in H | \pi(a)v = v\}$ contains that image. Therefore $W = 0$, so $S = 0$. On the other hand, $S = T^*T$, so for each $v \in H$

$$0 = \|Sv\| = \langle T^*Tv, v\rangle = \langle Tv, Tv\rangle = \|Tv\|$$

and T itself must be zero, which is what we wanted to show.

CHAPTER 4

Main classes of examples

In this chapter we review the principal constructions of actions of semisimple Lie groups and their lattices.

1. Homogeneous G-spaces

The first class of examples appears in the general context of homogeneous dynamics, i.e., as the restriction to a subgroup of the left action of a Lie group on its right homogeneous spaces, and projections of these restrictions to some special double coset spaces. Basics of homogeneous dynamics, as well as characteristic examples of cyclic homogeneous dynamical systems, appear in [**S-HK**, Sections 2.1b, 3.3c] and [**S-KSS**, Sections 1.4, 1.5].

a. Ergodicity for homogeneous actions. Let $\rho : G \to H$ be a group homomorphism and H_0 a closed subgroup of H. Then G acts on the quotient H/H_0 by left-translations:

$$\Phi(g, hH_0) = \rho(g)hH_0.$$

The Howe-Moore ergodicity theorem provides many nontrivial examples of Haar measure preserving ergodic homogeneous actions.

If G is a simple noncompact Lie group with finite center and Γ is a lattice in G, then by Corollary 3.3.2 any closed noncompact subgroup L of G acts ergodically on G/Γ by left-translations.

For example, if H is a semisimple subgroup of G without compact factors and Λ is a lattice in H the action of λ on G/Γ by left-translations is Haar measure preserving and ergodic. This is the first standard class of smooth ergodic actions by lattices in semisimple groups.

Using the next proposition we conclude that the action of Γ on G/L by translations is also ergodic, where L and Γ are as before.

PROPOSITION 4.1.1. *If H_1, H_2 are two closed subgroups of a locally compact group G, then H_1 acts on G/H_2 ergodically if and only if H_2 acts on G/H_1 ergodically.*

PROOF. Any Borel set in G/H_1 invariant under the left action of H_2 lifts to a Borel set in G invariant under both the right action of H_1 and the left action of H_2 and hence projects to H_2/G to a Borel set invariant under the right action of H_1. Since the latter is isomorphic to the left action of H_1 on G/H_2 the proposition follows. □

If G and H are unimodular groups, then the G-action on G/H by translations leaves invariant a Borel measure. If G is semisimple and Γ is a lattice of G (hence both are unimodular groups), the action of any noncompact closed subgroup $L \subset G$ leaves invariant a probability measure with respect to which the action is ergodic, as remarked above. On the other hand, the action of Γ on G/L need not preserve any measure at all. Consider the following example: let $G = SL(2,\mathbb{R})$ and L the subgroup of upper triangular matrices. The action $\Gamma = SL(2,\mathbb{Z})$ on G/L by translations does not preserve any Borel measure ([**S-HK**, Example 4.2.3]).

More generally, if G is a non-compact connected simple Lie group and L is a closed amenable subgroup, then the G-action on G/L does not preserve any probability measure. When L is a minimal parabolic subgroup (defined after Proposition 4.1.2 below) then L is amenable and G/L is compact. Moreover, this action is amenable. The standard example of a minimal parabolic subgroup is the subgroup of upper-triangular matrices in $SL(n,\mathbb{R})$. This example already appeared in Section 2.5b in connection with the notion of amenable action. See Subsection 4.1c and [**Bor**] for a more detailed discussion of parabolic subgroups.

These are all examples of *boundary actions*, which we now consider in more detail.

b. The Furstenberg boundary, boundary actions and parabolic subgroups. Let G denote a general locally compact second countable group and X a compact metric G-space with distance function d. We say that a sequence F_n of subsets of X converges to $x \in X$ if the diameter of $F_n \cup \{x\}$ approaches 0, where the diameter of a subset $L \subset X$ is defined as the supremum of $d(x,y)$ over $x, y \in L$.

The G-space X is called *proximal* if for any pair of points $x, y \in X$ there is a sequence $g_n \in G$ such that $d(g_n x, g_n y) \to 0$ as $n \to \infty$. This is equivalent to the existence, for each pair x, y, of an element $z \in X$ and a sequence $g_n \in G$ such that $g_n x$ and $g_n y$ converge to z as $n \to \infty$. For example, the natural action of $GL(n,\mathbb{R})$ on projective space $P^{n-1}(\mathbb{R})$ is a transitive proximal action.

It is not difficult to show that if X is a proximal G-space and L is any finite subset of X, then there exists a sequence $g_n \in G$ such that the diameter of $g_n L$ approaches 0. A set $L \subset X$ with the property that the diameter of $g_n L$ approaches 0 for some sequence $g_n \in G$ will be called *contractible*.

A G-space X is said to be *strongly proximal* if for any probability measure μ on X there exist both a sequence $g_n \in G$ and $x \in X$ such that $g_{n*}\mu$ converges to the probability measure δ_x supported on x. X will be called a *boundary* of G if X is a minimal strongly proximal G-space. It is immediate that strongly proximal implies proximal. The converse is not true in general, but it is not difficult to show that (see [**Mar2**, VI.1.6]) if X is proximal and each point $x \in X$ has a contractible neighborhood, then X is strongly proximal; and if X is a minimal, proximal G-space that contains a non-empty open contractible subset, then X is a boundary of G.

A compact G-space (not necessarily metrizable) is said to be a *boundary* of G if for every probability measure μ on X the closure of $\{g_*\mu \mid g \in G\}$ in the space of probability measures on X contains point measures. A compact G-space X is said to be a *universal boundary* of G if X is a boundary of G and every boundary of G is the image of X under some G-map. X is also referred to as the *maximal Furstenberg boundary* of G. It is shown in [**Fur4**] that every group G has a universal boundary, $B(G)$ (not necessarily metrizable), which is unique up to isomorphism.

Proposition 4.1.2 ([**Fur4**]). *The following properties hold:*

(1) *If G is amenable, then any boundary of G consists of a single point.*

(2) *If G contains a closed amenable subgroup H such that G/H is compact, then G/H is a universal boundary of G.*
(3) *If G is a connected Lie group, then G contains a closed subgroup H such that $B(G) = G/H$. Furthermore, H is uniquely determined up to conjugacy. For a connected semisimple Lie group with finite center and Iwasawa decomposition $G = K \cdot A \cdot N$, then H is the normalizer in G of the solvable subgroup $A \cdot N$.*

If G is a connected semisimple Lie group, then the subgroup H defined in the proposition is the *minimal parabolic subgroup* of G. We will denote a minimal parabolic subgroup of G by P. More generally, a closed subgroup Q of G is said to be a *parabolic* subgroup if it contains a minimal parabolic subgroup. The G-spaces of the form G/Q, where Q is a parabolic subgroup of G are all the G-factors of the universal boundary (maximal Furstenberg boundary) G/P. It can be shown that that are 2^r of these Q, if G has real rank r. Consequently, there are 2^r G-factors of the maximal Furstenberg boundary of G, which are all of the form G/Q.

c. Parabolic subgroups of $SL(n,\mathbb{R})$ and projective actions. As an example consider the group $SL(n,\mathbb{R})$ with its minimal parabolic subgroup of upper–triangular matrices. Then any parabolic subgroup is determined by a representation of n as the (ordered) sum of natural numbers $i_1 + \cdots + i_r$ and consists of upper block–triangular matrices with diagonal blocks of size i_k, $k = 1, \ldots, r$. The rank of $SL(n,\mathbb{R})$ is equal to $n-1$ and there are exactly 2^{n-1} such subgroups including the minimal parabolic (that corresponds to the maximal boundary), and the group $SL(n,\mathbb{R})$ itself (that corresponds to the trivial single–point boundary). Note that any parabolic subgroup other than the minimal one is not amenable since it contains a subgroup isomorphic to $SL(2,\mathbb{R})$.

The maximal boundary is identified with the *complete flag variety* in $\mathbb{R}^n$, which is defined as follows. A *complete flag* in $\mathbb{R}^n$ is a chain of subspaces

$$0 = V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = \mathbb{R}^n$$

with $\dim V_i = i$ for each i. The set of complete flags in $\mathbb{R}^n$ can be shown to have the structure of an algebraic variety. Note that $G = SL(n,\mathbb{R})$ acts transitively on the set of all complete flags in $\mathbb{R}^n$ and that the stabilizer subgroup, P, of the standard flag

$$\{0\} \subset \mathbb{R}^1 \subset \mathbb{R}^2 \subset \cdots \subset \mathbb{R}^{n-1} \subset \mathbb{R}^n$$

is a minimal parabolic subgroup (of upper triangular matrices). The complete flag variety for $\mathbb{R}^n$ is thus a homogeneous space of the form G/P.

Other boundaries are easily identified with spaces of partial flags where subspaces of certain dimensions are missing. For example, the representation $n = (n-k) + k$ produces the Grassmannian manifold $G_{n,k}$ of all k-dimensional subspaces as a boundary of $SL(n,\mathbb{R})$. Since the case $k = 1$ gives the projective space $\mathbb{R}P(n-1)$ and the corresponding boundary action is the standard action by projective transformations, the boundary actions of semisimple groups are sometimes called *projective actions*.

The restriction of a projective action to a lattice subgroup $\Gamma \subset G$ produces an action of Γ which is properly ergodic by Proposition 4.1.1, minimal, and has no finite Borel invariant measures. This is another standard class of smooth ergodic actions by lattices in semisimple groups. Such actions will also be referred to as projective actions.

We register here for later use the following fundamental fact.

THEOREM 4.1.3 (Furstenberg's lemma). *Suppose that G is a connected semisimple Lie group and P is a minimal parabolic subgroup. Let X be a compact metric Γ-space, where Γ is a lattice in G. Then, possibly after discarding an invariant Borel null set in G/P (with respect to the smooth measure class), there is a measurable Γ-map φ from G/P into the space of probability measures on X.*

PROOF. This follows from the discussion on amenable actions of section 2.5.b and the fact that the G-action on G/P is amenable (Proposition 2.5.2 (4)). □

d. Stationary measures and Furstenberg entropy. The boundary actions considered in the previous section are essential for the description of actions of semisimple Lie groups without invariant measures. We discuss here for later reference some further results, due to Furstenberg, concerning boundary actions. (See [**Fur1**], [**Fur2**], [**Fur3**], [**Fur4**].) For details on this, the reader should consult the survey [**S-F**].

Let X be a standard Borel G-space. Given a probability measure ν on X and a probability measure μ on G, define the *convolution* of μ and ν by

$$\mu * \nu = \int_G g_* \nu \; d\mu(g).$$

The ν is called *μ-stationary* if $\mu * \nu = \nu$. A measure class is called stationary if it contains a stationary measure.

Suppose now that G is a connected semisimple Lie group with finite center and let K be a maximal compact subgroup of G. A probability measure on G is called *admissible* if it is absolutely continuous with respect to a Haar measure, and its support generates G as a semigroup. Such measures are easily shown to exist, and may be assumed to be invariant under the right and left actions of K.

It can be shown that if μ is admissible and ν is any μ-stationary probability measure on X, then the measure class represented by ν is G-invariant. Moreover, by the Kakutani-Markov fixed point theorem, for any compact G-space, there exists at least one μ-stationary probability measure.

THEOREM 4.1.4 (Furstenberg). *Let Q be a parabolic subgroup of G, where G is a connected semisimple Lie group with finite center. Let μ be an admissible measure on G. Then there exists a unique μ-stationary measure ν on G/Q. This measure is in the measure class of the smooth measure and if μ is left K-invariant, for some maximal compact subgroup K, then ν is the unique K-invariant measure on G/Q.*

An important numerical invariant associated to a stationary measure is its *Furstenberg entropy.* Let (X, ν) be a G-space with a stationary measure, and μ an admissible measure on G. Define the Furstenberg entropy of (X, ν) as

$$h_\mu(X, \nu) = \int_G \int_X - \log \frac{d(g_*^{-1}\nu)}{\nu}(x) \; d\nu(x) d\mu(g).$$

It is not difficult to show that $h_\mu(X, \nu)$ is non-negative, and that it is zero exactly when ν is a G-invariant measure.

2. Automorphisms of compact groups and related examples

Actions by automorphisms of compact groups also yield many examples of ergodic G-spaces.

a. Inner and outer automorphisms. Let G be a compact group with its Haar probability measure. Denote by $\mathrm{Aut}(G)$ the group of automorphisms of G, i.e. of isomorphisms of G onto itself. Each $g \in G$ corresponds to an element of $\mathrm{Aut}(G)$ defined by conjugation by g and these automorphisms form a normal subgroup of $\mathrm{Aut}(G)$ of *inner automorphisms*, denoted $\mathrm{Inn}(G)$. The quotient $\mathrm{Aut}(G)/\mathrm{Inn}(G)$ is the group of outer automorphisms, denoted $\mathrm{Out}(G)$. If G is abelian, $\mathrm{Inn}(G)$ is trivial, and if G is a connected semisimple Lie group it can be shown that $\mathrm{Inn}(G)$ is a finite index subgroup in $\mathrm{Aut}(G)$.

The subgroup of inner automorphisms never acts ergodically on G. In fact, if it did, G would act on itself by conjugation transitively, which is impossible since the identity element is a fixed point.

b. Lattice actions on the torus. As a first example, let $G = \mathbb{T}^n$ – the n-dimensional torus. The group of volume preserving automorphisms of $\mathbb{T}^n$ is $GL(n, \mathbb{Z})$. The orientation preserving automorphisms comprise $SL(n, \mathbb{Z})$, which acts ergodically on $\mathbb{T}^n$. This follows from an analysis of the dual action on $\mathbb{Z}^n$: the orbit of any character other than the trivial one is infinite. In fact, by the same argument, a matrix $A \in GL(n, \mathbb{Z})$ acts on the torus ergodically with respect to the Haar measure if and only if there are no roots of unity among its eigenvalues. A simple index type argument shows that any ergodic automorphism of the torus has a fixed point. It also has a dense set of finite orbits since points with rational coordinates have finite orbits. (See [**KKS**].)

To obtain more general lattice actions on tori, first note that if Γ is a discrete group and $\rho : \Gamma \to SL(n, \mathbb{Z})$ is a homomorphism such that the corresponding representation of Γ on $\mathbb{R}^n$ is irreducible, then Γ acts on $\mathbb{T}^n$ ergodically. By the Margulis arithmeticity theorem 1.5.3, if G is a connected semisimple Lie group with trivial center, no compact factor and real rank at least 2 and Γ is an irreducible lattice in G, then Γ is arithmetic. It follows that Γ has a subgroup of finite index that admits an irreducible monomorphism into $SL(n, \mathbb{R})$ for some n and hence acts ergodically on $\mathbb{T}^n$. This is the third standard class of smooth actions by lattices after homogeneous (Section 4.1a) and projective (Section 4.1c) actions. Like the former and unlike the latter these actions preserve smooth measures.

The suspension construction (Section 2.3c) applied to an ergodic action of a lattice $\Gamma \subset G$ by automorphisms of a torus produces an ergodic action of the Lie group G on a smooth manifold M (which is compact if and only if the lattice Γ is cocompact). This manifold is a fiber bundle over G/Γ with the torus fiber. This is the second class of examples of ergodic volume preserving actions of semisimple Lie groups. (The first class we considered were the homogeneous examples of Section 4.1a.

c. Affine actions and actions on nilmanifolds. There are several ways to extend the constructions of the previous section. Two of them produce new examples of smooth volume preserving actions.

1. *Affine actions.* Let $\pi : \Gamma \to \mathrm{Aff}(\mathbb{T}^n)$ be a homomorphism of a discrete group Γ to the group of affine maps of the torus. An affine map can be written as a composition of a linear map and a translation. The linear parts form a subgroup of automorphisms. In general, of course, ergodic affine actions may have nonergodic linear parts (e.g., an action by pure translations, in which case Γ must be abelian if the representation is faithful; or when the linear part is unipotent. See [**S-HK**,

Section 8.3a]). It is interesting to note, however, that both for actions of $\mathbb{Z}^k,\ k \geq 2$ containing elements with ergodic linear part, and for a faithful action of an irreducible lattice in a higher-rank semisimple Lie group, the affine action has a finite index subgroup with a fixed point. As a result, it is isomorphic to an action by automorphisms. In the abelian case this follows from the observation that the orbit of a fixed point of an ergodic element consists of fixed points, so that it must be a finite orbit. In the semisimple case this is a corollary of the Margulis superrigidity theorem.

2. *Automorphisms of nilmanifolds.* Nontrivial outer automorphisms preserving lattices exist not only for $\mathbb{R}^n$ but also for some simply connected nilpotent groups. Lattices in such groups are always cocompact and the quotient of the nilpotent group by the lattice is called a (compact) nilmanifold. Nilmanifolds can be described as extensions of tori with tori for fibers. Accordingly, actions by automorphisms of nilmanifolds appear as extensions of actions by automorphisms of tori.

d. Bernoulli actions. Going beyond differentiable actions on manifolds there are other constructions, of which we mention a simple but very general one. Suppose K is a compact group, G is a discrete group and $X = K^G$ is the space of all sequences $\{x_g\}_{g\in G}$ such that $x_g \in K$. Let μ_0 be the normalized Haar measure on K and consider the product measure μ on X. Then X is a compact group and the action of G on itself by left-translations induces an action of G on X by automorphisms preserving μ. It will be called a *Bernoulli action* by analogy with Bernoulli shifts. Of course this action has other invariant measures, which can be also called Bernoulli, given by products of identical copies of other Borel measures on K.

A G-space X with invariant measure μ is said to have a *countable Plancherel spectrum* if the canonical unitary representation of G on $L^2(X,\mu)$ decomposes as a countable direct sum in which each factor is isomorphic to the regular representation of G. It can be shown that the spectrum of a Bernoulli action is countable Plancherel.

3. Isometric actions

a. Isometry group of a metric space. Let X be a compact metric space with metric $d(\cdot,\cdot)$. A continuous action of a topological group G on X is called *isometric* if

$$d(gx, gy) = d(x, y)$$

for all $g \in G$ and all $x, y \in X$.

One easily shows that closures of orbits of isometric actions are either disjoint or coincide. Therefore, X decomposes as a disjoint union of orbit closures. In particular, if the action is topologically transitive (i.e., has a dense orbit) then the action is minimal (all orbits are dense) ([**S-HK**, Section 2.2,4]). Furthermore, if Z_x is the closure of the orbit Gx, then the function $x \mapsto Z_x$ from X into the space of all compact subsets of X with the Hausdorff metric is continuous. If $R \subset X \times X$ is the equivalence relation on X that identifies two points if and only if their orbit closures coincide, then R is closed in $X \times X$. If X is a regular (T_3) space, it follows that the quotient space X/R (with the quotient topology) is a metric space.

In this case, the ergodic decomposition of a G-invariant Borel measure corresponds to the disintegration of the measure with respect to the quotient map $X \to X/R$.

