Sums of Powers of Natural Numbers

We'll use the symbol $S_k$ for the sum of the $k^{th}$ powers of the first $n$ natural numbers. In other words,

$$S_k = 1^k + 2^k + \ldots + n^k.$$ 

Of course, this is a “formula” for $S_k$, but it doesn't help you compute — it doesn't tell you how to find the exact value, say, of $S_3 = 1^3 + 2^3 + \ldots + 15^3$. We'd like to get what's called a closed formula for $S_k$, that is, one without the annoying “…” in it.

For $S_0 = 1^0 + 2^0 + \ldots + n^0$, this is easy: since there are $n$ terms, each equal to 1, so we get

$$S_0 = 1 + 1 + \ldots + 1 = 1 \cdot n = n$$

For $S_1$, it's already harder. Here's a slick way of finding a closed formula for $S_1$:

Write down $S_1$ twice, in two different orders:

$S_1 = 1 + 2 + 3 + \ldots + (n-1) + n$, and

$S_1 = n + (n-1) + (n-2) + \ldots + 2 + 1$ Then add to get:

$$2S_1 = (n + 1) + (n + 1) + (n + 1) + \ldots + (n + 1) + (n + 1).$$

Since there are $n$ terms on the right, each equal to $(n + 1)$, we get

$$2S_1 = n(n + 1),$$

so

$$S_1 = \frac{n(n + 1)}{2}$$

This is a “usable” closed formula: for example, $1 + 2 + 3 + \ldots + 15 = \frac{15(16)}{2} = 120$.

Here's a list of formulas for:

$$S_0 = 1^0 + 2^0 + \ldots + n^0 = n$$

$$S_1 = 1^1 + 2^1 + \ldots + n^1 = \frac{n(n+1)}{2}$$

$$S_2 = 1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$S_3 = \left(\frac{n(n+1)}{2}\right)^2$$ (Curious observation: $S_3 = [S_1]^2$)
Where do these formulas come from? There is a systematic way to get a formula for each \( S_k \) once you know the previous formulas for \( S_0, S_1, \ldots, S_{k-1} \). We'll illustrate here with just two examples:

i) If someone noticed the (easy) fact the \( S_0 = 1^0 + 2^0 + \ldots + n^0 = 1 + 1 + \ldots + 1 = n \) how could this fact be used to get a formula: \( S_1 = 1^1 + 2^1 + \ldots + n^1 = 1 + 2 + \ldots + n = \ ??? \)

Well, for any positive integer \( j \), we know that \( (j + 1)^2 - j^2 = 2j + 1 \). So we write this down substituting in each value \( j = 1, j = 2, \ldots, j = n \)

\[
\begin{align*}
2^2 - 1^2 &= 2(1) + 1 \\
3^2 - 2^2 &= 2(2) + 1 \\
4^2 - 3^2 &= 2(3) + 1 \\
&\vdots \\
(n + 1)^2 - n^2 &= 2(n) + 1
\end{align*}
\]

Adding up the columns on both sides (with lots of cancellations on the left-hand side) gives

\[
\frac{(n + 1)^2 - 1}{2} = 2(1 + 2 + \ldots + n) + n = 2S_1 + n \text{ so } \frac{n^2 + n}{2} = \frac{n(n+1)}{2} = S_1.
\]

ii) So now we know formulas for \( S_0 \) and \( S_1 \). How can use these to get a formula for \( S_2 = 1^2 + 2^2 + \ldots + n^2 = \ ??? \) It's the same idea, but just a little more algebra.

