The Intermediate Value Theorem (I.V.T.)

Suppose that $f(x)$ is continuous on the closed interval $[a, b]$
and that N is a number between $f(a)$ and $f(b)$

Then there must be a number c in the interval $[a, b]$ for which $f(c) = N$.

$\uparrow$

this is the same as saying

$\downarrow$

the equation $f(x) = N$ must have a solution $x = c$ in the interval $[a, b]$

Example: A Tibetan monk starts up a mountain to a monastery at 7am and arrives at the top at 7pm. The elevation of the monastery is 5000 ft.

The next day he starts back down at 7 am, following the same path, and arrives at the bottom at 7pm

Must there be a time between 7 am and 7pm (for example, perhaps 3:57 p.m.) when he is at exactly the same elevation when going done as he was when going up?

(for example, perhaps he was at elevation 2753 ft at 11 am both when going up and also when coming down)

Solution: For time t (hrs), $0 \leq t \leq 12$, let $u(t)$ be the monk's elevation after t hrs going up:

so $u(0) = 0$ and $u(12) = 5000$

Let $d(t)$ be his elevation at time after t hrs coming down: so $d(0) = 5000$ and $d(12) = 0$

$u(t)$ and $d(t)$ are continuous functions
defined on $[0, 12]$

Now let $f(t) = u(t) - d(t)$ also continuous, defined on $[0, 12]$

$$f(0) = u(0) - d(0) = 0 - 5000 = -5000$$

$$f(12) = u(12) - d(12) = 5000 - 0 = 5000$$

$N = 0$ is a number between $f(0)$ and $f(12)$

so

The Intermediate Value Theorem says that there must be a time c in $[0, 12]$ where

$$f(c) = 0$$

That means $u(c) - d(c) = 0$

so at time c $u(c) = d(c)$

$\lim_{x \to \infty} f(x) = L$ means: we can make $f(x)$ as close to the number L as we like by letting x get larger and larger positive

$\lim_{x \to -\infty} f(x) = L$ means: we can make $f(x)$ as close to the number L as we like by letting x get larger and larger negative

---

$\lim_{x \to \infty} f(x) = \infty$ means: we can make $f(x)$ get **arbitrarily large positive** (as large as we like)

by letting x get **larger and larger positive**

$\lim_{x \to \infty} f(x) = -\infty$ means: we can make $f(x)$ get **arbitrarily large negative** (as large negative as we like)

by letting x get **larger and larger positive**

$\lim_{x \to -\infty} f(x) = \infty$ means: we can make $f(x)$ get **arbitrarily large positive** (as large as we like)

by letting x get **larger and larger negative**

$\lim_{x \to -\infty} f(x) = -\infty$ means: we can make $f(x)$ get **arbitrarily large negative** (as large negative as we like)

by letting x get **larger and larger negative**

A horizontal line $y = c$ is called a horizontal asymptote for $y = f(x)$ if

either $\lim_{x \to \infty} f(x) = c$ or $\lim_{x \to -\infty} f(x) = c$

To try to find horizontal asymptotes, we need to check both limits. A function $f(x)$ might turn out to have 0, 1, or 2

Q1: What is $\lim_{x\to\infty} \frac{6x^5 - x^3 - 1}{2+4x^2-7x^5}$

A) $\frac{1}{2}$ B) $-\frac{1}{2}$ C) $\frac{6}{7}$ D) $-\frac{6}{7}$ E) $-\frac{1}{7}$

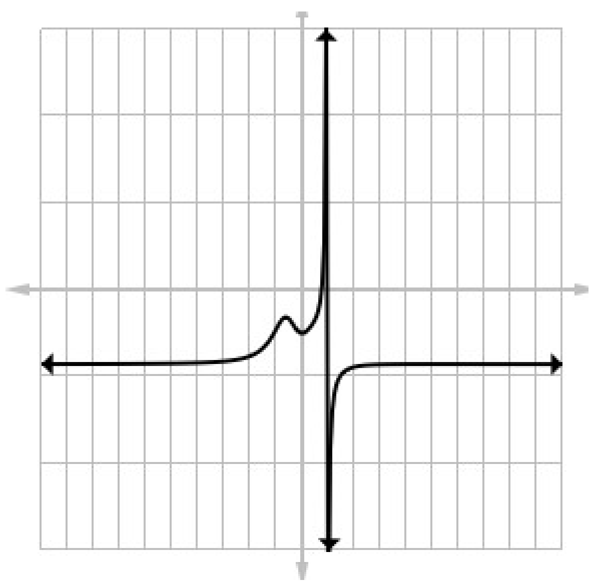
Ans: $\lim_{x\to\infty} \frac{6x^5 - x^3 - 1}{2+4x^2-7x^5} = \lim_{x\to\infty} \frac{6x^5/x^5 - x^3/x^5 - 1/x^5}{2/x^5 + 4x^2/x^5 - 7x^5/x^5}$