The following result is due to van Danzig and van der Waerden [**vDvW**]. (See also [**Kob**].)

THEOREM 4.3.1. *Let X be a connected, locally compact metric space with metric d and $Iso(X,d)$ the group of isometries of X. For each point $x \in X$, let $Iso(X,d)_x$ denote the isotropy subgroup of $Iso(X,d)$ at x. Then $Iso(X,d)$ is locally compact with respect to the compact-open topology and $Iso(X,d)_x$ is compact for every x. If X is compact, then $Iso(X,d)$ is compact.*

It follows from Theorem 4.3.1 that if a topological group G acts isometrically on a locally compact metric space X and the action is topologically transitive, then the group $\mathrm{Iso}(X,d)$ acts on X transitively and X is homeomorphic to the homogeneous space $\mathrm{Iso}(X,d)/\mathrm{Iso}(X,d)_x$.

When X is a Riemannian manifold, by a result of Myers and Steenrod its group of isometries is a Lie transformation group with respect to the compact-open topology (see [**Kob**]). There are, *a priori*, two definitions of isometry for a Riemannian manifold. In one, a diffeomorphism of X is called an isometry if it preserves the metric tensor; in the other definition, the transformation preserves the Riemannian distance function. It can be shown that the two definitions are equivalent. (See also [**S-HK**, Section 7.1c,d] for further information on this subject.)

If G is a Lie group and X is a compact Riemannian manifold on which G acts topologically transitively by (smooth) isometries, then X is isometric to a homogeneous space of the form K/K_0, where K is a compact Lie group (the connected component of the identity of the group of isometries of X), K_0 is a closed subgroup and the G-action on M is isomorphic to a G-action on K/K_0 by left-translations via a homomorphism from G into a dense subgroup of K. Conversely, given a homomorphism of Lie groups $\rho : G \to K$, where K is compact, one obtains an action of G on K by translations, which is ergodic if and only if $\rho(G)$ is dense. In this case, the action of G by translations on any quotient $M := K/K_0$ by a closed subgroup K_0 will be ergodic and isometric, for any K-invariant Riemannian metric on M.

b. Isometric lattice actions. Homomorphisms with dense images from lattices in noncompact semisimple Lie groups into compact Lie groups produce the fourth standard class of smooth ergodic lattice actions (after the left actions on right homogeneous spaces of Section 4.1a, the projective actions of Section 4.1c and actions by automorphisms of torus and related constructions, studied in Section 4.2b. All these classes except for projective actions have natural smooth invariant measures.

Now we discuss one example of homomorphism of this kind that is essentially arithmetic in nature. Fix positive integers N, n such that $n < N$ and consider the quadratic form

$$Q(x) = \sum_{i=1}^{N-n} x_i^2 - \sqrt{2} \sum_{i=N-n+1}^{N} x_i^2.$$

Let $m = N - n$ and define $O(m,n)$ to be the subgroup of $GL(N,\mathbb{C})$ that leaves $Q(x)$ invariant, that is,

$$O(m,n) = \{A \in GL(N,\mathbb{C}) \mid Q(Ax) = Q(x) \text{ for all } x \in \mathbb{C}^N\}.$$

Then $O(m,n)$ is an algebraic group defined over the field $\mathbb{Q}(\sqrt{2})$ and the set of real points $O(m,n)(\mathbb{R}) = O(m,n) \cap GL(m,\mathbb{R})$ is a real algebraic simple group of real rank equal to the smaller number between m and n. Let σ be a $\mathbb{Q}$-automorphism of $\mathbb{C}$ such that $\sigma(\sqrt{2}) = -\sqrt{2}$. This induces a map $\sigma : GL(m,\mathbb{C}) \to GL(m,\mathbb{C})$. Note that $\sigma(O(m,n))$ is the group leaving invariant the quadratic form

$$Q^+(x) = \sum_{i=1}^{N-n} x_i^2 + \sqrt{2} \sum_{i=N-n+1}^{N} x_i^2.$$

Therefore $\sigma(O(m,n))(\mathbb{R})$ is a compact Lie group.

Let $\Gamma = O(m,n)(\mathbb{Z}[\sqrt{2}])$. Γ is a lattice in $O(m,n)(\mathbb{R})$. (See, for example, [**Z1**].) The restriction, $\sigma : \Gamma \to \sigma(O(m,n))(\mathbb{R})$, is an injective isomorphism into a compact Lie group. Let K be the closure of $\sigma(\Gamma)$. We have constructed in this way an injective homomorphism from a lattice $\Gamma \subset O(m,n)(\mathbb{R})$ into a compact Lie group K with dense image. Therefore, Γ acts (by left-translations) ergodically on K, as well as on any quotient K/K_0, for any closed subgroup K_0 of K.

4. Gaussian dynamical systems

We give now another construction of finite measure preserving ergodic actions which works for a very general class of groups. A consequence of this construction and of the Cocycle Superrigidity Theorem (Chapter 6) will be that a higher rank semisimple group G has uncountably many mutually non-orbit equivalent finite measure ergodic preserving actions. This should be contrasted with Theorem 2.5.5 (Chapter 2), for actions of amenable groups.

Unlike the constructions discussed earlier in the chapter, the Gaussian construction is infinite-dimensional and does not produce smooth actions on finite dimensional manifolds. In many cases it is also not isomorphic to *any* smooth action since, for example, individual elements have infinite entropy. For $\mathbb{Z}$ actions, however, there are certain Gaussian actions which can be realized as diffeomorphisms of compact finite dimensional manifolds in a nonstandard way using a version of the conjugation–approximation method discussed in [**S-HK**, Section 7.2f].

Let ν be the Gaussian measure on $\mathbb{R}$. Recall that its density with respect to the Lebesgue measure on the line is

$$\frac{d\nu}{dx} := (2\pi)^{-1/2} e^{-x^2/2}.$$

Let $X = \mathbb{R}^{\mathbb{N}} = \mathbb{R} \times \mathbb{R} \times \cdots$ be the space of all sequences of real numbers. Denote by $\mu = \nu^{\mathbb{N}} = \nu \otimes \nu \otimes \cdots$ the product measure on X and by $\alpha_i : X \to \mathbb{R}$ the projection onto the i-th coordinate normalized so that the integral of α_i^2 on X is 1.

The α_i form an orthonormal basis of a Hilbert subspace $\mathcal{H}_0 \subset L^2(X,\mu)$ and by considering polynomials in the α_i one shows that $L^2(X,\mu)$ is the direct sum of the symmetric powers $S^n(\mathcal{H}_0)$. Moreover, it can be shown that for each orthogonal operator U on $\mathcal{H}_0$ there is an essentially unique almost everywhere defined measure-preserving transformation of X inducing the transformation U on $L^2(X,\mu)$.

Let now $\mathcal{H}$ be a real separable infinite dimensional Hilbert space and let $\{e_i\}$ be an orthonormal basis. There is then a linear map $T : \mathcal{H} \to L^2(X,\nu)$ defined by extending the correspondence $T(e_i) = \alpha_i$. T is an orthogonal isometry from $\mathcal{H}$ to $T\mathcal{H} = \mathcal{H}_0$ and for each orthogonal operator U on $\mathcal{H}$ there is an essentially unique almost everywhere defined measure-preserving transformation $f_U : X \to X$ such that the induced map

$$f_U^* : L^2(X, \mu) \to L^2(X, \mu)$$

satisfies $f_U^* \circ T = T \circ U$.

By means of the above correspondence between orthogonal operators and measure preserving transformations of X it is possible to construct a measure preserving action of a group G for each orthogonal representation of G on a separable (real) Hilbert space. More precisely, suppose that G is locally compact and π is an orthogonal representation of G on $\mathcal{H}$. Then for each $\pi(g)$ we obtain an almost everywhere defined transformation $f_{\pi(g)}$. These transformations can be chosen so as to define a measurable action of G on X with invariant measure μ. (See [**Z1**], Appendix B.)

The action of G on X thus constructed has the following property. Let ρ be the unitary representation of G on the complexification of $L^2(X, \mu)$ and let $T^{\mathbb{C}} : \mathcal{H} \to L^2(X, \mu) \otimes \mathbb{C}$ be the complexification of T. Then the complexification of the original representation π is a subrepresentation of ρ; in fact, ρ is isomorphic to the direct sum of the symmetric powers $S^n(\pi^{\mathbb{C}})$, $n \geq 1$.

PROPOSITION 4.4.1 (Segal [**Se**]). *The measure preserving action of G on X is ergodic if and only if π has no finite dimensional invariant subspaces.*

Let π_1 and π_2 be infinite dimensional irreducible unitary representations of G, which we view as orthogonal representations, by restriction of scalars. By the above construction we obtain two finite measure preserving ergodic actions of G. If these actions are equivalent, we can regard the complexification of π_1 as a direct summand of some symmetric power $S^m(\pi_2^{\mathbb{C}})$. It follows that for any given π_1 there can be at most countably many irreducible representations that define, by the above construction, the same ergodic action. Consequently, if G is a locally compact group having an uncountable number of inequivalent infinite dimensional unitary representations, then G has uncountably many inequivalent measure-preserving ergodic actions.

Consider now the case in which the actions defined by π_1 and π_2, as before, are equivalent modulo an automorphism A of G. More precisely, if we represent one of the actions by $(g, x) \mapsto gx$, the other has the form $(g, x) \mapsto A(g)x$, where A is an automorphism of G. In this case, $\pi_2^{\mathbb{C}} \circ A$ is a direct summand of $S^n(\pi_1^{\mathbb{C}})$, for some n. If A is an inner automorphism, then $\pi_2 \circ A$ is equivalent to π_2.

If G is a connected noncompact semisimple Lie group, its group of outer automorphisms is finite. Moreover, G has uncountably many inequivalent infinite dimensional irreducible unitary representations (see, for example, [**Ma2**]). This leads to the following observation.

PROPOSITION 4.4.2. *If G is a connected noncompact semisimple Lie group, then G has uncountably many measure preserving ergodic actions, no two of which are equivalent modulo an automorphism of G.*

5. Examples of actions obtained by suspension

The suspension of an action of a subgroup of a given group G was defined in Section 2.3c. It is a very flexible tool to obtain new actions from old ones. We give now a few examples for $G = SL(n, \mathbb{R})$.

Let $H \subset SL(n,\mathbb{R})$ denote the stabilizer of a point in projective space $P^{n-1}(\mathbb{R})$. H consists of matrices of the form

$$H = \left\{ \begin{pmatrix} a & * \\ 0 & A \end{pmatrix} : A \in GL(n-1,\mathbb{R}),\ a \det A = 1 \right\}.$$

Then $\rho : H \to \mathbb{R}$, $\rho(h) := \log(a^2)$, is a homomorphism onto $\mathbb{R}$. Any continuous flow $\varphi : \mathbb{R} \times M \to M$ on a compact manifold M gives rise to a continuous action of H on M, via the composition of ρ and φ. The H-action now yields an action of $SL(n,\mathbb{R})$ by suspension: form the quotient $N = (G \times M)/H$, where H acts on $G \times M$ by $(g,x)h := (gh, \varphi(-\rho(h),x))$. The element of N represented by the pair (g,x) will be denoted $[g,x]$. Define an action of $SL(n,\mathbb{R})$ on N by $g[g_1,x] := [gg_1,x]$. This example shows that all the complications that arise for actions of $\mathbb{R}$ are also present in $SL(n,\mathbb{R})$-actions.

To describe the second example, first note that it is possible to find a cocompact lattice Λ in $SL(2,\mathbb{R})$ and a surjective homomorphism from Λ into the nonabelian free group F_2 on two generators. Denote by $\rho : \Lambda \to F_2$ such a homomorphism. Then any continuous action of F_2 on a compact manifold yields by suspension an action of $SL(2,\mathbb{R})$ on another compact manifold. We can regard $SL(2,\mathbb{R})$ to be embedded as a subgroup in a parabolic subgroup Q of $SL(n,\mathbb{R})$. The suspension from $SL(2,\mathbb{R})$ to Q and then to $SL(n,\mathbb{R})$ yields a continuous action of $SL(n,\mathbb{R})$ on a compact manifold. Therefore, actions of $SL(n,\mathbb{R})$ may exhibit any of the complications that are present in actions of the free group F_2.

6. Blowing up

Another construction of higher rank actions was introduced by A. Katok and J. Lewis in [**KL1, KL2**] and later extended by J. Benveniste. (See [**Benv**].) Before describing one such example, we recall the notion of *blowing up* a point $o \in \mathbb{R}^n$. We may assume without loss of generality that o is the origin. Let S be the subset of $\mathbb{R}^n \times P^{n-1}(\mathbb{R})$ defined by

$$S = \{(x,l) \in \mathbb{R}^n \times P^{n-1}(\mathbb{R}) : x \in l\}.$$

Then S is a smooth manifold. It is, in fact, a smooth vector bundle over projective space with respect to the natural projection onto $P^{n-1}(\mathbb{R})$. Note that the projection onto the first factor, $\pi : S \to \mathbb{R}^n$, is one-to-one at all points except at those that are mapped to o. These exceptional points constitute the zero section of the vector bundle just mentioned and is, therefore, diffeomorphic to $P^{n-1}(\mathbb{R})$. The construction of S can be viewed as "blowing up" the origin of $\mathbb{R}^n$ and gluing on the resulting boundary sphere a copy of projective space (by means of the antipode map). S is clearly an invariant submanifold for the product action of $GL(n,\mathbb{R})$ on $\mathbb{R}^n \times P^{n-1}(\mathbb{R})$.

This is a local construction, and can be done at points of $\mathbb{T}^n$ as well. Let now Γ be a finite index subgroup of $SL(n,\mathbb{Z})$. Then Γ acts on the n-dimensional torus $\mathbb{T}^n$ by affine automorphisms. Since Γ fixes the origin, the blow-up construction at 0 yields a new manifold and a new action of Γ on it. A variation of the above procedure could be to blow-up two distinct fixed points of Γ and glue together the two boundary spheres. More complicated surgeries are also possible. After performing an appropriate change of the smooth structure of the manifold, the resulting action can be shown to preserve a smooth volume form.

CHAPTER 5

Smooth actions and geometric structures

1. Local properties

Smooth $\mathbb{R}$-actions on manifolds arise classically in the context of integrating ordinary differential equations: a smooth vector field X on a manifold M gives rise to a local flow on M whose infinitesimal generator is X. If the vector field is complete, one has an $\mathbb{R}$-action. More generally, the infinitesimal generator of a smooth action of a connected Lie group G corresponds, as defined later in the section, to an action of the Lie algebra of G and, conversely, a smooth action of the Lie algebra yields a local action of the group.

The precise definitions are as follows. Let M be a smooth manifold. A smooth *local action* on M of a connected Lie group G is a smooth map Φ from an open subset $\mathcal{D}$ of $G \times M$ into M such that for each $x \in M$ the set

$$\mathcal{D}_x := \{g \in G \mid (g, x) \in \mathcal{D}\}$$

is a connected open neighborhood of $e \in G$, $\Phi(e, x) = x$, and if (h, x), $(g, \Phi(h, x))$ and (gh, x) are in $\mathcal{D}$, then

$$\Phi(gh, x) = \Phi(g, \Phi(h, x)).$$

Φ is a G-action on M if $\mathcal{D} = G \times M$.

Let $\mathfrak{X}(M)$ denote the linear space of smooth vector fields on M. Equipped with the Lie bracket of vector fields, $\mathfrak{X}(M)$ is a Lie algebra. A local action Φ of G on M gives rise to a homomorphism of Lie algebras

$$\theta : \mathfrak{g} \to \mathfrak{X}(M)$$

where $\mathfrak{g}$ is the Lie algebra of G, as follows: for each $X \in \mathfrak{g}$ and each $x \in M$, write $\Phi^x : \mathcal{D}_x \to M$ so that $\Phi^x(g) = \Phi(g, x)$. Then

$$\theta(v)(x) := -(d\Phi^x)_e v.$$

Recall that if $\gamma(s)$ is a differentiable path in G with $\gamma(0) = e$ and derivative $\gamma'(0) = v$, then $(d\Phi^x)_e v = \frac{d}{ds}|_{s=0}(\Phi^x \circ \gamma)(s)$.

We call θ the *infinitesimal generator* of Φ. It may be regarded as an action of the Lie algebra $\mathfrak{g}$ on M.

Conversely, the following holds ([**Pa**]):

THEOREM 5.1.1. *Let G be a Lie group, M a differentiable manifold and suppose it is given a homomorphism of Lie algebras $\theta : \mathfrak{g} \to \mathfrak{X}(M)$. Then there exists a differentiable local action of G on M whose infinitesimal generator is θ. If, furthermore, G is connected, simply connected, and each $\theta(v)$, $v \in \mathfrak{g}$, is a complete vector field, then there exists a (global) differentiable action of G on M whose infinitesimal generator is θ.*

Already at this level of generality one sees clear differences between actions of $\mathbb{R}$ and actions of groups having a more complicated structure. Consider, for example, smooth vector fields X_1, X_2 and X_3 on a compact manifold M. Suppose that these fields satisfy the Lie bracket relations:

$$\begin{aligned}[X_1, X_2] &= X_3\\ [X_2, X_3] &= X_1\\ [X_3, X_1] &= X_2.\end{aligned}$$

These relations determine an action of the Lie algebra of the rotation group $SO(3)$ (equivalently, an infinitesimal action of $SO(3)$). It follows from the previous theorem that there exists a unique global action of the compact group $SU(2)$ (the double covering of $SO(3)$) whose infinitesimal generator is the given one. Since every one-parameter subgroup of the latter is periodic it follows that each flow line of each X_i is closed.

2. Actions preserving a geometric structure

a. Geometric structures and Lie transformation groups. In some geometric applications, one considers the action of the group of automorphisms (or isometries) of some geometric structure. The interaction between the dynamical properties of the action and properties of the geometric structure invariant under that action often leads to interesting results. Many results of this nature are described in [**Gro**]. Some basic facts about transformation groups in differential geometry can be found in [**Kob**]. This section describes a few results in this direction.

A subgroup G of the group Diff(M) of all diffeomorphisms of a smooth manifold M is a *Lie transformation group* if there exists a Lie group structure on G such that the natural action of G on M is differentiable and any flow $\{\varphi_t\}$ on M contained in G corresponds to the action of a one-parameter subgroup of G.

THEOREM 5.2.1 (Bochner-Montgomery, [**Mon**]). *If a subgroup G of Diff(M), where M is a differentiable manifold, is locally compact, then G is a Lie transformation group.*

There is a large class of geometric structures whose automorphism groups are Lie transformation groups. A very broad characterization of such structures is given by Gromov [**Gro**] and are called *rigid geometric structures*, about which we will have more to say later. These include the so called *structures of finite type* of Cartan. Some of the most classical examples, such as pseudo-Riemannian metrics and the conformal class of a pseudo-Riemannian metric in dimension 3 or greater, correspond to Cartan's structures.