For any \( j \) we know that \( (j + 1)^3 - j^3 = 3j^2 + 3j + 1 \). So we write this down substituting each value \( j = 1, j = 2, \ldots, j = n \)

\[
\begin{align*}
2^3 - 1^3 &= 3(1^2) + 3(1) + 1 \\
3^3 - 2^3 &= 3(2^2) + 3(2) + 1 \\
4^3 - 3^3 &= 3(3^2) + 3(3) + 1 \\
&\vdots \\
(n + 1)^3 - n^3 &= 3(n^2) + 3(n) + 1.
\end{align*}
\]

Adding up the columns on both sides (with lots of cancellations on the left-hand side) gives

\[
\frac{(n + 1)^3 - 1}{3} = 3(1^2 + 2^2 + \ldots + n^2) + 3(1 + 2 + \ldots + n) + (1 + 1 + \ldots + 1) = 3S_2 + nS_1 + S_0,
\]

so

\[
n^3 + 3n^2 + 3n + 1 - 1 = 3S_2 + 3S_1 + S_0. \quad \text{Then we solve to get } S_2.
\]

\[
n^3 + 3n^2 + 3n - 3S_1 - S_0 = 3S_2, \quad \text{so } S_2 = \frac{n^3 + 3n^2 + 3n - 3S_1 - S_0}{3}
\]

We can substitute the formulas we already know for \( S_0 \) and \( S_1 \) and simplify to get

\[
S_2 = \frac{n^3 + 3n^2 + 3n - 3\frac{n(n+1)}{2} - n}{3} = \frac{n(n + 1)(2n + 1)}{6}
\]
Optional Material

If you know the binomial formula (from high school) and can therefore expand $(j + 1)^k$, then the same idea works for any natural number $k$. But the bigger $k$ is, the more algebra is involved. An outline goes like this.

The formula for the “binomial coefficients” : \[ \binom{k}{l} = \frac{k!}{l!(k-l)!} \]

Suppose we have figured out formulas for $S_0, S_1, S_2, \ldots, S_{k-1}$. We know (from the binomial theorem) that for any $j$,

\[(j + 1)^{k+1} - j^{k+1} = \binom{k+1}{1} j^k + \binom{k+1}{2} j^{k-1} + \binom{k+1}{3} j^{k-2} + \ldots + 1\]

Write this out for each value $j = 1, 2, \ldots, n$.

\[
\begin{align*}
2^{k+1} &- 1^{k+1} = \binom{k+1}{1} 1^k + \binom{k+1}{2} 1^{k-1} + \binom{k+1}{3} 1^{k-2} + \ldots + 1 \\
3^{k+1} &- 2^{k+1} = \binom{k+1}{1} 2^k + \binom{k+1}{2} 2^{k-1} + \binom{k+1}{3} 2^{k-2} + \ldots + 1 \\
&\vdots \\
(n + 1)^{k+1} - n^{k+1} &= \binom{k+1}{1} n^k + \binom{k+1}{2} n^{k-1} + \binom{k+1}{3} n^{k-2} + \ldots + 1.
\end{align*}
\]

Add the columns:

\[
(n + 1)^{k+1} - 1 = \binom{k+1}{1} (1^k + 2^k + \ldots + n^k) + \binom{k+1}{2} (1^{k-1} + 2^{k-1} + \ldots + n^{k-1})
+ \binom{k+1}{3} (1^{k-2} + 2^{k-2} + \ldots + n^{k-1}) + \ldots + n
\]

\[
= \binom{k+1}{1} S_k + \binom{k+1}{2} S_{k-1} + \binom{k+1}{3} S_{k-2} + \ldots + S_0.
\]

Then we solve for what we want:

\[
S_k = \left[ (n + 1)^{k+1} - 1 - \binom{k+1}{2} S_{k-1} - \binom{k+1}{3} S_{k-2} - \ldots - S_0 \right] / \binom{k+1}{1}
\]

\[
= \left[ (n + 1)^{k+1} - 1 - \binom{k+1}{2} S_{k-1} - \binom{k+1}{3} S_{k-2} - \ldots - S_0 \right] / (k + 1)
\]

We are assuming that we already have formulas for $S_{k-1}, S_{k-2}, \ldots, S_1, S_0$ – which we then substitute into this formula to get one closed, if complicated, formula for $S_k$ in terms of $n$. Try it to find a formula for $S_4$:

\[
S_4 = 1^4 + 2^4 + \ldots + n^4 = \ldots
\]