$= \lim_{x\to\infty} \frac{6 - 1/x^2 - 1/x^5}{2/x^5 + 4/x^3 - 7} = \frac{6-0-0}{0+0-7} = -\frac{6}{7}$

If we look at $\lim_{x\to-\infty} \frac{6x^5 - x^3 - 1}{2+4x^2-7x^5}$, we can use the same steps to get the same answer:

$\lim_{x\to-\infty} \frac{6 - 1/x^2 - 1/x^5}{2/x^5 + 4/x^3 - 7} = -\frac{6}{7}$

The only horizontal asymptote is $y = -\frac{6}{7}$. The graph is shown below (*don't worry right now about the other feathures in the graph: the point is just to show you the horizontal asymptote*).



Example: 1) $\lim_{x\to\infty} \frac{\sqrt{2x^2+3}}{x+4} = \lim_{x\to\infty} \frac{\frac{\sqrt{2x^2+3}}{x}}{\frac{x+4}{x}} =$

$$\lim_{x\to\infty} \frac{\sqrt{\frac{2x^2+3}{x^2}}}{\frac{x+4}{x}} = \lim_{x\to\infty} \frac{\sqrt{2+\frac{3}{x^2}}}{1+\frac{4}{x}} = \frac{\sqrt{2+0}}{1+0} = \sqrt{2}$$

2) $\lim_{x\to-\infty} \frac{\sqrt{2x^2+3}}{x+4} = \lim_{x\to\infty} \frac{\frac{\sqrt{2x^2+3}}{x}}{\frac{x+4}{x}} =$

↓ **caution**

$$= \lim_{x\to\infty} \frac{-\sqrt{\frac{2x^2+3}{x^2}}}{\frac{x+4}{x}} = \lim_{x\to\infty} \frac{-\sqrt{2+\frac{3}{x^2}}}{1+\frac{4}{x}} = \frac{-\sqrt{2+0}}{1+0} = -\sqrt{2}$$

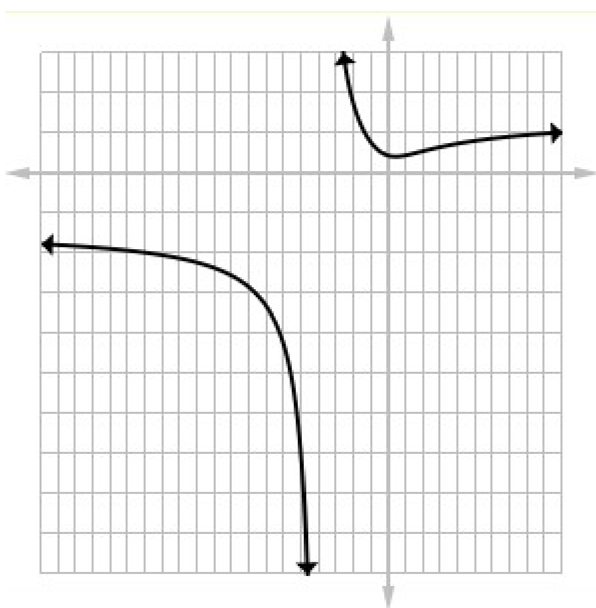
The horizontal asymptotes for $y = f(x) = \frac{\sqrt{2x^2+3}}{x+4}$

are

$y = \sqrt{2}$ and

$y = -\sqrt{2}$ as shown in the figure below

(Note: there is also a vertical aymptote at $x = -4$, but you should know about that from earlier material)



Example: 1) $\lim_{x\to\infty} \sqrt{x^2 + 1} - x = \lim_{x\to\infty} \frac{\sqrt{x^2+1}-x}{1} \cdot \frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x}$

$\lim_{x\to\infty} \frac{x^2+1-x^2}{\sqrt{x^2+1}+x} = \lim_{x\to\infty} \frac{1}{\sqrt{x^2+1}+x}$. The denominator $\to \infty$