Thus, any faithful action of a group G by diffeomorphisms preserving such a structure $\mathcal{G}$ corresponds to an embedding of G into the Lie transformation group of automorphisms, $\mathrm{A}ut(M, \mathcal{G})$. It is quite natural to expect that the structure of such embeddings is much more special than embeddings of G into the the group of diffeomorphisms of M, which correspond to arbitrary smooth G-actions, or even of actions preserving volume, symplectic structure or complex structure since automorphisms groups of such structures are infinite-dimensional.

b. H**–structures.** Let H be a Lie subgroup of the general linear group $GL(n, \mathbb{R})$. Let M be an n-dimensional differentiable manifold and let $F(M)$ denote the $GL(n, \mathbb{R})$-principal bundle of frames over M. An *H-reduction* of $F(M)$ is defined as a principal subbundle of $F(M)$ with the structure group H (the H-action being the restriction to H of the right-action of $GL(n, \mathbb{R})$ on $F(M)$). For example, a pseudo-Riemannian metric on M can be characterized as an $O(k, n-k)$-reduction, for some k.

Let now $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{gl}(n, \mathbb{R})$ and denote by $\mathfrak{h}_r$ the space of all symmetric $(r+1)$-linear mappings $\beta : (\mathbb{R}^n)^{r+1} \to \mathbb{R}^n$ such that for any $v, v_1, \ldots, v_r$ in $\mathbb{R}^n$ the linear transformation

$$v \mapsto \beta(v, v_1, \ldots v_r)$$

belongs to $\mathfrak{h}$. If $\mathfrak{h}_r = 0$ and $\mathfrak{h}_{k-1} \neq 0$ for some r, $\mathfrak{h}$ is said to be *of finite order* r. If $\mathfrak{h}$ has finite order r, then $\mathfrak{h}_s = 0$ for all $s \geq r$. The subalgebra $\mathfrak{h}$ is called *elliptic* if it does not contain matrices of rank 1. It can be shown that every subalgebra of finite order is elliptic.

Given a geometric structure $\mathcal{G}$ on M, let $\mathrm{Aut}(M, \mathcal{G})$ denote the group of diffeomorphisms of M that leave $\mathcal{G}$ invariant. For example, if $\mathcal{G}$ is an H-reduction of $F(M)$, $\mathrm{Aut}(M, \mathcal{G})$ is the group of automorphisms of the principal H-subbundle. The following results can be found in [**Kob**].

THEOREM 5.2.2. *Let $\mathcal{G}$ be an H-structure on an n-dimensional differentiable manifold M. Suppose that the Lie algebra $\mathfrak{h}$ of H has finite order r. Then Aut$(M, \mathcal{G})$ is a Lie transformation group of dimension not greater than*

$$n + \sum_{i=1}^{r-1} dim\ \mathfrak{h}_i.$$

Furthermore, if $\mathcal{G}$ is an H-structure on a compact differentiable manifold M and the Lie algebra $\mathfrak{h}$ is elliptic, then Aut$(M, \mathcal{G})$ is a Lie transformation group.

We denote by $\mathrm{Aut}(M, \mathcal{G})^{loc}$ the pseudo-group of local isometries of $\mathcal{G}$. (A local isometry of $\mathcal{G}$ is a diffeomorphism between open subsets of M that pulls-back the geometric structure to itself.) If $\mathfrak{h}$ has finite order and $\mathcal{G}$ is an H-structure, $\mathrm{Aut}(M, \mathcal{G})^{loc}$ can be given the structure of a *Lie pseudo-group*.

c. Extension of local isomorphisms of H**-structures.** A remark made by M. Gromov, which lies behind a number of geometric applications of dynamics, is that if the automorphism group of a rigid structure, such as an H-structure of finite order, acts topologically transitively on M, then M is, in a sense, close to being a locally homogeneous space. The precise statement, stated in the very special case of H-structures, is as follows. A more general form of the theorem will be given later in the chapter. (See also [**Gro**].)

THEOREM 5.2.3. *Suppose that $\mathcal{G}$ is an H-structure on a differentiable manifold M and that the Lie algebra $\mathfrak{h}$ is of finite order. If Aut$(M, \mathcal{G})$ has a dense orbit in M, the pseudo-group Aut$(M, \mathcal{G})^{loc}$ has an open and dense orbit in M.*

In particular, when the action of $\mathrm{Aut}(M, \mathcal{G})$ (where $\mathcal{G}$ is as in the theorem) is minimal, M is a locally homogeneous space in the following sense: the Lie pseudo-group $\mathrm{Aut}(M, \mathcal{G})^{\mathrm{loc}}$ acts transitively on M.

d. Groups of automorphisms of Lorentz manifolds. Even very mild assumptions about the dynamics of the automorphism group of a given geometric structure may impose severe restrictions on the structure of the full Lie group of automorphisms. We select one result concerning Lorentz manifolds that illustrates this point well. (The dynamical property needed in the proof is essentially the lack of properness.) The result is due to S. Adams, G. Stuck and A. Zeghib. (The statement below is taken from [**AS**]. See also [**Ze1**].) A key idea in their approach is due to N. Kowalsky [**Kow**]. The theorem extends earlier results by Zimmer and Gromov. (See [**Z5**] and [**Gro**].)

THEOREM 5.2.4 (Adams-Stuck, Zeghib). *Let G be a connected Lie group. Then the following are equivalent:*

(1) *There exists a compact connected Lorentz manifold M on which G acts locally faithfully by isometries.*
(2) *The universal covering $\tilde{G}$ of G is isomorphic to the product $L \times K \times \mathbb{R}^d$, where K is compact and semisimple (or trivial), $d \geq 0$, and L is in the following list:*
    (a) *the universal covering group of $SL(2, \mathbb{R})$*
    (b) *the two dimensional group of affine motions of $\mathbb{R}$*
    (c) *a Heisenberg group H_n*
    (d) *a certain countable family of semidirect products $\mathbb{R} \ltimes H_n$*
    (e) *the trivial group.*

Moreover, if L is in the above list, then any locally faithful action of L by isometries of a compact Lorentz manifold is locally free.

e. Gromov's rigid geometric structures. The results of this and the next subsection are mainly concerned with general properties of pseudo-groups of local isometries of a given geometric structure, and are mostly local in nature. These properties will be needed in Section 5.4, when we discuss the so-called Gromov representation, which will allow in certain situations to connect the topology of a manifold with (semisimple) group actions.

We give a brief account of Gromov's theory of rigid structures and rigid transformation groups. A map f will be called a (smooth) *local diffeomorphism* from a manifold M to a second manifold N, at $x \in M$, if there are open sets $U \subset M$, $V \subset N$, such that $x \in U$ and $f : U \to V$ is a diffeomorphism. The r-jet at x of such an f is, by definition, the equivalence class of smooth local diffeomorphisms at x, from M to N, so that f, g are equivalent if $f(x) = g(x)$ and all the derivatives up to order r of $g^{-1} \circ f$ at x (in an arbitrary coordinate chart containing x) coincide with the corresponding derivatives of the identity map. An r-jet at 0 of a local diffeomorphism from $\mathbb{R}^n$ to M that takes 0 to $x \in M$ will be called a *frame of order r at x*. The collection of all frames of order r at $0 \in \mathbb{R}^n$ forms a Lie group, denoted G^r. The collection of all frames of order r at all points of M forms a principal G^r-bundle. Note that, for $r = 1$, this corresponds to the ordinary frame bundle. The frame bundle of order r of M will be written $F^r(M)$ and $F^r(M)_x$ will denote the fiber above $x \in M$.

A (smooth) map, φ, between manifolds M and N induces a map, φ_r, from $F^r(M)$ to $F^r(N)$, by taking the r-jet (at 0) of the composition of φ with the representative of an r-frame.

A (smooth) geometric structure of order r on a manifold M of dimension n can now be defined as a G^r-equivariant map

$$\mathcal{G} : F^r(M) \to V,$$

where V is some G^r-space.

For example, a Riemannian metric on M corresponds to a $GL(n,\mathbb{R})$-equivariant map $\mathcal{G} : F^1(M) \to GL(n,\mathbb{R})/O(n)$, where the action of $GL(n,\mathbb{R})$ on the quotient is by left-translations. (In this example, note that the preimage under $\mathcal{G}$ of the identity coset of $GL(n,\mathbb{R})/O(n)$ defines an $O(n)$-reduction of $F^1(M)$.) When V is, as in this example, an algebraic variety with an algebraic action of G^r, $\mathcal{G}$ is called (after Gromov) an A-structure.

A (local) *isometry* of $\mathcal{G}$ is a (local) diffeomorphism φ of M whose induced map, φ_r, on $F^r(M)$ satisfies $\mathcal{G} \circ \varphi_r = \mathcal{G}$. If the latter equation is satisfied up to i-jet at each $\xi \in F^r(M)_x$, we say that φ represents an infinitesimal isometry of order i at x. The collection of all infinitesimal isometries of order i at x, sending x to y, will be denoted by $\mathrm{Iso}^i(x,y)$. Similarly, one defines the space of local isometries from x to y, denoted $\mathrm{Iso}^{\mathrm{loc}}(x,y)$. The relation that identifies x and y if and only if $\mathrm{Iso}^i(x,y)$ in non-empty is an equivalence relation. The same remark holds for $\mathrm{Iso}^{\mathrm{loc}}$.

We are now ready to define rigid geometric structures. Let $\mathcal{G}$ be a smooth geometric A-structure of order r on a manifold M. Then $\mathcal{G}$ is said to be *rigid* (or $r+i$-rigid) if the homomorphism $\mathrm{Iso}^{i+1}_{x,x} \to \mathrm{Iso}^i_{x,x}$ is injective for all $x \in M$. Here are a few examples.

Immersions. A C^1 0-order structure of type V is simply a C^1 map $\mathcal{G}$ from M into V. In this case, $\mathcal{G}$ is 0-rigid if and only if $\mathcal{G}$ is an immersion.

Anosov map. Let $T : M \to M$ be an Anosov map of a compact manifold M. Then it can be shown that $T_1 : F^1(M) \to F^1(M)$ generates a properly discontinuous $\mathbb{Z}$-action on $F^1(M)$. The map $\mathcal{G} : F^1(M) \to V := F^1(M)/\mathbb{Z}$ corresponds to a geometric structures in the general sense define above, although not an A-structure. A map φ is an infinitesimal isometry of $\mathcal{G}$ of order i exactly when it is the $i+1$-jet of some iterate of T. In particular, it is 0-rigid.

Structures of finite type. It can be shown that Cartan's structures of finite type (defined earlier) are all rigid in the present sense. In particular, complete parallelisms, affine connections, pseudo-Riemannian metrics are all examples of rigid structures.

f. The Iso-relations for rigid structures. We describe the general properties of the equivalence relation defined by Iso^i and $\mathrm{Iso}^{\mathrm{loc}}$. The main result is described in the next theorem. (This is stated in a somewhat more restricted form than in [**Gro**].)

THEOREM 5.2.5 (Gromov). *Let M be a smooth manifold and $\mathcal{G}$ a smooth rigid A-structure on M of order r. Then, there exists a positive integer s_0 and, for each $s \geq s_0$, there is an open subset $W_s \subset M \times M$ such that the following holds:*

(1) *The equivalence classes of the Iso^s-relation are closed smooth submanifolds of W_s.*
(2) *For each $(x,y) \in W_s$ and each $\xi \in \mathrm{Iso}^s(x,y)$, there exists a unique (germ of) local isometry of $\mathcal{G}$ that sends x to y. Furthermore, the correspondence from infinitesimal to local isometries is continuous.*

(3) *The set U_s of all $x \in M$ for which $(x, y) \in W_s$ for some $y \in M$ is open dense, and the collection of $y \in M$ such that $(x, y) \in W_s$, for each $x \in U_s$, is an open neighborhood of x.*

(4) *Under the extra assumption that $Iso^s(x, y) \neq \varnothing$ for some big enough s and all $x, y \in M$ (in other words, Iso^s is transitive for some big enough s), then every $x \in M$ has a neighborhood U_x such that $Iso^{loc}(x, y)$ is non-empty for all $y \in U_x$. In particular, if the group of isometries of $(M, \mathcal{G})$ has a dense orbit in M, then $(M, \mathcal{G})$ is locally homogeneous.*

COROLLARY 5.2.6. *If $\mathcal{G}$ is a rigid A-structure on M, then $Iso(M, \mathcal{G})$ is a Lie group such that the natural action on M is smooth. The Lie algebra of $Iso(M, \mathcal{G})$ is the space of Killing fields, that is, the vector fields on M whose natural lift to $F^r(M)$ lies in the kernels of the tangent maps $d\mathcal{G}$.*

g. The algebraic hull and the $\mathbf{Iso}^i$-relations. The interest in understanding the algebraic hull of a G-action becomes apparent in the next proposition, which says, in part, that the algebraic hull of the action sets a "lower bound" on the size of the groups $\mathrm{Iso}^i_{x,x}(M, \mathcal{G})$ for any A-structure $\mathcal{G}$ invariant under the action, for almost all $x \in M$.

PROPOSITION 5.2.7. *Let G be a Lie group that acts smoothly on a manifold M so as to preserve a geometric A-structure of order r. Let $L_i \subset G^{r+i}$ be (a representative) of the measurable algebraic hull of the G-action (by automorphisms on $F^{r+i}(M)$) with respect to an ergodic quasi-invariant measure on M. Then for almost all $x \in M$, and each $i \geq 0$, the group $Iso^i_{x,x}(M, \mathcal{G})$ contains a subgroup isomorphic to L_i.*

PROOF. Recall that each $\mathcal{G}^i$ is an equivariant map from $F^{r+i}(M)$ into some algebraic variety V^i upon which G^{r+i} acts algebraically. By Proposition 2.4.4, over a set of full measure in M, $\mathcal{G}$ takes values into a single G^{r+i}-orbit, $G^{r+i} \cdot v_0$, in V^i. Therefore, $\mathcal{G}^i$ can be described as an L_{v_0}-structure, where L_{v_0} is the isotropy subgroup of v_0 in G^{r+i}. An element of $\mathrm{Iso}^i_{x,x}(M, \mathcal{G})$ can be described as a pair (ξ, η) (modulo G^{r+i}), where ξ, η lie in the fiber of $F^{r+i}(M)$ above x and $\mathcal{G}^i(\xi) = \mathcal{G}^i(\eta)$. It follows that $\mathrm{Iso}^i_{x,x}(M, \mathcal{G})$ is isomorphic to L_{v_0}. On the other hand, the algebraic hull L_i is contained in L_{v_0}, by Proposition 2.4.6. □

COROLLARY 5.2.8. *Suppose that $\mathcal{G}$ is an analytic rigid A-structure. For i sufficiently large $Iso^{loc}_{x,x}(M, \mathcal{G})$ contains a group isomorphic to L_i for each $x \in M$. Furthermore, for each $x \in M$ and each x' in the universal covering $\tilde{M}$ in the fiber of x, the space of (globally defined) Killing vector fields on $\tilde{M}$ (of the lift of $\mathcal{G}$ to $\tilde{M}$) vanishing at x' contains a Lie algebra isomorphic to the Lie algebra of L_i.*

PROOF. The first assertion follows from Theorem 5.2.9(1). The fact about Killing fields is a consequence of 5.2.9(2). □

h. Analytic structures. We state here some special properties enjoyed by real analytic rigid structures. For the proof of the next theorem, the reader is referred to [**Gro**].

THEOREM 5.2.9 (Gromov). *Suppose that M is a connected analytic manifold, and that $\mathcal{G}$ is an analytic rigid A-structure.*

(1) *Let M be compact. Then, there exists an integer k and, for each $x \in M$, there exists a neighborhood U_x of x such that an infinitesimal isometry of order k or greater, taking x into U_x, extends to a local isometry (whose germ is uniquely determined).*
(2) *If M is simply connected, then every local Killing field of $\mathcal{G}$ defined on a connected nonempty open set extends uniquely to a global Killing field.*

THEOREM 5.2.10 (Gromov). If $\mathcal{G}$ is a rigid analytic A-structure and M is compact and simply connected, then $\mathrm{Iso}(M, \mathcal{G})$ has finitely many connected components. The same holds for each isotropy subgroup $\mathrm{Iso}(M, \mathcal{G})_x$, $x \in M$.

3. Smooth actions of semisimple Lie groups

This section considers the general interaction of local differential geometry (from the viewpoint of the previous section) and the dynamics of actions of general semisimple Lie groups (without compact factors). There are two main reasons for studying this interaction. On one hand, smooth dynamics and ergodic theory can be invaluable tools in the geometric study of structures with large symmetry groups. On the other hand, and somewhat more in line with our purposes, having some knowledge about which geometric structures are invariant under a given group action often provides useful information about the dynamical properties of the action. As an obvious example, if the action on a compact manifold preserves a Riemannian metric, then all the Lyapunov exponents of every element of the acting group are zero. More generally, the existence of invariant tensor fields forces on the Lyapunov exponents *resonance relations* as described, for example, in [**Kan1**] and [**FK**].

Using the concept of algebraic hull and its basic properties, we prove the following two theorems that illustrate some of the special properties enjoyed by smooth actions of semisimple Lie groups. For simplicity we assume that G is a (noncompact, connected) simple Lie group. The first theorem shows that a smooth nontrivial action of such a group, preserving a probability measure, is essentially locally free; that is, almost every isotropy subgroup G_x is a discrete subgroup of G. (Recall that G_x is the subgroup of all $g \in G$ such that $gx = x$.)

THEOREM 5.3.1 (Zimmer). *Let G be a noncompact connected simple Lie group that acts on a connected manifold M with a finite G-invariant measure positive on open sets. Denote by G_x the isotropy subgroup of $x \in M$. Then, if the action is not trivial, there exists an open dense set of full measure on which G_x is discrete.*

The second theorem shows that if the action preserves some geometric structure, then the Lie group associated to that structure must contain a group locally isomorphic to G.

THEOREM 5.3.2 (Zimmer). *Let G be a connected, noncompact, simple Lie group, acting nontrivially on a compact n-dimensional manifold M. Suppose that the action preserves an H-structure on M where H is a real algebraic subgroup of $GL(n, \mathbb{R})$ consisting of matrices of determinant ± 1. Then there is a Lie algebra embedding $\pi : \mathfrak{g} \to \mathfrak{h}$ such that the representation π of $\mathfrak{g}$ on $\mathbb{R}^n$ contains $ad(\mathfrak{g})$ as a subrepresentation.*

For example, it follows from this second theorem that a group such as $SL(3, \mathbb{R})$ cannot act (smoothly) nontrivially, by measure preserving transformations on a compact Lorentz manifold, since the Lie algebra of $SL(3, \mathbb{R})$ cannot be realized as a subalgebra of the Lie algebra of the Lorentz group in any dimension. It can also be shown (see [**Z2**]) that if $\mathfrak{g}$ is the Lie algebra of a noncompact simple Lie group and $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$ is a Lie algebra homomorphism such that on V there is a nondegenerate symmetric bilinear form of signature $(1, \dim V - 1)$, invariant under $\pi(\mathfrak{g})$, and $\mathrm{ad}(\mathfrak{g})$ is a subrepresentation of π, then $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$. Therefore, by the previous theorem, if a connected noncompact simple Lie group G acts nontrivially on a compact manifold preserving a Lorentz metric, then G is locally isomorphic to $SL(2, \mathbb{R})$. (This remark goes a long way in justifying Theorem 5.2.4.)

It is also interesting to note the general fact contained in the next proposition. We say that a geometric structure is *unimodular* if it determines an L-reduction of the frame bundle $F^1(M)$ where L is the group of matrices with determinant 1 or -1. In other words, the structure is unimodular if it incorporates a volume density.

PROPOSITION 5.3.3. *Suppose that G is a noncompact simple Lie group with finite center acting nontrivially on a compact manifold M so as to preserve a unimodular A-structure $\mathcal{G}$. Then for i sufficiently large and almost all $x \in M$, $Iso^{loc}_{x,x}(M, \mathcal{G})$ contains a group locally isomorphic to G.*

PROOF. This is due to Proposition 5.3.5 and Theorem 5.2.5. □

Although these theorems are not used again in the survey, many of the arguments that go into their proofs will be needed again and are of independent interest. For that reason the proofs of Theorems 5.3.1 and 5.3.2 will be shown in detail at the end of the section. The most important of those arguments are: the *cocycle reduction lemma* as well as its principal bundles formulation, described in the next two subsections; and the concept of algebraic hull and its main properties, described in Subsections 2.4c and 5.2g. The theorems are proved in the last subsection.

a. Proofs of Theorems 5.3.1 and 5.3.2.

PROOF OF THEOREM 5.3.1. We first show the following claim: for each ergodic component of the measure, either $G_x = G$ at almost every $x \in M$ or G_x is discrete at almost every x.

Let $\mathfrak{g}_x$ be the Lie algebra of G_x and $\mathrm{Gr}(\mathfrak{g})$ the union of the Grassmann varieties of subspaces of $\mathfrak{g}$. Define $\phi : M \to \mathrm{Gr}(\mathfrak{g})$ by $\phi(x) = \mathfrak{g}_x$. Then ϕ is easily seen to be measurable and for each $g \in G$ and $x \in M$

$$\phi(gx) = \mathrm{Ad}(g)\phi(x).$$

If μ is the G-invariant probability measure on M, $\phi_*\mu$ is an $\mathrm{Ad}(G)$-invariant probability measure on $\mathrm{Gr}(\mathfrak{g})$. Since $\mathrm{Ad}(G)$ is a finite index subgroup of its Zariski closure H in $GL(\mathfrak{g})$, we obtain an H-invariant probability measure on $\mathrm{Gr}(\mathfrak{g})$. That invariant measure must be supported on the set of H-fixed points (this is essentially the Borel Density Theorem), so that over a set of full measure in M, the map ϕ takes values in the set of H-fixed points. In particular, there is a conull G-invariant subset $S \subset M$ such that $\phi|_S$ is a G-invariant function. By ergodicity, ϕ is constant almost everywhere. Call $\mathfrak{l}$ the constant value of ϕ. Then $\mathfrak{l} = \mathrm{Ad}(g)\mathfrak{l}$ for all $g \in G$, hence $\mathfrak{l}$ is an ideal of $\mathfrak{g}$. But the only ideals of $\mathfrak{g}$ are $\mathfrak{g}$ and 0, so that G_x is either G or a discrete subgroup at almost every x.

We now drop the assumption that the measure is ergodic, and suppose that it is positive on open sets. Let $\Lambda \subset M$ be the (measurable) subset where $G_x = G$. We want so show that Λ has measure 0. Suppose for a contradiction that this is not the case and let K be the maximal compact subgroup of G. The action of K can be linearized at each of its fixed points. (Choose a K-invariant Riemannian metric on M and consider a normal neighborhood near the point. Then in exponential coordinates the local action will be linear.) Therefore, each density point of Λ has a neighborhood where K acts trivially. It is clear that if x is in the closure of an open set where K is trivial, then x is fixed by K and has a neighborhood of fixed points. (Again by linearization.) Therefore, K acts trivially on all of M. On the other hand, the subgroup of G that acts trivially on M is a normal subgroup. Since G is simple we obtain the desired contradiction. □

The next lemma, needed for Theorem 5.3.2, is of independent interest. It gives an example for which the algebraic hull can be exactly determined.

LEMMA 5.3.4. *Let G be a noncompact connected simple Lie group and let M be a G-space with an ergodic G-invariant probability measure μ. Denote by H the Zariski closure of $Ad(G)$ in $GL(\mathfrak{g})$. Then H is the algebraic hull of the G-action by bundle automorphisms on the (trivial) principal H-bundle $P = M \times H$ by $g(x,h) := (gx, Ad(g)h)$.*

PROOF. ([**Z2**].) The lemma will be proved once we verify the following. Let M be a G-space with an ergodic G-invariant probability measure μ. Suppose that $\rho : G \to GL(V)$ is a representation of G on a finite dimensional (real) vector space V. Denote by H the Zariski closure of $\rho(G)$ in $GL(V)$ and suppose that $\rho(G)$ is a subgroup of finite index in H. We assume moreover that H is generated by algebraic 1-parameter subgroups. It will be shown that H is the algebraic hull of the G-action by bundle automorphisms of the (trivial) principal H-bundle $P = M \times H$ given by $g(x,h) := (gx, \rho(g)h)$.

Let $L \subset H$ denote the algebraic hull and let Q be a G-invariant measurable L-reduction of P. The reduction is, in effect, a G-invariant measurable assignment of an L-orbit (for the right-translation L-action on H) at each $x \in M$, i.e., a G-invariant measurable section of the fiber bundle P/L, whose standard fiber is H/L. In the present situation, where P is already a product, having an L-reduction is equivalent to having a measurable map $\phi : M \to H/L$ such that for each $x \in M$, $g(x, \phi(x)) = (gx, \phi(gx))$. Therefore, ϕ has the property

$$\phi(gx) = \rho(g)\phi(x)$$

for all $g \in G$ and all $x \in M$. The probability measure $\phi_*\mu$ on H/L is $\rho(G)$-invariant since $\phi_*\mu = \phi_* g_* \mu = \rho(g)_* \phi_* \mu$, for each $g \in G$. By averaging $\phi_*\mu$ over the finite group $H/\rho(G)$ we obtain an H-invariant probability measure on H/L. We can now apply the Borel Density Theorem (Theorem 2.2.11) to conclude that $H = L$. □

PROOF OF THEOREM 5.3.2. If an action preserves an H-structure on M such that H consists of matrices of determinant 1, then the action also preserves a nonvanishing alternating n-form on M. Similarly, if we allow the elements of H to have determinant either 1 or -1, then the action preserves a nonvanishing n-form which is only well-defined up to sign, i.e. a *volume density*. This allows us to define a smooth G-invariant measure on M. The total measure of M is finite since M is compact, so after normalization we may assume that M admits a G-invariant probability measure μ whose support is the entire M.

For each ergodic component of μ we can apply the previous theorem (Theorem 5.3.1) and conclude that G_x is either discrete or equal to G for μ-a.e. $x \in M$. If

$G_x = G$ for μ-a.e. x, we have by continuity that the action is trivial, contrary to the hypothesis. Therefore, there must be a G-invariant measurable subset $S \subset M$ of positive μ-measure such that G_x is discrete for all $x \in S$.

At each $x \in S$, the differential of the orbit map $\tau_x : G \to M$, $\tau_x(g) := gx$, yields an identification of the tangent space at x of the G-orbit of x with the Lie algebra of G, as indicated by the following arrow $F_x := (D\tau_x)_e : \mathfrak{g} \to V_x := T_x(G \cdot x)$. Moreover, with respect to this identification, the derivative action of each $g \in G$ on the G-invariant subbundle V of TM with fibers V_x is given by $\mathrm{Ad}(g)$, i.e. $Dg_x : V_x \to V_{gx}$ and $F_{gx} \circ \mathrm{Ad}(g) = Dg_x \circ F_x$.

Let m be the dimension of $\mathfrak{g}$ and view $\mathfrak{g}$ as the subspace $\mathbb{R}^m \subset \mathbb{R}^n$, corresponding to setting equal to 0 the last $n-m$ coordinates of $\mathbb{R}^n$. Let H_1 denote the image of G in $GL(m, \mathbb{R})$ under the adjoint representation. Then by the above discussion, we obtain over S a measurable $\overline{H}$-reduction of the frame bundle $F(M)|_S$ where $\overline{H}_1$ is a subgroup of $GL(n, m, \mathbb{R})$ that restricts to H on the invariant subspace $\mathbb{R}^n$. We can now apply Lemma 5.3.4 to conclude that the algebraic hull of the action contains $\mathrm{Ad}(G)$. But some conjugate of the algebraic hull is contained in H. Since the Lie algebra of $\mathrm{Ad}(G)$ is isomorphic to $\mathfrak{g}$, we obtain that some conjugate of $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{h}$. □

Proposition 5.3.3 is now a corollary of the following.

PROPOSITION 5.3.5. *Suppose that G is a noncompact simple Lie group with finite center acting nontrivially on a compact manifold M so as to preserve a unimodular A-structure $\mathcal{G}$. Then, for almost all $x \in M$ and each $i \geq 1$, the algebraic hull L_i (and, therefore, also $Iso^i_{x,x}(M, \mathcal{G})$) contains a group locally isomorphic to G.*

PROOF. This is a consequence of (the proof of) Theorem 5.3.2 together with Proposition 5.2.7. □

4. Dynamics, rigid structures, and the topology of M

The present section is concerned with the relationship between G-actions and the topology of a compact manifold supporting the action. The main result here is Theorem 5.4.1, due to M. Gromov. More on the subject of the section will be discussed later in connection with higher real-rank semisimple Lie groups and superrigidity.

In all of the section, M will be a compact real analytic manifold, equipped with a real analytic, rigid, unimodular A-structure $\mathcal{G}$. G will denote a connected noncompact simple Lie group with finite center that acts analytically by isometries of $\mathcal{G}$. The action in assumed to be nontrivial.

a. The Gromov representation. The main result of the section is the following. (The assumptions of the previous paragraph are in force.)

THEOREM 5.4.1 (Gromov). *There exists an integer m and a representation $\rho : \pi_1(M) \to GL(m, \mathbb{R})$ such that the Zariski closure of the image of ρ contains a group locally isomorphic to G.*

As a simple example, the theorem implies that $SL(2, \mathbb{R})$ cannot act nontrivially on a sphere S^n so as to preserve the volume form and an analytic connection. More generally, COROLLARY 5.4.2. *G cannot act analytically and nontrivially on a manifold M with amenable fundamental group leaving invariant an analytic rigid A-structure.*

PROOF. This is due to the fact that the Zariski closure of an amenable group is amenable. See theorem 4.1.15 in [**Z1**]. □

Similar results hold for lattices, as the next theorem shows [**FZ**].

THEOREM 5.4.3 (Fisher-Zimmer). *Let Γ be a lattice in G, where G has real-rank at least 2. Suppose that Γ acts analytically and ergodically on a compact manifold M preserving a unimodular rigid geometric structure. Then either*

(1) *the action is isometric and $M = K/K_0$, where K is a compact group, K_0 is a closed subgroup, the fundamental group of M is finite, and the action is by right translation via an injective homomorphism with dense image $\rho : \Gamma \to K$, or*
(2) *there exists a linear representation $\sigma : \pi_1(M) \to GL(n, \mathbb{R})$ with infinite image such that the automorphism group of the Zariski closure of $\sigma(\pi_1(M))$ contains a group locally isomorphic to G.*

For actions of higher real rank semisimple groups, the Gromov representation can be used in combination with Zimmer's Cocycle Superrigidity Theorem (Theorem 6.2.1) and a fundamental theorem of M. Ratner to produce much sharper results relating the fundamental group of a manifold and (higher rank) groups that can (or cannot) act on M. We refer the interested reader to [**Z8**], as well as [**Z9**] and [**LZ1**] for further details on this subject.

b. Construction of the Gromov representation. The representation ρ will be obtained by considering an action of G on the space of global Killing fields, for the analytic rigid structure, on the universal covering space of M.

Let $\mathfrak{g}$ be the Lie algebra of G. Since G acts isometrically on M, $\mathfrak{g}$ can be identified with a Lie algebra of Killing fields on the universal covering $\tilde{M}$. Let s be large enough so that the conclusion of Theorem 5.2.9 holds. For almost all $x \in M$, let $L = L(x)$ be the algebraic hull at x of the action on $F^{r+s}(M)$. Let $x' \in \tilde{M}$ be in the fiber of x and denote by $\mathfrak{l}$ the Lie algebra isomorphic to the Lie algebra of L consisting of (global) Killing fields on $\tilde{M}$ that vanish at x'. (See Corollary 5.2.8.)

LEMMA 5.4.4. *The Lie algebra $\mathfrak{l}$ normalizes $\mathfrak{g}$. Furthermore, the Lie algebra homomorphism $X \mapsto [X, \cdot]$ from $\mathfrak{l}$ into the algebra of derivations of $\mathfrak{g}$ is onto $ad(\mathfrak{g})$.*

PROOF. Let $J^{r+s}TM$ denote the vector bundle of $r+s$-jets of local vector fields on M. Denote by $\mathfrak{g}(x)$ the subspace of the fiber of $J^{r+s}TM$ at x consisting of the $k+s$-jets of Killing fields induced from elements of $\mathfrak{g}$ by the action. According to Lemma 5.3.1, $\mathfrak{g}(x)$ is isomorphic to $\mathfrak{g}$ for almost all x. Call the isomorphism $\varphi_x : \mathfrak{g} \to \mathfrak{g}(x)$. It is also clear that $g\mathfrak{g}(x) = \mathfrak{g}(gx)$ for each $g \in G$ and almost all x and that

$$g\varphi_x(X) = \varphi_{gx}(\mathrm{Ad(g)}).$$

(Ad : $G \to GL(\mathfrak{g})$ denotes the adjoint action of G.) In particular, the algebraic hull L may be chosen so that it also leaves the (measurable) subbundle $x \mapsto \mathfrak{g}(x)$ invariant. Arguing as in the proof of Theorem 5.3.2, we also obtain that the connected component L^0 of L will act on $\mathfrak{g}(x) \cong \mathfrak{g}$ by Ad(G). We now use the fact that Killing fields are uniquely determined by their $r+s$-jets. This shows that $\mathfrak{g}$ is normalized by L^0, hence also by the Lie algebra $\mathfrak{l}$, and that $\mathfrak{l}$ acts on $\mathfrak{g}$ according to the adjoint representation. □

COROLLARY 5.4.5. *Let* $\mathfrak{g}$ *be the Lie algebra of* G*, viewed as the algebra of Killing fields on* $\tilde{M}$*. Let* $\mathfrak{z}$ *be the centralizer of* $\mathfrak{g}$ *in the Lie algebra of all Killing fields on* $\tilde{M}$*. For each* $x \in \tilde{M}$*, let* $\mathfrak{z}(x)$ *and* $\mathfrak{g}(x)$ *be the images under the evaluation map at* x*. Then,* $\mathfrak{g}(x)$ *is contained in* $\mathfrak{z}(x)$ *for almost all* x*.*

PROOF. Let $x \in \tilde{M}$ be any point for which Lemma 5.3.1 holds. If $g \in G$ is sufficiently close to the identity, it is possible to choose $u \in L = L(x)$, also close to the identity, such that u acts on $\mathfrak{g}$ by $\mathrm{Ad}(g)^{-1}$. The composition $g \circ u$ is a local diffeomorphism near x that acts trivially on $\mathfrak{g}$ (recall that we can identify $\mathfrak{g}(x)$ with $\mathfrak{g}$) and sends x to gx. Let Z be the local group near x with Lie algebra $\mathfrak{z}$. Then, we have just shown that the H-orbit of x contains an open neighborhood of x that also lies in the local Z-orbit of x. It follows that $\mathfrak{g}(x) \subset \mathfrak{z}(x)$, as claimed. □

The following example may help illuminate the proof. Suppose that G is a subgroup of a simply connected Lie group H, acting on $M = H/\Gamma$ by left translations, where Γ is a discrete subgroup of H. On the universal covering H of M there is also a right action of G, which commutes with the left action. The corresponding vector fields are the elements of $\mathfrak{z}$ constructed above. The geometric structure in this case can be taken to be an invariant affine connection on H/Γ.

We now conclude the proof of Theorem 5.4.1. There is no loss of generality in supposing that that G is simply connected, so that we can lift the G-action on M to a global action of G on $\tilde{M}$. Let $\mathfrak{z}$ be as in Corollary 5.4.5. Note that $\mathfrak{z}$ is invariant under $\pi_1(M)$. In fact, if $\gamma \in \pi_1(M)$, $X \in \mathfrak{z}$ and $Y \in \mathfrak{g}$, then $\gamma_* Y = Y$ (since Y is the lift of a vector field on M) and

$$[\gamma_* X, Y] = [\gamma_* X, \gamma_* Y] = \gamma_*[X, Y] = 0$$

so that $\gamma_* X$ is also in $\mathfrak{z}$. Therefore, we obtain a representation

$$\eta : \pi_1(M) \to GL(\mathfrak{z}).$$

Form the associated vector bundle

$$E := \tilde{M} \times_\eta \mathfrak{z} := (\tilde{M} \times \mathfrak{z})/\pi_1(M) \to M.$$

G acts on E as follows: $g(x, Z)\mathrm{mod}(\pi_1(M)) = (gx, Z)\mathrm{mod}(\pi_1(M))$.

The evaluation map $\tilde{M} \times \mathfrak{z} \to T\tilde{M}$ is easily seen to be $\pi_1(M)$-equivariant, and $T\tilde{M}/\pi_1(M) = TM$, so we obtain a vector bundle map $eval : E \to TM$. The map *eval* is easily seen to commute with the G-actions on E and on TM. By Corollary 5.4.5, the image of *eval* contains the measurable subbundle $\mathfrak{g}(x)$ of tangent spaces to the G-orbits.

Since the algebraic hull for the action of G on the frame bundle associated to the vector bundle $x \mapsto \mathfrak{g}(x)$ contains $\mathrm{Ad}(G)$, it can be shown that the algebraic hull for the G-action on the $GL(\mathfrak{z})$-principal bundle $P := (\tilde{M} \times GL(\mathfrak{z}))/\pi_1(M))$ also contains $\mathrm{Ad}(G)$. On the other hand, we claim that the algebraic hull of the G-action on P is contained in the Zariski closure, H, of the image of $\pi_1(M)$ under the representation η. In fact, it is clear that P contains a G-invariant H-reduction, given by $(\tilde{M} \times GL(\mathfrak{z}))/\pi_1(M)$. This concludes the proof of Theorem 5.4.1.

CHAPTER 6

Actions of semisimple Lie groups and lattices of higher real-rank

1. Preliminaries

a. Zimmer's program. In this section G will denote a connected higher (real-) rank semisimple Lie group and Γ a lattice in G.