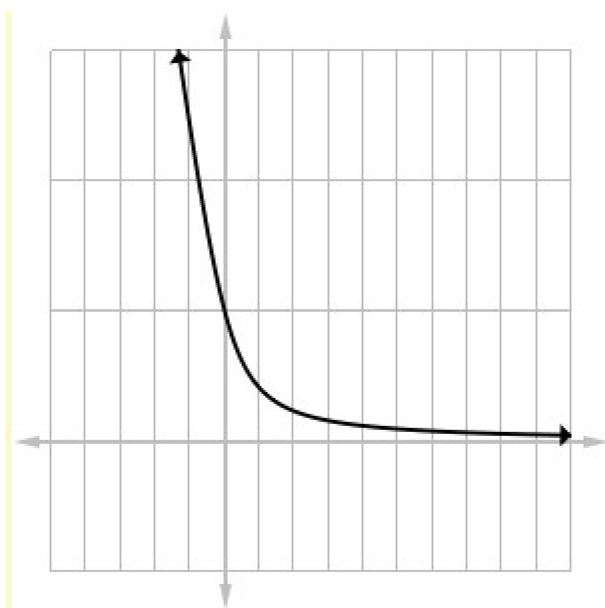
as $x \to \infty$, so since the numerator is constant 1, $\lim_{x\to\infty} \frac{1}{\sqrt{x^2+1}+x} = 0$

2) $\lim_{x\to-\infty} \sqrt{x^2 + 1} - x$. Here, the answer is clear: no real work is needed. As $x \to -\infty$, the quantity $\sqrt{x^2 + 1}$ gets larger and larger (arbitrarily large!) and the "$-x$" term (since x itself is negative) simply makes the function larger still.

In other words, $\lim_{x\to-\infty} \sqrt{x^2 + 1} - x = \infty$

Thus, $f(x) = \sqrt{x^2 + 1} - x$ has only one horizontal asymptote, $y = 0$.

See the graph below:



Example $\lim_{x\to\infty} \frac{1}{x}\sin x$

Intuitively, $\sin x$ oscillates up and down between $\pm 1\, a$ as $x \to \infty$, but the $\frac{1}{x}$ multiplier shrinks to 0 and "damps down" the oscillations in the sin function.

To be more precise, we use a Squeeze Theorem:

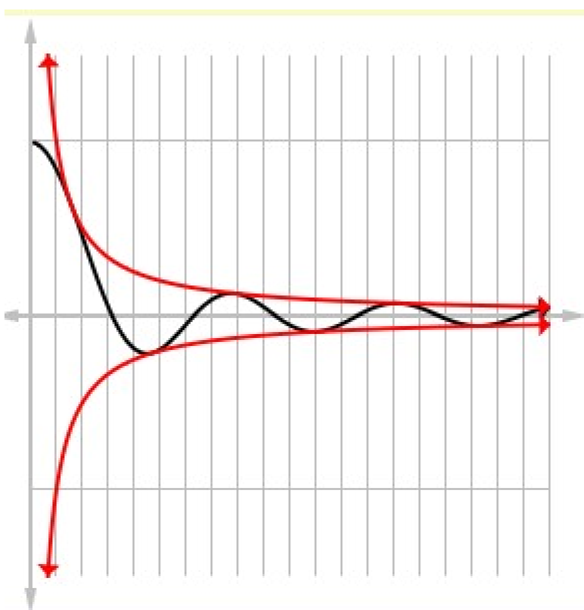
$$-1 \quad \leq \sin x \qquad \leq 1$$

so for $x > 0$ $$-\frac{1}{x} \leq \frac{1}{x}\sin x \leq \frac{1}{x}$$

Since $-\frac{1}{x}$ and $\frac{1}{x}$ both $\to 0$ as $x \to \infty$, the function $\frac{1}{x}\sin x$, being squeezed between them, must also $\to 0$. In other words, $\lim_{x\to\infty} \frac{1}{x}\sin x = \lim_{x\to\infty} \frac{\sin x}{x} = 0$

See the picture below:

so $\lim_{x\to\infty} \frac{1}{x}\sin x = \lim_{x\to\infty} \frac{\sin x}{x} = 0$



Notice $\lim_{x\to 0^+} \frac{\sin x}{x} = 1$, as we already knew should happen.

You check what happens with $\lim_{x\to -\infty} \frac{1}{x}\sin x$. The picture is similar.

Q2 Consider $\lim_{x \to \infty} \frac{x^k - 3x^2 + 5}{x^n + x - 3}$ where k and n are positive integers

A) If $k = n$, this limit must exist

B) If $k < n$, this limit must exist

C) If $k > n$, this limit must exist

D) None of A), B)., C) are true

E) All of A), B), C) are true

Answer:

A) is false: If $k = n = 1$, for example, $\lim_{x \to \infty} \frac{x - 3x^2 + 5}{x + x - 3}$ does not exist (it $\to \infty$)

B) is true: n and k are supposed to be positive integers.

$k < n$ is an impossible condition if $n = 1$

if $n = 2$ or more, and $k < n$, then the numerator has a lower degree than the denominator, so (using the technique of the first clicker question Q1), the limit of the fraction will be 0.