We have seem before (Chapter 4) a number of classes of examples of actions of G or Γ. For smooth actions on manifolds the main examples were the projective actions, actions on locally homogeneous spaces by translations, and actions by automorphisms of nilmanifolds. A question posed by R. Zimmer [**Z4**], which has guided a large body of work on actions of higher rank groups is: to what extent is every continuous action by continuous transformations on a compact manifold built out of these relatively simple types of examples? An affirmative answer to this question would in particular imply that actions of higher rank lattices on compact manifolds of "small dimensions" are *finite*, that is, the kernel of the action contains a subgroup of finite index in Γ. Here "small" can be understood relative to the dimension of the lowest dimensional vector space that supports a nontrivial representation of G.

In the present chapter we state a number of results that validate the spirit of Zimmer's question. Nevertheless, one should bear in mind the examples of Sections 4.5 and 4.6, which make it clear that the problem as posed is too general and that we must impose further restrictions on the actions.

2. The measurable theory

a. The Cocycle Superrigidity Theorem. The central result in the ergodic theory of actions of semisimple Lie groups or lattices of higher real-rank is the Cocycle Superrigidity Theorem. It is an extension to G-spaces of Margulis Superrigidity Theorem [**Mar2**]. The proof described in the present section is that of [**Z1**], which in turn is based on a proof of Margulis's theorem that first appeared in the appendix of a Russian translation of Raghunathan's book [**Rag**]. An English translation of that appendix appears in [**Mar1**].

The theorem provides an answer to the following general question: given a cocycle $\alpha : G \times M \to H$, when is α cohomologous to a ρ-simple cocycle, that is, to a constant cocycle of the form $\beta(g, x) = \rho(g)$, where $\rho : G \to H$ is a homomorphism.

In the theorem, $\overline{G}$ will denote an algebraic linear group in $GL(n, \mathbb{C})$ defined by polynomials with real coefficients (an $\mathbb{R}$-group). The notation $\overline{G}(\mathbb{R})$ stands for the subgroup of real matrices in $\overline{G}$ and $G = \overline{G}(\mathbb{R})^0$ is the connected component of the identity element of $\overline{G}(\mathbb{R})$.

THEOREM 6.2.1 (Cocycle Superrigidity). *Suppose $\overline{G}$ is a connected semisimple algebraic $\mathbb{R}$-group, of real rank at least 2, and assume $G = \overline{G}(\mathbb{R})^0$ has no compact factors. Suppose X is an irreducible ergodic G-space with finite invariant measure. Let H be a connected algebraic k-group, almost simple over k, where k is a local field of characteristic 0. Suppose*

$$\alpha : G \times X \to H(k)$$

is a cocycle such that α is not equivalent to a cocycle taking values in a subgroup $L(k)$ where L is a proper algebraic k-subgroup of H. Then

(1) *If $k = \mathbb{R}$, H is $\mathbb{R}$-simple (equivalently center free), and $H(\mathbb{R})$ is not compact, then there is a rational homomorphism $\pi : \overline{G} \to H$ defined over $\mathbb{R}$ such that α is equivalent to the cocycle $(g, x) \mapsto \pi(g)$.*
(2) *$k = \mathbb{C}$ and H is simple (equivalently, center free), then either α is equivalent to a cocycle taking values in a compact subgroup of H, or there is a rational homomorphism $\pi : \overline{G} \to H$ such that α is equivalent to the cocycle $(g, x) \mapsto \pi(g)$.*
(3) *If k is totally disconnected, then α is equivalent to a cocycle taking values in a compact subgroup of $H(k)$.*

PROOF. We give here an overview of the proof along the lines of [**Z1**].

Let P be a minimal $\mathbb{R}$-parabolic subgroup of $\overline{G}$ and write $P_0 = P \cap G$. By the Howe-Moore Ergodicity Theorem (Theorem 3.3.1) P_0 also acts ergodically on X. An application of Fubini's theorem shows that X being an ergodic P_0-space implies that the product action of G on $X \times G/P_0$ is also ergodic. The product action is also amenable, by Proposition 2.5.2.

As a first step, one shows that there exists a proper algebraic subgroup $L \subset H$ defined over $\mathbb{R}$ and a measurable map $\varphi : X \times G/P_0 \to V$, where $V = H(\mathbb{R})/L(\mathbb{R})$, such that for each g and almost all $(x, [g']) \in X \times G/P_0$

$$\varphi(gx, g[g']) = \alpha(g, x)\varphi(x, [g']).$$

This can be seen as follows. Let $Q \subset H$ be a proper parabolic subgroup defined over $\mathbb{R}$. Then H/Q is a compact metrizable H-space. The cocycle $\alpha : G \times X \to H$ can be regarded as a cocycle over the product G-action on $X \times G/P_0$ that is constant on the second factor. Since the product G-action is amenable, one obtains by the very definition of amenability (see the definition and examples given prior to Proposition 2.5.2) an α-invariant measurable function from $X \times G/P_0$ into the space of probability measures $\mathcal{M}(H/Q)_1$. In other words, there is a measurable map $\varphi : X \times G/P_0 \to \mathcal{M}(H/Q)_1$ such that

$$\varphi(gx, g[g']) = \alpha(g, x)\varphi(x, [g']).$$

Using that the action on $\mathcal{M}(H/Q)_1$ is tame (Theorem 2.2.13), the cocycle reduction lemma (Proposition 2.4.2), and that the G-action on $X \times G/P_0$ is ergodic we conclude that φ can be regarded as an α-invariant map into H/L for a proper real algebraic subgroup L of H.

For each $x \in X$, consider the map $\varphi_x : G/P_0 \to H/L$ defined by

$$\varphi_x(u) = \varphi(x, u).$$

The next claim is that for almost all x the map φ_x is essentially rational. It will become apparent that it is here that the rank condition becomes essential.

Let t be a noncentral element of G that belongs to an $\mathbb{R}$-split torus A and let Z be the set of real points of a connected $\mathbb{R}$-group contained in the centralizer of t. Let $\mathcal{F}$ denote the space of measurable maps from Z into H/L equipped with the topology of convergence in measure. Let T be the one-parameter subgroup of P_0 containing t and define the map $\Phi : X \times G/T \to \mathcal{F}$ by $\Phi(x,u)(z) = \varphi(x, zu)$. By the α-invariance property of φ, we have that for each $g \in G$

$$\Phi(g(x,u)) = \alpha(x,g)\Phi(x,u)$$

for almost all (x,u).

Let Π be the natural projection from $\mathcal{F}$ onto $\mathcal{F}/H$. Then $\Pi \circ \Phi$ is essentially G-invariant. It can be shown that the action of H on $\mathcal{F}$ is tame. (See Chapter 3 of [**Z1**]). Moreover, the same argument used above (used for P_0 where we now have T) gives that the G-action on $X \times G/T$ is ergodic. Therefore, $\Pi \circ \Phi$ is constant almost everywhere, so that there is a single H-orbit in $\mathcal{F}$ that contains the map $\Phi(x, gT)$ for almost all $(x,g) \in X \times G$.

It is now not hard to show that for almost all (x,g), there is $v_{(x,g)} \in H/L$ and a measurable homomorphism $h_{(x,g)}$ from Z into an algebraic subgroup of H such that $\Phi_{(x,g)}(z) = h_{(x,g)}(z)v_{(x,g)}$. In this way, proving that φ_x is essentially rational for almost all x reduces to proving that certain homomorphisms are rational. The details can be found in [**Z1**]. The rank condition is needed in order to have "sufficiently many" centralizers of $t \in A$. (The key point here is that measurable homomorphisms are smooth, and their restriction to a unipotent subgroup is algebraic.)

Denote by $R = R(G/P_0, H/L)$ the space of essentially rational maps from G/P_0 into H/L. At this stage, we have obtained a measurable map $x \in X \mapsto \varphi_x \in R$ such that $\varphi_x(u) = \varphi(x,u)$. It can be shown that the natural action of $G \times H$ on R is tame and the isotropy subgroups of points are real algebraic groups. (See [**Z1**, 3.3.2].) This allows one to apply once again the argument involving the cocycle reduction lemma (as well as the Borel density theorem) to conclude (after a number of steps): by changing α to an equivalent cocycle one gets a φ such that φ_x is (essentially rational and) independent of x, for x in a conull set. In other words, there exists a cocycle β equivalent to α, proper real algebraic subgroup L of H, and a rational map $\varphi : G/P_0 \to H/L$ such that for each $g \in G$ and almost all $x \in X$,

$$\varphi(gu) = \beta(g,x)\varphi(u).$$

It is now easy to show that β (modulo a normal subgroup of H contained in L, which must be trivial by the assumptions on H) is independent of x, so it defines a measurable homomorphism from G into H. As is done for h above, one shows that the homomorphism is rational. □

By inducing from a lattice action (for a lattice in G) to an action of G itself, the cocycle superrigidity theorem can be shown to have the following corollary.

THEOREM 6.2.2. *Let Γ be a lattice in $G(\mathbb{R})^0$, where G is a connected almost $\mathbb{R}$-simple $\mathbb{R}$-group of real rank at least 2. Let X be an ergodic Γ-space with finite invariant measure. Let H be a connected $\mathbb{R}$-simple Lie group (with trivial center) with $H(\mathbb{R})$ not compact. Suppose $\alpha : \Gamma \times X \to H(\mathbb{R})$ is a cocycle with algebraic hull $H(\mathbb{R})$. Then there is an $\mathbb{R}$-rational homomorphism $\pi : G \to H$ such that α is equivalent to $(\gamma, x) \to \pi(\gamma)$.*

b. Algebraic hulls of higher rank actions. One assumption of the Cocycle Superrigidity Theorem is that the target group, H, is a simple group. Because of this condition, it becomes essential in any application of the theorem to be able to describe as precisely as possible the algebraic hull of the cocycle α. The next theorem is, therefore, of fundamental importance.

We describe the next result in the principal bundles setting, namely, suppose that a locally compact group G acts by principal bundle automorphisms of a (continuous) principal L-bundle $P \to M$, where L is a real algebraic group and M is separable and metrizable. Suppose further that the G-action on M is ergodic with respect to a finite invariant measure.

Recall that H is said to be *reductive* if $H/Z(H)$ is semisimple, where $Z(H)$ is the center of H. The next theorem says that for higher real-rank actions on compact spaces preserving a probability measure, the algebraic hull is reductive with compact center. Therefore, when applying the Cocycle Superrigidity Theorem, one typically obtains the conclusion that a given cocycle α over a higher rank action is cohomologous to a constant cocycle modulo a cocycle into a compact abelian group. For many dynamical problems, such as the determination of entropy and Lyapunov exponents, the compact cocycle is unimportant.

THEOREM 6.2.3 (Zimmer [**Z3**]). *Suppose that M is compact and that G is a connected semisimple Lie group with finite center and all simple factors of real-rank at least 2. Also suppose that G acts ergodically on M preserving a finite measure. If $P \to M$ is a principal L-bundle, where L is a real algebraic group, then the algebraic hull of the action is a reductive group with compact center. The same conclusion holds if instead of a G-action we have a Γ-action, where Γ is a lattice in G.*

We refer the reader to the paper [**Z3**]. Once one knows that the algebraic hull is reductive, the fact that the center is compact is a consequence of Theorem 3.2.3 of Chapter 3. The assertion about lattices is proved in [**Z3**] in the cocompact case. The result for non-uniform lattices was shown by J. Lewis [**Lew**]. When M is a transitive G-space the theorem is due to Margulis [**Mar2**].

c. Rigidity and orbit equivalence. We showed earlier in Chapter 2 (Proposition 2.3.1) how the problem of deciding whether two given orbit equivalent actions are isomorphic amounts to checking whether a given cocycle is cohomologous to a π-simple one. Therefore, the Cocycle Superrigidity Theorem bears directly on this problem. That theorem yields the following.

THEOREM 6.2.4 ([**Z1**], Theorem 5.2.1). *Let G, G' be connected semisimple Lie groups with finite center and no compact factors. Suppose X (respectively, X') is an essentially free ergodic irreducible G-space (respectively, G'-space) with finite invariant measure, and assume that the actions are orbit equivalent. Assume that the real rank of G is at least 2. Then*

(1) *G and G' are locally isomorphic.*
(2) *If the groups are center-free, then G and G' are isomorphic, and identifying G and G' via this isomorphism, the actions of G on X and on X' are isomorphic.*

One consequence of the previous theorem and Proposition 4.4.2 is that if G is a connected semisimple Lie group without compact factors and real rank at least 2, then G has uncountably many mutually non-orbit equivalent finite measure preserving ergodic actions.

A similar result holds for lattices.

THEOREM 6.2.5 ([**Z1**], Corollary 5.2.2). *Let* Γ *be a lattice in a higher real-rank simple Lie group* G *and* Γ' *a lattice in a semisimple Lie group* G'*, where* G *and* G' *have finite centers. Suppose that* Γ *and* Γ' *have orbit equivalent essentially free finite measure preserving actions on* X *and* X'*, respectively. Then* G *and* G' *are locally isomorphic.*

d. More on orbit equivalence. We describe in this section some further results concerning orbit equivalence rigidity, due to A. Furman.

Let $(X, \mathcal{B}, \mu)$ be a non-atomic probability space, where $(X, \mathcal{B})$ is a standard Borel space. Given a measure preserving ergodic action of a countable group Γ on $(X, \mathcal{B}, \mu)$, one defines an equivalence relation, $\mathcal{R}_\Gamma$, on X as the subset of $X \times X$ consisting of pairs (x, y) such that the Γ-orbits of x and y coincide. Note that a Γ-space X with invariant measure μ and a Λ-space Y with invariant measure ν are orbit equivalent if there are conull subsets $X' \subset X$ and $Y' \subset Y$ and a Borel isomorphism $\theta : X' \to Y'$ such that $\theta_*\mu$ and ν are equivalent measures and $(x_1, x_2) \in \mathcal{R}_\Gamma$ if and only if $(\theta(x_1), \theta(x_2)) \in \mathcal{R}_\Lambda$.

A general question that can be asked about orbit equivalences is the following. Suppose that X is an essentially free Γ-space with finite invariant measure μ. What properties of the group Γ and the action are determined by the equivalence relation $\mathcal{R}_\Gamma$?

We have already encountered a few results in this direction: all actions of (say, countable) amenable groups are mutually orbit equivalent (Theorem 2.5.5); an essentially free finite measure preserving action of a non-amenable group is never orbit equivalent to an action of an amenable group (Theorem 2.5.3); therefore, amenability of a group acting essentially freely is determined by $\mathcal{R}_\Gamma$. Furthermore, if the group is amenable, $\mathcal{R}_\Gamma$ contains no further information. Theorem 3.2.5 shows that, for Γ, having the Kazhdan property T is also determined by $\mathcal{R}_\Gamma$ (for essentially free finite measure preserving actions). For actions of higher real-rank semisimple groups and lattices, one has the theorems of the previous subsection.

It was shown by Furman that there are group actions for which the equivalence relation $\mathcal{R}_\Gamma$ on X determines the group Γ and the Γ-space X (with finite invariant measure) uniquely, up to finite groups. Some of Furman's results will be described below.

First consider the following definition. Two countable groups Γ_1 and Γ_2 will be called *virtually isomorphic* if there exist finite normal subgroups $N_i \subset \Gamma_i$ such that the quotient groups $\Gamma_i' := \Gamma_i/N_i$ contain isomorphic subgroups of finite index: $\Lambda_1 \cong \Lambda_2$. Let (X_i, μ_i) be an ergodic Γ_i space with finite invariant measure μ_i, for $i = 1, 2$, where Γ_1 and Γ_2 are virtually isomorphic. Then Γ_i' naturally acts on the quotient probability space (X_i', μ_i') of N_i orbits. Consider the action of the finite index subgroup Λ_i on (X_i', μ_i') and let (Y_i, ν_i) be one of the (at most $|\Gamma_i' : \Lambda_i|$) mutually isomorphic Γ_i-ergodic components. Then the Γ_1-action on (X_1, μ_1) and the Γ_2-action on (X_2, μ_2) will be said to be *virtually isomorphic actions* if Γ_1 and Γ_2 are virtually isomorphic groups and for some choice of N_i and Λ_i as above the actions of Λ_1 on (Y_1, ν_1) and of Λ_2 on (Y_2, ν_2) are isomorphic.

The following weaker version of orbit equivalence will also be needed. A Γ_1-action on X_1 preserving a finite measure μ_1, and a Γ_2-action on X_2 preserving a finite measure μ_2 will be called *weakly orbit equivalent* if there exist subsets $A_i \subset X_i$ of positive measure such that the restricted relations $\mathcal{R}_{\Gamma_i}|_{A_i} := \mathcal{R}_{\Gamma_i} \cap (A_i \times A_i)$, on

$(A_i, \mu_i|_{A_i})$, for $i = 1, 2$, are isomorphic. (Virtual isomorphism is a particular case of weakly orbit equivalence).

We state now two theorems by A. Furman. For other of his results in the same subject we refer the reader to [**Furm1**].

THEOREM 6.2.6 (Furman). *Let Γ be a lattice in a simple, connected, non-compact Lie group G with finite center and real rank at least 2. Suppose that Γ acts ergodically by measure preserving transformations on a probability space $(X, \mathcal{B}, \mu)$. Assume that X does not have Γ-equivariant measurable quotients of the form $Ad(G)/\Delta$, where Δ is a lattice in $Ad(G)$. Then the Γ-action is* strongly orbitally rigid *in the following sense: if Λ is an arbitrary countable group which has an essentially free finite measure preserving ergodic action on (Y, ν) which is weakly orbit equivalent to the Γ-action on (X, μ), then Γ and Λ are virtually isomorphic groups and the actions of Γ on (X, μ) and of Λ on (Y, ν) are virtually isomorphic.*

For example, it follows from the theorem that the natural action of $SL(n, \mathbb{Z})$ on the n-torus, $n > 2$, is strongly orbitally rigid, since this action does not admit $SL(n, \mathbb{Z})$-equivariant quotients of the form $(\mathrm{Ad}(SL(n, \mathbb{R})), \mathrm{Haar})$. This can be seen by the following observation. For $\gamma \in SL(n, \mathbb{R})$ with at least one eigenvalue not on the unit circle, the entropy $h(\gamma, \mathbb{T}^n)$ of the action on the torus is strictly less than the entropy $h(\gamma, PSL(n, \mathbb{R})/\Delta)$, since these entropies can be computed by the formulas

$$h(\gamma, \mathbb{T}^n) = \sum_{\lambda_i \geq 0} \lambda_i, \quad h(\gamma, PSL(n, \mathbb{R})/\Delta) = \sum_{\lambda_i \geq \lambda_j} (\lambda_i - \lambda_j)$$

where $\exp \lambda_i$ is the absolute value of the i-th eigenvalue of γ.

Consider now the case of an ergodic action of a higher rank lattice $\Gamma \subset G$ that admits G/Δ quotients. The Γ-action on G/Δ is itself such an example and it is known to be weakly orbit equivalent to the action of Δ on G/Γ. By the next theorem, this is essentially the only source of actions which are weakly orbit equivalent to an action of a higher rank lattice.

THEOREM 6.2.7 (Furman). *Let $\Gamma \subset G$ be a higher rank lattice as in Theorem 6.2.6, and let Γ act ergodically on a probability space (X, μ) by measure preserving transformations. Then given any Γ-equivariant measurable quotient map $\pi : X \to Ad(G)/\Delta$, where $\Delta \subset Ad(G)$ is a lattice, there exists a canonically defined essentially free measure preserving ergodic action of Δ on a probability space (X_π, μ_π) that is weakly orbit equivalent to the first action.*

Suppose now that Λ is a countable group that acts ergodically and essentially freely by measure preserving transformations on a probability space (Y, ν), and that the Λ-action is weakly orbit equivalent but not virtually isomorphic to the Γ-action on (X, μ). Then there exists a lattice $\Delta \subset Ad(G)$ and a measurable Γ-equivariant quotient map $\pi : X \to Ad(G)/\Delta$ such that Λ and Δ are virtually isomorphic groups and the actions of Λ on (Y, ν) and of Δ on (X_π, μ_π) are virtually isomorphic.

e. Rigidity of entropy. The Cocycle Superrigidity Theorem says, in essence, that for ergodic actions of higher real-rank semisimple Lie groups and lattices, their dynamical behavior, at least in what concerns measurable ergodic theory, is mainly determined by a linear finite dimensional representation of the acting group. (The homomorphism π of Theorem 6.2.1.) One particularly interesting illustration of this point has to do with the range of values that the measure theoretical entropy can attain for actions of higher real rank groups. The following is Theorem 9.4.12 in [**Z1**].

THEOREM 6.2.8 (Furstenberg [**Fur5**]). *Let G be a connected almost simple real group with real rank at least 2. Suppose that G acts by C^2 volume preserving diffeomorphisms on a compact manifold M, and suppose that the action is ergodic. For $g \in G$, let $h(g)$ denote the measure theoretic entropy of the action of g on M. Then either $h(g) = 0$ for all g, or for each $g \in G$, there is an $\mathbb{R}$-rational representation $\pi : \tilde{G} \to SL(r, \mathbb{C})$, for some r, such that*

$$h(g) = max\{\log(|\lambda|) : \lambda \text{ is an eigenvalue of } \pi(\tilde{g})\},$$

where $\tilde{G}$ is the (algebraic) universal covering group of G and $\tilde{g}$ is any element of $\tilde{G}$ that projects to g. Furthermore, if n is the dimension of M, then π is an exterior power of a representation σ of $\tilde{G}$, where the representation space of σ has dimension at most n. In particular, for a fixed n, and fixed $g \in G$, the set of possible values of $h(g)$ over all volume preserving, ergodic, C^2 actions of G on compact manifolds of dimension at most n is finite.

PROOF. The key ingredient here besides the Cocycle Superrigidity Theorem is the Pesin Entropy Formula. In the case of ergodic actions, that formula gives that $h(g)$ can be described as follows. Let α be the derivative cocycle, for the $\mathbb{Z}$-action on M generated by g. Write $\alpha^p := \Lambda^p(\alpha)$ for the p-th exterior derivative of α and define

$$e(\alpha^p)(x) := \lim_{n\to\infty} \frac{1}{n} \log^+ \|\alpha^p(g^n, x)\|$$

if the limit exists. (It does exist at almost every point.) Then the entropy formula implies that $h(g) = \max_p e(\alpha^p)$. One then shows that if α is cohomologous to the constant cocycle α_π, then $e(\alpha^p) = e(\alpha_\pi^p)$, from which one obtains the expression for $h(g)$ given in the theorem. □

f. Superrigidity and Ratner's theorem. Zimmer's classification program (mentioned at the beginning of this chapter) is based on the conjecture that manifolds that support actions of higher real-rank semisimple Lie groups and lattices are derived from building block actions consisting of the boundary actions described in Chapter 4, and *arithmetic actions*, defined as follows: let H be a real algebraic group, Γ an arithmetic subgroup of H, $K \subset H$ a possibly trivial compact subgroup, and $\rho : G \to H$ a continuous homomorphism so that K centralizes $\rho(G)$. Then G naturally acts on the left on $K\backslash H/\Gamma$ through ρ. In the category of measurable G-spaces, and in the presence of a finite G-invariant measure, the essential building block actions should be the arithmetic ones only.

It was noted in [**Z16**] that the Cocycle Superrigidity Theorem, in combination with M. Ratner's celebrated theorem solving a conjecture of Raghunathan on invariant measures on locally homogeneous spaces (which is discussed in much greater length in [**S-KSS**]), can be used to obtain certain (measurable) arithmetic factor actions for ergodic finite measure preserving actions of higher rank groups on compact manifolds. Before explaining this point, we recall Ratner's result. (See [**Ra**].)

THEOREM 6.2.9 (Ratner). *Let H be a connected Lie group, Γ a discrete subgroup of H and $G \subset H$ a connected simple non-compact subgroup. Let ν be a finite G-invariant ergodic measure on H/Γ, where the action is given by the embedding* *of G in H. Then there is a closed connected subgroup L of H containing G and a point $x = h\Gamma \in H/\Gamma$ for which the following hold:*

(1) *$L \cap h\Gamma h^{-1}$ is a lattice in L, and*
(2) *ν is the measure on H/Γ corresponding to the invariant volume on $L/L \cap h\Gamma h^{-1}$, where the quotient is naturally identified with the orbit $Lx \subset H/\Gamma$.*

To understand what role Ratner's theorem has to play in this subject, we first state a proposition that contains one key remark of [**Z16**] (but stated in a more restricted form). The proposition will require the notion of a finite *ergodic extension* of a G-action on a manifold M with measure μ. This is an ergodic G-space M', where M' is a finite covering space of M and the covering map is a G-map. We also need the following remark. If a connected group G acts continuously on a connected manifold M, then the G-action lifts to an action of the universal covering group $\tilde{G}$ on the universal covering $\tilde{M}$ of M. The covering map $\tilde{M} \to M$ can be regarded as a principal Γ-bundle over M, where Γ is the group of deck transformations of $\tilde{M}$. If Γ is contained in a Lie group H, one can form the associated principal H-bundle $P = (\tilde{M} \times H)/\Gamma$.

PROPOSITION 6.2.10. *Let G be a connected, higher real-rank simple Lie group, with finite center, acting continuously on a compact connected manifold M so as to preserve a finite ergodic measure μ. We make the following assumptions:*

(1) *The group Γ of deck transformations of M is a subgroup of a real algebraic group H; and*
(2) *the algebraic hull of the $\tilde{G}$-action on $P = (\tilde{M} \times H)/\Gamma$ is not compact.*

Then, there exists a non-trivial continuous homomorphism $\rho : \tilde{G} \to H$ (where $\tilde{G}$ is the universal covering group of G) and a measure preserving G-map

$$\theta : M' \to K\backslash L/\Gamma',$$

where M' is a finite ergodic extension of the G-space M, L is a closed subgroup of H containing $\rho(\tilde{G})$, K is a compact subgroup of L centralizing $\rho(\tilde{G})$, and Γ' is a lattice of L isomorphic to a subgroup of Γ. The G-action on the double coset space $K\backslash L/\Gamma'$ is the natural one associated to the homomorphism ρ, and the measure on $K\backslash L/\Gamma'$ derives from the projection of the L-invariant volume form on L/Γ'.

PROOF. If a connected Lie group G acts on a connected smooth manifold M, then the G-action lifts to an action of $\tilde{G}$ on $\tilde{M}$. Let $P = (\tilde{M} \times H)/\Gamma$ and denote by $\mathcal{G} : P \to H/\Gamma$ the G-invariant, H-equivariant map that defines the Γ-reduction, $\tilde{M}$, of P.

Let $Q \subset P$ be a G-invariant H'-reduction of P, where $H' \subset H$ is the algebraic hull of the G-action on P. It will be assumed that H' is algebraically connected, that is, $H' \otimes \mathbb{C}$ is a connected group. (If this is not the case, it can be shown that there exists a finite ergodic extension M' of M such that the G-action on the pull-back of P to M' has algebraically connected algebraic hull. It will be supposed for the rest of the proof that $M' = M$.)

We can now apply Theorems 6.2.1 and 6.2.3 to the action on P. The conclusion is that the $H'/Z(H')$-bundle $\overline{Q} := Q/Z(H')$ (where $Z(H')$ is the compact center of H') admits a measurable section, $\sigma : M \to P/Z(H')$, such that $g\sigma(x) = \sigma(gx)\rho(g)$, for $x \in M$ and $g \in G$, where ρ is a nontrivial surjective (rational) homomorphism from G to $H'/Z(H')$.

Denote by $\overline{\mathcal{G}} : \overline{Q} \to Z(H')\backslash H/\Gamma$ the G-invariant, $H'/Z(H')$-equivariant map obtained from $\mathcal{G}$ by factoring out the center of H' and define

$$\theta = \overline{\mathcal{G}} \circ \sigma : M \to Z(H')\backslash H/\Gamma.$$

The first claim is that θ is a G-map. In fact, $\theta \circ g = \rho(g) \circ \theta$ for each $g \in G$, since

$$\begin{aligned}\theta(x) &= \overline{\mathcal{G}}(\sigma(x)) \\ &= \overline{\mathcal{G}}(g\sigma(x)) \\ &= \overline{\mathcal{G}}(\sigma(gx)\rho(g)) \\ &= \rho(g)^{-1}\overline{\mathcal{G}}(\sigma(gx)) \\ &= \rho(g)^{-1}\theta(gx).\end{aligned}$$

In particular, the measure $\nu := \theta_*\mu$ is $\rho(G)$-invariant. The result of the proposition is now a consequence of Ratner's theorem. □

g. The engaging conditions. Proposition 6.2.10 can be used to study the fundamental group of manifolds that support actions of higher rank groups. (See [**Z9, Z16, LZ1, LZ2**].) In order to do that, it is desirable to have dynamical or geometric conditions that imply hypothesis 1 and 2 of the proposition.

We first discuss Hypothesis 2. (We use here the same notation employed there.) It is not difficult to show that the existence of an invariant (measurable) reduction of the H-bundle P with compact group is equivalent to an invariant (measurable) reduction with finite group of $\tilde{M}$, the latter being regarded as principal Γ-bundle over M. The *engaging conditions* are conditions on the action that prevent the possibility of such invariant reductions.

The following definitions are due to Zimmer. A G-action on M is *totally engaging* if any measurable $\tilde{G}$-invariant reduction is trivial; that is, if Λ is the group of the reduction, then $\Lambda = \Gamma$. The action is called *engaging* if there is an invariant reduction to $\Lambda \subset \Gamma$ only if Λ surjects into every finite quotient of Γ; that is, Λ is *profinitely dense* in Γ. A *topologically engaging* action is one for which the reduction cannot exist unless Λ is infinite.

It can be shown that if the G-action on M is ergodic, then the action is engaging if and only if $\tilde{G}$ acts ergodically on every finite covering space of M. Therefore, if G contains an Anosov diffeomorphism of M, then the action on M is engaging. On the other hand, the example described in the last paragraph of Section 4.6 is not engaging.

The next proposition contains a general class of examples.

Proposition 6.2.11. *Let $M = H/\Gamma$, where Γ is a lattice in H and H is a noncompact simple Lie group with finite center. Let G be a closed noncompact subgroup of H, acting on M by left-translations. Then the action is totally engaging. (Hence it is also engaging and topologically engaging.)*

Proof. Having a $\tilde{G}$-invariant reduction of $\tilde{M}$ with group Λ is equivalent to the existence of an G-equivariant section $\sigma : M \to H/\Lambda$. Since Γ is a lattice in H, there exists a $\tilde{G}$-invariant probability measure, μ, on M, which is ergodic due to the Howe-Moore ergodicity theorem. (Theorem 3.3.1.) The measure $\sigma_*\mu$ is then a $\tilde{G}$-invariant ergodic probability measure on H/Λ. Ratner's theorem implies that there exists a closed connected subgroup L of H such that $\sigma_*\mu$ is supported on a single L-orbit and is L-invariant. But $\sigma_*\mu$ must project to μ, which gives measure

0 to any immersed submanifold of lower dimension. Therefore $L = H$ and $\sigma_*\mu$ is H-invariant. Consequently H/Λ has an H-invariant probability measure, so Λ is a lattice in H. By the Howe-Moore ergodicity once more, the G-action (hence also $\tilde{G}$-action) on H/Λ is ergodic. This implies that the section σ is surjective, hence $\Lambda = \Gamma$. □

In the next theorem M can be any second countable, metrizable topological space that satisfies the usual conditions for existence of covering spaces.

THEOREM 6.2.12 (Zimmer, Lubotzky-Zimmer). *Let G be a connected, higher real-rank simple Lie group with finite center. Suppose that it is given a continuous action of G on a compact manifold M such that the action preserves a probability measure μ. Let $\rho : \pi_1(M) \to GL(n, \mathbb{C})$ be a homomorphism and assume that $\Gamma := \rho(\pi_1(M))$ is infinite. Then the following hold:*

(1) *If the action is totally engaging, then Γ is arithmetic in some $\mathbb{Q}$-group H that contains G.*
(2) *If the action is engaging, then Γ is $\mathfrak{s}$-arithmetic in some $\mathbb{Q}$-group H that contains G.*
(3) *If the action is topologically engaging, and ρ is injective with discrete image, then Γ contains an arithmetic subgroup of some $\mathbb{Q}$-group H that contains G.*

The ideas contained in the proof of Proposition 6.2.10 together with the topologically engaging hypothesis essentially yield part 3. We refer the reader to [**LZ1, LZ2**] for the proof with the other engaging conditions. For the definition of $\mathfrak{s}$-arithmetic groups, which generalizes the notion of arithmetic groups, the reader is referred to [**Mar2**].

Under the conditions of Gromov's Theorem 5.4.1 (having, in particular, an analytic invariant rigid A-structure) one obtains the representation of $\pi_1(M)$ needed for the application of Theorem 6.2.12. Furthermore, the arithmetic quotient for the higher real-rank action obtained by Proposition 6.2.10 can be shown to be large in a dynamical sense (as measured by the measure theoretic entropy). This is made precise in the next theorem, from [**Z9**]. For each $g \in G$, $h(g, M)$ denotes the entropy of the action of g on M, and $h(g, K\backslash H/\Gamma)$ is the entropy of the action of g on the arithmetic quotient.

THEOREM 6.2.13 (Zimmer). *Let M be a compact real analytic manifold with a real analytic unimodular (i.e. containing a volume density) rigid geometric A-structure. Let G be a connected simple Lie group with real-rank at least 2 and suppose that G acts analytically on M preserving the structure. Assume further that the action is engaging. Then there is a measurable fully entropic arithmetic quotient $K\backslash H/\Gamma$ for the G-action on M; that is, $h(g, M) = h(g, K\backslash H/\Gamma)$ for all $g \in G$, where $K\backslash H/\Gamma$ is as in Proposition 6.2.10.*

3. Topological superrigidity

It turns out that the Cocycle Superrigidity Theorem can be, at least in part, formulated in a purely topological or differentiable setting. The results described in this subsection are from [**FL**].

The general setting will be a C^s ($s \geq 0$) principal H-bundle $\pi : P \to M$ over a manifold M, where H is a real algebraic group. Suppose that a Lie group G acts on P by principal bundle automorphisms, the action being C^s. For a given subgroup

$L \subset H$ and an open subset $U \subset M$, a C^s *reduction* of P over U is a C^s L-subbundle Q of $P|_U$. The L-reduction is said to be G-invariant if U is a G-invariant set and G acts by automorphisms of Q, that is, Q is a G-invariant subset of P.

We know from Proposition 2.4.7 that if the action of G on M is topologically transitive, there exists an algebraic subgroup L of H, uniquely determined up to conjugacy, such that there is a C^s G-invariant L-reduction of P over some G-invariant dense open set $U \subset M$, but there does not exist such a reduction for some proper real algebraic subgroup L_1 of L. We call L the C^s algebraic hull of the G-action on P.

A 1-parameter subgroup T of G is said to be $\mathbb{R}$-*semisimple* if for each linear representation ρ of G $\rho(a)$ is diagonalizable with real eigenvalues for all $a \in T$.

THEOREM 6.3.1 (Topological superrigidity [**FL**]). *Let G denote the connected component of a semisimple real algebraic group with real rank at least 2. Suppose that G acts by H-bundle automorphisms on some C^s principal H-bundle P over a manifold M such that the action is also C^s. Assume that*

(i) *H is the C^s algebraic hull of the G-action;*
(ii) *every $\mathbb{R}$-semisimple 1-parameter subgroup of G acts topologically transitively on M.*

Assume furthermore that there is a subgroup $K \subset G$ with the following properties:

(iii) *K acts topologically transitively on M,*
(iv) *K commutes with some hyperbolic 1-parameter subgroup of G,*
(v) *the C^s algebraic hull of the K-action does not contain a nontrivial normal subgroup of H.*

Then, there exists a continuous surjective homomorphism $\rho : G \to H$ and a C^s section σ of $P|_U$, for some open dense G-invariant subset U of M, such that $g\sigma(x) = \sigma(gx)\rho(g)$ for all $g \in G$ and $x \in U$,

Note that if the G-action on M preserves an ergodic probability measure whose support is M, an application of the Howe-Moore ergodicity theorem and Poincaré recurrence gives the following corollary, in which topological transitivity and recurrence are replaced by ergodicity of G.

THEOREM 6.3.2. *Suppose that a connected semisimple group G of real-rank at least 2 acts by H-bundle automorphisms on some C^s principal H-bundle P over a manifold M such that the action is also C^s. Assume that the action preserves an ergodic probability measure whose support is M and that H is the C^s algebraic hull of the G-action. Assume moreover that there is a noncompact subgroup $K \subset G$ such that K commutes with some $\mathbb{R}$-semisimple 1-parameter subgroup of G and the C^s algebraic hull of the K-action does not contain a nontrivial normal subgroup of H. Then, there exists a continuous surjective homomorphism $\rho : G \to H$ and a C^s section σ of $P|_U$, for some open dense G-invariant subset U of M, such that $g\sigma(x) = \sigma(gx)\rho(g)$ for all $g \in G$ and $x \in U$,*

The next result uses an observation due to Zimmer [**Z15**] that complements the previous theorem by giving conditions for the section σ to be defined on the whole manifold. We first define the concept of a *parabolic invariant*, introduced by Zimmer.

Let V be a smooth real algebraic variety equipped with a real algebraic left action of H. By a C^s *geometric structure* on P of type V we mean a C^s section of the associated V bundle $P_V = (P \times V)/H$. Suppose that G is the identity component of a real algebraic semisimple group. A geometric structure φ on P is called a *parabolic invariant* if it is invariant under some parabolic subgroup of G.

The G-action on the H-bundle P is said to be *effective* relative to the geometric structure $\sigma : M \to P_V$ if, for some $x \in M$, the group of automorphisms of P_x fixing $(g_*\sigma)(x)$ for all $g \in G$ is trivial, where E is associated to φ as indicated earlier.

THEOREM 6.3.3. *In addition to the assumptions of Theorem 6.3.1, suppose there exists a C^s parabolic invariant relative to which the G-action is effective. Then the C^s section σ obtained in Theorem 6.3.1 is defined over the whole manifold.*

We describe an application of Theorem 6.3.1 to smooth actions of a lattice group Γ in G. We recall that an action on M by a lattice Γ in a Lie group G induces by suspension an action of G on the manifold $N := (G \times M)/\Gamma$. The Γ-action on the product is given by the quotient of $G \times M$ by the action of Γ given by

$$(g, x) \cdot \gamma := (g\gamma, a(\gamma)^{-1}x).$$

G acts locally freely on N, so that the foliation of N by fibers $M_{[g]} := p^{-1}([g])$ is preserved by the action and is everywhere transverse to the G-orbits. We denote the transversal foliation by $\mathcal{M}$. Also note that each fiber of $\mathcal{M}$ is diffeomorphic to M.

Since N may fail to be compact (when the lattice is not uniform), it will be necessary to assume the existence of a Riemannian metric on N with norm $\|\cdot\|$ for which $\|g_*\|$ is uniformly close to 1 for all g sufficiently close to e. In particular, as G is connected, $\|g_*\|$ is uniformly bounded for each $g \in G$. (This is clearly satisfied for the model actions, for example, the suspension of the affine action of $SL(n, \mathbb{Z})$ on the n-torus.)

Also with respect to $\|\cdot\|$, we say that $k \in G$ is an *Anosov* element if $T\mathcal{M}$ decomposes as a continuous direct sum of subbundles

$$T\mathcal{M} = E^- \oplus E^+$$

such that k (resp., k^{-1}) is uniformly contracting on E^- (resp., E^+), i.e., there is λ, $0 < \lambda < 1$, such that $\|k_*|_{E^-}\| \leq \lambda$ (resp., $\|(k^{-1})_*|_{E^+}\| \leq \lambda$). We call E^- (resp., E^+) the *stable* (resp., *unstable*) subbundle of k.

THEOREM 6.3.4. *We assume that a lattice Γ of $G = SL(n, \mathbb{R})$, $n \geq 3$, acts smoothly on a compact manifold M of dimension n. Suppose, moreover, that for the induced G-action on $N = (G \times M)/\Gamma$, (i) every $\mathbb{R}$-semisimple 1-parameter subgroup of G acts topologically transitively on N and (ii) some regular element k of G is Anosov. Then M is a flat torus for some smooth Riemannian metric and the Γ-action is a standard affine action with respect to that metric.*

We point out that the standard actions on tori satisfy the above conditions. We refer the reader to [**FL**] for further applications of Theorem 6.3.1.

4. Actions on low-dimensional manifolds

a. Lattice actions on the circle. We discuss now a number of results that provide support for the conjecture that actions of higher rank groups on low dimensional compact manifolds must be finite. For the most part we consider actions of higher rank lattices on the circle.

A conjecture of Zimmer states that if Γ is a lattice in a simple Lie group G with real rank at least 2, then every continuous Γ-action on a circle is finite. (That is, a finite index subgroup of Γ must act trivially.) Note that the rank condition is essential since $PSL(2,\mathbb{R})$, as well as any lattice in it, acts non-trivially on S^1. (If we regard S^1 as the quotient of $PSL(2,\mathbb{R})$ by the subgroup of upper-triangular matrices, then $PSL(2,\mathbb{R})$ acts on the circle by left-translations.)

The first result giving partial confirmation to the conjecture was obtained by D. Witte and is described in the next theorem. An arithmetic lattice Γ of an algebraic subgroup of $GL(n,\mathbb{R})$ is said to have $\mathbb{Q}$-rank k if it contains an abelian subgroup of rank k each of whose elements can be diagonalized by conjugation with some matrix in $GL(n,\mathbb{Q})$. A finite index subgroup of $SL(n,\mathbb{Z})$, $n \geq 3$, has $\mathbb{Q}$-rank $n-1$, whereas any cocompact lattice in a simple Lie group has $\mathbb{Q}$-rank 0.

THEOREM 6.4.1 (Witte [**Wi**]). *Suppose that Γ is an arithmetic lattice with $\mathbb{Q}$-rank at least 2. Then every Γ-action by homeomorphisms on a circle is finite.*

If the actions are allowed to be C^1, then the conjecture is confirmed by theorems due to E. Ghys [**Gh1**] and, independently, M. Burger and N. Monod. (Burger and Monod also require the second cohomology group $H^2(\Gamma,\mathbb{R})$ to vanish. [**BM**].) The proof of Ghys has been streamlined and extended to actions on circle bundles by Zimmer and Witte in [**WZ**].

THEOREM 6.4.2 (Ghys, Burger-Monod). *Any C^1-action of a higher real-rank lattice on the circle is finite.*

Results of a similar type using very different techniques have been proved by B. Farb and P. Shalen, when the action is real analytic. [**FS**].

b. Outline of Ghys's proof of Theorem 6.4.2. We explain some of the main ideas in Ghys's proof, for $\Gamma = SL(3,\mathbb{R})$.

LEMMA 6.4.3. *Suppose that every 1-dimensional linear representation of every finite index subgroup of the higher rank lattice Γ is finite. If a differentiable action of Γ on the circle has a finite orbit, then the action is finite.*

PROOF. By possibly having to replace Γ with a finite index subgroup, it can be assumed that action has a fixed point x. By taking derivatives one obtains a linear action of Γ on the tangent space T_xM, which by assumption must be finite. By passing to yet another finite index subgroup it can be assumed that this linear action on T_xM is trivial. Note that $[\Gamma,\Gamma]$ is an infinite normal subgroup of Γ. By the Normal Subgroup Theorem of Margulis (Theorem 6.7.6 discussed at the end of the chapter) $\Gamma/[\Gamma,\Gamma]$ must be finite. The proof follows now from Lemma 6.4.4, given next. □

LEMMA 6.4.4 (Thurston). *Let Γ be a subgroup of the group of orientation preserving C^1 diffeomorphisms of a circle fixing a point. Assume that Γ is finitely generated, and that $\Gamma/[\Gamma,\Gamma]$ is finite. Then Γ is the trivial group.*

See also [**Schw**] for a simple proof of the lemma. The lemma is a very special case of a remarkable result of Thurston's, generalizing a stability theorem for codimension-one foliations, known as Reeb's stability theorem. We state the following version of Thurston's result. For related results the reader should consult [**Thur**].

THEOREM 6.4.5 (Thurston). *Let G be a topological group generated by a compact neighborhood of the identity. Suppose G acts continuously in the C^1 topology as a group of C^1 transformations on a connected manifold (of arbitrary dimension n), having a fixed point p. Also suppose that the linear action at p (on the tangent space* *at p) is trivial and that G does not admit a continuous nontrivial homomorphism into $\mathbb{R}$. Then the G-action is trivial.*

LEMMA 6.4.6 (Ghys). *An action of a higher rank lattice by homeomorphisms of a circle must preserve a probability measure.*

PROOF. The proof of the lemma relies on Theorem 4.1.3 (Furstenberg's lemma). The theorem says that one can find a Γ-equivariant measurable map $\Psi : G/P \to \mathcal{P}$, where $\mathcal{P}$ is the space of probability measures on the circle and P is a parabolic subgroup of G. Equivariance means that $\Psi(\gamma gP) = \gamma\Psi(gP)$ for $\gamma \in \Gamma$ and $gP \in G/P$. A careful study of this map shows that it is essentially constant. But being Γ-equivariant implies that its value is a Γ-invariant probability measure on the circle. □

When the probability measure obtained by the previous lemma has atoms, one goes on to show that Γ must have a finite orbit, and Thurston's theorem can be used (after passing to a subgroup of finite index). If the probability measure does not have atoms, it is possible to introduce new coordinates on the circle (by integrating the measure) with respect to which the action is by rotations. In particular, Γ maps into an abelian group of homeomorphisms. Since $\Gamma/[\Gamma, \Gamma]$ is finite we conclude that the Γ-action is finite.

c. Low dimensional actions preserving rigid structures. When one imposes the condition that the G-action (possibly a lattice action) preserves a rigid structure, the classification problem becomes more tractable since the action now corresponds to homomorphisms of G into finite dimensional groups. We indicate here a small number of representative results, in which the invariant geometric structure is a linear connection on the tangent bundle of M.

Let G be a connected semisimple Lie groups with finite center such that every simple factor has real rank at least 2. Let $\mathfrak{g}$ be the Lie algebra of G and $d(\mathfrak{g})$ the minimal dimension of a non-trivial (real) representation of $\mathfrak{g}$. Suppose that $\Gamma \subset G$ is a lattice subgroup. Then the following holds. (For the proof, we refer the reader to the papers [**Z11, Z12, Z13**].)

THEOREM 6.4.7 (Zimmer). *Let M be a compact manifold with dimension $n < d(\mathfrak{g})$. Then every smooth action of Γ on M which preserves a volume density and a connection must also preserve a Riemannian metric.*

It is interesting to note that the assumption that a higher-rank action preserves a connection is natural in the following sense. It can be shown that Theorem 6.2.3, which says that the algebraic hull of a higher rank action is reductive (with compact center), implies the existence of a *measurable* invariant connection. The space of invariant measurable connections (for ergodic actions of semisimple Lie groups without compact factors) can be shown to be finite dimensional, by using the ideas of Section 5.3a, and it often reduces to a single point (a unique invariant connection). Therefore the real assumption is about the regularity of a very special geometric structure, rather than its existence. This point is explored further in [**Fe1**]. See also [**Fe2**].

The structure of the compact Lie groups K admitting a dense image homomorphism $\Gamma \to K$ has been described by Margulis. (See, for example, [**Z13**].) In combination with Margulis work, the previous theorem yields the following corollary.

COROLLARY 6.4.8. *Let* Γ *be a lattice in* $SL(n, \mathbb{R})$ *(*$n \geq 3$*). Let* M *be a compact manifold with dimension strictly less than* n*. Then any action of* Γ *on* M *preserving a volume density and a connection is finite, that is, is an action by a finite quotient of* Γ*.*

Therefore, the standard action of $SL(n, \mathbb{Z})$ on the torus $\mathbb{T}^n$ is an affine (connection preserving) non-finite action of minimal dimension. The next theorem gives a characterization of this action. (A similar result was obtained by Zimmer under the assumption that the connection is Riemannian.)

THEOREM 6.4.9 ([**Fe1**]). *Let a lattice* Γ *in* $SL(n, \mathbb{R})$ *act on a compact smooth manifold* M *of dimension* n *so as to preserve a Lipschitz continuous affine connection and an ergodic volume density. Assume moreover that the action does not preserve a Riemannian metric. Then the manifold is diffeomorphic to an* n*-dimensional torus,* Γ *is commensurable to* $SL(n, \mathbb{Z})$ *and the action is* C^∞ *isomorphic to the standard affine action on the torus.*

It was pointed out earlier that an Anosov diffeomorphism can be regarded as a rigid geometric structure in the sense of Gromov, although it is not an A-structure (that is, of algebraic type), such as a connection. Nevertheless, Anosov maps are often associated to connections of low regularity. (This was first observed by M. Kanai in [**Kan1**].) See also [**Fe2**].) The next theorem is a corollary of the previous one and the result established in [**Fe2**] about the existence of invariant connections for Anosov diffeomorphisms under a pinching condition. We recall that an Anosov diffeomorphism $f : M \to M$ (with expanding and contracting subbundles denoted by E^+, E^-) is said to satisfy the half-pinching condition if, for some Riemannian metric on M, there are constants $C > 1$, $0 < a < A$, with $A < 2a$, such that for all $x \in M$, all $m \in \mathbb{N}$ and all $v \in E^\pm(x)$,

$$\frac{1}{C}\|v\|e^{-mA} \leq \|df_x^{\mp m} v\| \leq C\|v\|e^{-ma}.$$

THEOREM 6.4.10 ([**Fe2**]). *Let* Γ *be a lattice in a higher rank semisimple Lie group* G*. Assume that* Γ *acts on a compact,* n*-dimensional smooth manifold* M *so as to preserve a volume density and so that some element of* Γ *is a half-pinched Anosov diffeomorphism. Also assume that the dimension of the first nontrivial representation of the universal covering group of* G *is* n*. Then the action is smoothly isomorphic to an affine action of* Γ *on the* n*-torus.*

5. Local differentiable rigidity of volume preserving actions

A smooth action of a group G is called *locally rigid* if a sufficiently small perturbation of it (in a given topology) is an isomorphic action. In the context of smooth actions of higher real-rank semisimple Lie groups and lattices, 'isomorphic' is usually understood as 'C^∞ conjugate'.

We continue to assume that G denotes the connected component of the identity element of a semisimple real algebraic group having real rank at least 2 and no simple factor of G is compact. This is what we have been calling a 'higher real rank group.' Lattices in such groups are referred to here as 'higher real-rank lattices'.

a. Standard actions. The measure preserving actions mentioned in Zimmer's conjecture (described at the beginning of the chapter) are built out of the following main building blocks:

(1) actions that preserve a Riemannian metric (see Section 4.3, on isometric actions);
(2) actions on nilmanifolds by automorphisms (an automorphism of a nilmanifold N/Λ is a diffeomorphism of N/Λ that lifts to an isomorphism $N \to N$ mapping Λ to itself);
(3) actions on quotients of the form H/Λ, where Λ is a lattice in a connected Lie group H and the action is given by left-translations via a homomorphism from G into H.

Local rigidity has been established for all actions by isometries of finitely generated groups with the Kazhdan property T. ([**Benv**] for cocompact higher rank lattice actions and [**FM**] in full generality.)

It was observed in [**MQ**] that proving local rigidity for actions of types 2 and 3 amounts to proving that certain homomorphisms (regarded as constant cocycles over a group action) are locally rigid, that is, a cocycle that is sufficiently close to a π-simple cocycle is, in some appropriate topology, cohomologous to it. In this subsection we explain this simple but crucial idea for actions of G of the third kind. Proving that a given cocycle over a higher real rank action is (measurably) cohomologous to a π-simple one is precisely the content of the Cocycle Superrigidity.

Consider a manifold of the form H/Λ, where H is a connected Lie group and Λ is a discrete subgroup of H. The acting group, G, will be a connected Lie group and we suppose that $\Phi : G \times H/\Lambda \to H/\Lambda$ is a smooth action. We will say that the action is *ρ-homogeneous* if there exists a homomorphism $\rho : G \to H$ such that Φ_g is left translation on H/Λ by $\rho(g)$.

Suppose that $\Phi : G \times M \to M$ and $\Psi : G \times N \to N$ are two smooth actions of G on diffeomorphic manifolds M and N. Ψ will be called a (smooth) *factor* of Φ if there exists a (smooth) map $F : M \to N$ such that $F \circ \Phi_g = \Psi_g \circ F$ for all $g \in G$. Note that the two actions will be smoothly conjugate if there is a smooth factor map $F : M \to N$ which has a smooth inverse and the inverse map F^{-1} makes Φ a factor of Ψ. We defined earlier the notion of a ρ-simple cocycle as a cocycle of the form $\alpha_\rho(g, x) = \rho(g)$. The main remark is now the following.

PROPOSITION 6.5.1. *Suppose that G and H are connected, simply connected groups and that $\Phi : G \times H/\Lambda \to H/\Lambda$ is a smooth action. Then, a certain H-valued cocycle over Φ naturally associated to the action (whose definition will be given in a moment) is (smoothly) cohomologous to a ρ-simple cocycle if and only if Φ contains a ρ-homogeneous action of G on H/Λ as a (smooth) factor.*

PROOF. Both M and V will be the same set, H/Λ, while σ will be the original action, Φ. The universal cover of M is $\tilde{M} = H$, and $G \times H$ is a connected simply connected manifold. Consequently, Φ lifts to a unique $\tilde{\Phi} : G \times H \to H$ that sends (e, e) to e, where e is the identity element of both H and G. It is immediate to check that $\tilde{\Phi}$ is also a group action and that for each (g, h) in $G \times H$ there is a unique $\tilde{c}(g, h) \in H$ such that $\tilde{\Phi}_g(h) = \tilde{c}(g, h)h$. Note that $\tilde{c}$ is a smooth function if the action is smooth and $\tilde{c}(g, h\lambda) = \tilde{c}(g, h)$ for all $\lambda \in \Lambda$. Therefore, $\tilde{c}$ gives rise to a well-defined smooth map $c : G \times M \to H$ such that $\Phi_g(x) = c(g, x)x$ for all $g \in G$ and $x \in M$. The map c is easily seen to be a cocycle over Φ taking values in H.

We first show that if Φ contains a ρ-homogeneous action on M, with factor map F that induces the identity on Λ, then c is cohomologous to a ρ-simple cocycle. F is a smooth self-map of M such that $F \circ \Phi_g = \rho(g)F$ for all $g \in G$. Let $\tilde{F}$

be any lift of F to $\tilde{M}$ and define, for each $x = [h] \in H/\Lambda$ $\varphi(x) = \tilde{F}(h)h^{-1}$. Clearly φ is well-defined and $F(x) = \varphi(x)x$ for all $x \in M$. Therefore $c(g,x)x = \varphi(\Phi_g(x))^{-1}\rho(g)\varphi(x)x$ for all $x \in M$ and $g \in G$, so $c(g,x)$ and $\varphi(\Phi_g(x))^{-1}\rho(g)\varphi(x)$ are equal modulo Λ for all g and x. Since G is connected and c is continuous we conclude that $c(g,x) = \varphi(\Phi_g(x))^{-1}\rho(g)\varphi(x)$, hence c and ρ are cohomologous as cocycles over Φ.

For the converse, if c is cohomologous to a ρ-simple cocycle, then there is a (smooth) map $\varphi : M \to H$ such that $c(g,x) = \varphi(\Phi_g(x))^{-1}\rho(g)\varphi(x)$ holds for all g and x. By defining $F(x) = \varphi(x)x$, we obtain a smooth map $F : M \to M$ such that $F(\Phi_g(x)) = \rho(g)F(x)$ for all g and x. □

b. Local rigidity of weakly hyperbolic actions. The following notion of hyperbolicity will be needed. A C^1 action Φ of F on a smooth compact manifold M will be called *weakly hyperbolic* if there exists a finite number of elements $g_1, \ldots, g_k \in F$ for which the following holds:

(1) For each $i = 1, \ldots, k$, the diffeomorphisms $\Phi(g_i)$ are *partially hyperbolic*; that is, there exists continuous $\Phi(g_i)$-invariant subbundles E_i^+, E_i^0, E_i^- of the tangent bundle TM, and real numbers $C_i \geq D_i \geq 1$, $a_i > b_i > 1$, such that $TM = E_i^+ \oplus E_i^0 \oplus E_i^-$ and for all $v^+ \in E_i^+$, $v^0 \in E_i^0$, $v^- \in E_i^-$, and positive integers n, the following inequalities hold:

$$\begin{aligned} D_i^{-1} b_i^{-n} \|v^0\| &\leq \|d\Phi(g_i^n)v^0\| \leq D_i b_i^n \|v^0\| \\ C_i^{-1} a_i^n \|v^+\| &\leq \|d\Phi(g_i^n)v^+\| \\ & \quad\quad \|d\Phi(g_i^n)v^-\| \leq C_i a_i^{-n} \|v^-\| \end{aligned}$$

(2) TM is the linear span of the E_i^+, $i = 1, \ldots, k$.

If, for example, F contains an Anosov diffeomorphism, g, then the F-action is weakly hyperbolic. (Take, $g_1 = g$ and $g_2 = g^{-1}$.) In this case, the F-action will be called an *Anosov* action.

Standard actions of the first type (in the enumeration of the previous subsection) are, of course, never weakly hyperbolic. As for standard actions of types 2 and 3, there are simple algebraic criteria to decide whether or not they are weakly hyperbolic. Consider first actions of type 3. Let $\pi : G \to H$ be a continuous homomorphism. Then it can be shown that the action of G on H/Γ via π is weakly hyperbolic if and only if the centralizer $Z_H(\pi_0(G))$ of $\pi_0(G)$ in H is discrete. For actions of type 2, the algebraic condition is as follows: denote by $\pi : \Gamma \to \mathrm{Aut}(N/\Lambda)$ a homomorphism of Γ into the group of automorphisms of the nilpotent Lie group N that preserve the lattice Λ. Denote by $d\pi$ the associated representation of Γ on the Lie algebra $\mathfrak{n}$ of N. Then the Γ action on N/Λ define by π is weakly hyperbolic if and only if there does not exist a Γ-invariant subspace $\mathfrak{n}_0$ such that the restriction of $d\pi(G)$ to $\mathfrak{n}_0$ is precompact in the group of linear isomorphisms of $\mathfrak{n}_0$.

We denote by $\mathrm{Diff}^r(M)$ the group of C^r diffeomorphisms of a smooth manifold M. The group of homeomorphisms of M will be written $\mathrm{Diff}^0(M)$. Two actions $\Phi_0, \Phi : F \to \mathrm{Diff}^r(M)$, of a (possibly discrete) Lie group F, are called C^r conjugate if there exists a C^r diffeomorphism $\phi : M \to M$ – the C^r *conjugacy* – such that $\Phi(g) = \phi^{-1} \circ \Phi_0(g) \circ \phi$ for all $g \in F$. A C^r action Φ_0 of F on a compact manifold M will be called $C^{r,k,l}$-locally rigid ($r \geq k$) if for any C^r action Φ of F on M sufficiently C^k-close to Φ_0, then Φ is C^l-conjugate to Φ_0.

The first theorem refers to actions of the third kind.

THEOREM 6.5.2 (Margulis-Qian [**MQ**]). *Let G be the connected component of the identity element of a semisimple algebraic group such that each simple factor has real-rank at least 2. Let Γ be a lattice in G. Let H be the connected component of the identity of a real algebraic (Lie) group and Λ a cocompact lattice in H. Let $\pi_0 : G \to H$ be a continuous homomorphism. Denote by Φ_0 the action of either G or Γ on H/Λ by left-translations via π_0, and suppose that Φ_0 is weakly hyperbolic. Then Φ_0 is both $C^{1,1,0}$ and $C^{\infty,1,\infty}$-locally rigid.*

For actions of the second kind, Margulis and Qian prove the following.

THEOREM 6.5.3 (Margulis-Qian [**MQ**]). *Let Γ be a lattice in G, for G as in Theorem 6.5.2. Let N/Λ be a compact nilmanifold, where N is a connected, simply-connected nilpotent Lie group and $\Lambda \subset N$ is a lattice. Let $\pi_0 : \Gamma \to Aut(N/\Lambda)$ be a homomorphism into the group of automorphisms of the nilmanifold such that π_0 extends to a continuous homomorphism $\pi_0 : G \to Aut(N)$, and denote by Φ_0 the associated Γ-action on N/Λ. Suppose that Φ_0 is weakly hyperbolic. Then Φ_0 is both $C^{1,1,0}$ and $C^{\infty,1,\infty}$-locally rigid.*

The proofs of Theorems 6.5.2 and 6.5.3 consist of three essential steps.

Step 1. An application of Zimmer's Cocycle Superrigidity Theorem, which yields that the cocycle implicit in the local rigidity theorem (described in the proof of Proposition 6.5.1) is measurably cohomologous to a π-simple one.

Step 2. Proof that the measurable cohomology is in fact continuous For that one needs to use the weak hyperbolicity properties that are present in the standard examples (of types 2 and 3.)

Step 3. Proof that the continuous conjugacy is in fact C^∞ . For that one uses the theory of nonstationary normal forms developed in [**GK**]. The procedure is very similar to that used in the proof of local differentiable rigidity of standard Anosov actions of higher–rank abelian groups which is discussed in the next chapter. See [**KS4**] for details.

6. Global differentiable rigidity with standard models

With the exception of the circle actions of Section 6.4.2, the global rigidity results known to-date for higher rank actions involve strong assumptions in addition to what is expected by Zimmer's conjecture. These assumptions are generally of two types. Either one supposes that the action leaves invariant some rigid geometric structure (we discussed some results of this kind in Chapter 5 and earlier in the present chapter), or one supposes that the action has "enough hyperbolicity." (These two sets of hypothesis are not at all unrelated as we saw, for example, in the discussion prior to Theorem 6.4.10.) We state in this section some global rigidity results of the second kind.

a. Weakly hyperbolic actions. The following global rigidity theorem is also proved in [**MQ**].

THEOREM 6.6.1 (Margulis-Qian). *Let G, Γ, N, and Λ be as in Theorem 6.5.3. Let Φ be a C^1-action of Γ on N/Λ such that*

(1) *Φ is Anosov;*
(2) *Φ has a finite orbit;*
(3) *Φ preserves a probability measure with support N/Λ.*

Then Φ *is* C^0*-conjugate to an affine* Γ*-action on* N/Λ *(that is, a standard action of type 2).*

b. Cartan and trelissed actions. An *Anosov action* is a locally faithful action of a (not necessarily connected) Lie group which contains an element that acts normally hyperbolically to the orbit foliation. If the group is discrete, an Anosov action is simply one that contains Anosov elements.

S. Hurder conjectured in [**Hu**] that *Anosov actions* of higher rank lattices on compact manifolds are essentially algebraic. In the same paper he introduced special types of Anosov actions, which he called *Cartan* and *trelissed* actions. A Cartan action is characterized by the property that suitable intersections of stable manifolds of certain commuting elements of the action are one-dimensional. (See [**Hu**] or [**GS2**, Definition 3.8] for the precise definition.) The action of a group G is said to be trelissed (with respect to an abelian subgroup $A \subset G$) if there is a sufficiently large collection of one-dimensional foliations invariant under the action of A. (Again, the precise definition can be found in [**Hu**] or [**GS2**, Definition 3.3].) We note that Cartan actions are trelissed.

In [**GS1, GS2**], E. Goetze and R. Spatzier prove a number of results that validate Hurder's conjecture for Cartan actions and certain trelissed actions. We only state here one of their results, concerning lattice actions. It requires the following definition. Let H be a connected, simply connected Lie group with $\Lambda \subset H$ a cocompact lattice. Define Aff(H) to be the set of diffeomorphisms of H that map right invariant vector fields on H to right invariant vector fields. Define Aff(H/Λ) to be the group of diffeomorphisms of H/Λ that lift to elements of Aff(H). An action of a group G on H/Λ is called in [**GS2**] an *affine algebraic action* if it comes from a homomorphism from G into Aff(H/Λ).

THEOREM 6.6.2 (Goetze-Spatzier). *Let* G *be a connected semisimple Lie group without compact factors such that each simple factor has real rank at least* 2*, and let* $\Gamma \subset G$ *be a lattice. Let* M *be a compact manifold without boundary and* μ *a smooth volume form on* M*. Let* Φ *be a volume-preserving Cartan action. Then, on a subgroup of finite index,* Φ *is* C^∞ *conjugate to an affine algebraic action.*

More specifically, on a subgroup of finite index, Φ *lifts to an action of a finite cover* $M' \to M$ *which is* C^∞ *conjugate to the standard algebraic action on the nilmanifold* $\tilde{\pi}_1(M')/\pi_1(M)$*, where* $\tilde{\pi}_1(M')$ *denotes the Malcev completion of the fundamental group of* M'*, i.e., the unique, simply connected, nilpotent Lie group containing* $\pi_1(M')$ *as a cocompact lattice.*

We also refer the reader to the article [**KL2**] for examples of of rigidity theorems without the assumption of existence of invariant measures.

7. Actions without invariant measures

a. Measurable rigidity. For most of the present chapter till now we have considered higher real-rank actions that preserve a finite measure. One important example of actions for which this condition is not satisfied are the boundary actions discussed in Chapter 4. One can, after Furstenberg, try to examine the structure of such non-measure preserving actions by considering the stationary measures, defined in Subsection 4.1d , which exist under very general conditions. We describe here a number of general results in this direction that were obtained by A. Nevo and R. Zimmer. (See [**NZ1, NZ2, NZ3**].)

The reader should refer to Subsection 4.1d for the definitions of stationary measures (which requires the notion of an admissible measure), maximal boundary, and the Furstenberg entropy. We recall, in particular that if G has real rank r, there are 2^r G-factors of the maximal Furstenberg boundary of G, which are all of the form G/Q, where Q is a parabolic subgroup of G.

Let G be a connected non-compact semisimple Lie group with finite center. Let μ denote an admissible probability measure on G. Let $(X, \mathcal{B}, \nu)$ denote a standard Borel space, with probability measure ν, and assume that G has a Borel measurable action on $(X, \mathcal{B})$. Also assume that ν is a μ-stationary measure. The pair (X, ν) will be referred to as a (G, μ)-space. The measure ν is G-quasi-invariant, so it makes sense to also require ν to be ergodic.

A G-factor of the form $(G/Q, \nu_0)$ of a (G, μ)-space (X, ν) will be called a *projective factor.*

PROPOSITION 6.7.1. [**NZ1**, Lemma 0.1].

Let G be a connected non-compact semisimple Lie group with finite center, μ an admissible probability measure on G, and (X, ν) an ergodic (G, μ)-space. Then there exists a parabolic subgroup Q of G, satisfying the following:

(1) *There exists a G-equivariant measurable factor map*

$$\varphi : (Z, \nu) \to (G/Q, \nu_0),$$

where ν_0 is the unique μ-stationary measure on G/Q. (The map φ is defined on a G-invariant ν-conull set $X' \subset X$.)

(2) *G/Q is the unique maximal projective factor of (X, ν). That is, any other projective factor of (X, ν) is a factor of G/Q. In particular, Q is uniquely determined up to conjugation.*

(3) *The map φ is uniquely determined up to ν-null sets.*

THEOREM 6.7.2 (Nevo-Zimmer). *Let G be a connected non-compact semisimple Lie group with finite center and no factors of real rank one. Let μ be an admissible probability measure on G, and let (X, ν) be an ergodic (G, μ)-space. Then the maximal projective factor G/Q is trivial if and only if the stationary measure ν is G-invariant. (If G contains factors of rank 1, the conclusion still holds as long as one assumes that the G-action on (X, ν) is* irreducible*, that is, every normal subgroup of G with positive dimension acts ergodically.)*

By Proposition 6.7.1, if $Q \subset G$ is a parabolic subgroup, then the μ-stationary measure ν_0 on G/Q is unique. Let $H(\mu) := h_\mu(G/P, \nu_0)$ denote the Furstenberg entropy of the maximal boundary G/P. An admissible measure μ on G will be called a measure of finite Furstenberg entropy if $H(\mu)$ is finite. Also define: $H_{na}(\mu)$ as the maximal value of $h_\mu(G/Q, \nu_0)$ as Q varies over the non-amenable parabolic subgroups of G; and $H_{\min}(\mu)$, then minimum value of $h_\mu(G/Q, \nu_0)$ as Q varies over the proper parabolic subgroups of G. With these definitions, the previous theorem has the following corollary.

COROLLARY 6.7.3. *Let (G, μ) and (X, ν) be as in Theorem 6.7.2. If $h_\nu(X, \nu) < H_{min}(\mu)$, then ν is G-invariant and $h_\nu(X, \nu) = 0$.*

Using the suspension construction (Section 2.3.c) one also obtains the following corollary for lattice actions.

COROLLARY 6.7.4. *Let G be as in Theorem 6.7.2 and let Γ be a lattice of G. Let Y be a compact metric Γ-space and assume that Y does not have a Γ-invariant probability measure. Then there exists a measurable Γ-equivariant factor map $\varphi : (Y, \eta) \to (G/Q, \eta_0)$, where Q is a proper parabolic subgroup of G and η and η_0 are Γ-quasi-invariant measures.*

It is possible, in general, that $0 < h_\nu(G/Q), \nu_0 < h_\mu(X, \nu)$, for the maximal projective factor $(G/Q, \nu_0)$ of (X, ν). The next theorem gives a sufficient condition which insures that the maximal projective factor has full Furstenberg entropy. Before stating the theorem we note the following. It can be shown that there exists a P-invariant probability measure λ on X such that $\nu = \nu_0 * \lambda$. This factorization is described in [**NZ1**].

THEOREM 6.7.5 (Nevo-Zimmer). *Let G be a connected semisimple Lie group of real rank at least two and finite center. Let μ be an admissible measure on G of finite entropy, and (X, ν) a (G, μ)-space. Let $\nu = \nu_0 * \lambda$, as in the previous paragraph. Let S denote a maximal $\mathbb{R}$-diagonalizable abelian subgroup of G contained in P. If every $s \in S$ is ergodic on (X, λ), then the maximal projective factor has full Furstenberg entropy, namely $h_\nu(G/Q, \nu_0) = h_\mu(X, \nu)$, λ is Q-invariant, and (X, ν) is induced from (i.e. the suspension of) a probability measure preserving action of Q. (In fact, the unique projective factor $\varphi : (X, \nu) \to (G/Q, \nu_0)$ described in Theorem 6.7.2 is an extension with relatively invariant measure.)*

b. The normal subgroup theorem. The theorems of Nevo and Zimmer about projective factors are closely related to the celebrated *Normal Subgroup Theorem* of Margulis [**Mar3**]. Although the Normal Subgroup Theorem might be regarded as a purely algebraic result about lattices in higher rank semisimple Lie groups, its proof is a striking application of the ergodic theoretic ideas related to projective actions. Note that we have used it already in the proof of Theorem 6.4.2.

THEOREM 6.7.6 (The normal subgroup theorem). *Let Γ be an irreducible lattice in a group G, where G is a connected semisimple Lie group of real rank at least 2 with finite center and no compact factors. Let N be a normal subgroup of Γ that is not contained in the center of G. Then Γ/N is a finite group.*

The original proof of the previous theorem relies on a fundamental discovery made by Margulis regarding the measure theoretic properties of lattice actions on the maximal boundary of the ambient Lie group that we present next. (See [**Mar3**], as well as [**Z1**]. For the definition of parabolic groups, see Section 4.1d.)

THEOREM 6.7.7 (Margulis' quotient theorem). *Let Γ be an irreducible lattice in G, where G is a connected semisimple Lie group with no compact factors, trivial center, and real rank at least 2. Let P be a minimal parabolic subgroup of G. Suppose that (X, μ) is a standard measurable Γ-space and that there is a measure class preserving Γ-map $\varphi : G/P \to X$ (possibly defined only almost everywhere.) Then there is a parabolic subgroup Q containing P so that, as Γ-spaces, G/Q and X are isomorphic in such a way that φ corresponds (a.e.) to the natural Γ-map $G/P \to G/Q$.*

The reader is referred to [**Z1**] or [**Mar2**] for the proof. We only indicate how the Normal Subgroup Theorem follows from the Quotient Theorem. Here we assume for simplicity that G is a simple non-compact real algebraic group without center. It follows from this assumption that N is a Zariski-dense subgroup of G. In fact, the Zariski closure of N in G is a normal subgroup of the Zariski closure of Γ, which is G by Theorem 4.1.2 (the Borel Density Theorem). Since G is simple and N is not trivial, the Zariski closure of N is G.

Suppose that Γ/N is not amenable. Then by the characterization (3), section 2.5.a, of amenability for groups, there is a compact Γ/N-space X (hence a Γ-space on which N acts trivially) that does not admit an invariant probability measure. By Proposition 2.5.2 (3,4), the action of G and of Γ on G/P are amenable. It follows from Furstenberg's Lemma (Theorem 4.1.3) that there exists a measurable Γ-map φ from G/P into the space of probability measures on X. We can now apply the Quotient Theorem to φ and conclude that $\varphi(G/P)$ is isomorphic to G/Q for some parabolic subgroup Q of G containing P. Note that the restriction to N of the Γ-action on G/Q is trivial, and since N is Zariski closed in G, then $G = Q$, implying that the image of φ is a point. But this contradicts that fact (derived from the assumption that Γ/N is not amenable) that X does not have Γ-invariant probability measures. This contradiction shows that Γ/N is an amenable group. Then by Theorem 3.2.3 (applied to the natural homomorphism $\Gamma \to \Gamma/N$, viewed as a cocycle over the trivial action of Γ on a point) it would follow that Γ/N is compact, hence finite, hence the Normal Subgroup theorem is proved.

The Normal Subgroup Theorem can also be deduced from Corollary 6.7.4.

c. Local differentiable rigidity of boundary actions for cocompact lattices. The following theorem is from [**KS4**].

THEOREM 6.7.8 (Katok-Spatzier). *Let G be a connected semisimple Lie group with finite center and without compact factors. Suppose that the real rank of G is at least 2. Then the action of Γ on G/P by left-translations, where P is a minimal parabolic subgroup of G, is locally C^∞-rigid.*

The main idea in the proof is to reduce the problem to the transversal rigidity of the orbit foliation of a suitable normally hyperbolic action on $\Gamma\backslash G$. This idea was first used by E. Ghys to obtain a smooth classification of boundary actions of Fuchsian groups [**Gh2**].

By completely different methods, M. Kanai established in [**Kan2**] a local rigidity theorem for the projective action of a cocompact Lattice in $SL(n+1,\mathbb{R})$ on the n sphere S^n, for all $n \geq 21$ and for small perturbations in the C^4-topology.

Bibliography

SURVEYS IN THIS AND COMPANION VOLUME

[S-BK] Victor Bangert, Anatole Katok: *Variational methods II: Twist maps, Lagrangian systems, and closed geodesics*

[S-BKP] Luis Barreira, Anatole Katok, Yakov Pesin: *Introduction to smooth theory and non-uniformly hyperbolic dynamics*

[S-B] Vitaly Bergelson: *Ergodic theorems and combinatorial ergodic theory*

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