Example We proved in class (and its in the textbook) that for a square upper or lower triangular matrix, the eigenvalues are the numbers that are on the diagonal of the matrix.

If $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix}$, then the eigenvalues of A are $\lambda = 1, 3, 6$.

We can immediately argue: there are 3 different eigenvalues. If we pick eigenvectors for each one, say $\boldsymbol{b_1}$, $\boldsymbol{b_2}$, and $\boldsymbol{b_3}$ then we know from a theorem (in the text, proved in class) that these eigenvectors must be lilnearly independent vectors (in $\mathbb{R}^3$). Therefore these eigenvectors will form a basis for $\mathbb{R}^3$. Therefore A is diagonalizable.

To actually diagonalize A, we need these eigenvectors.

This requires solving each homogenerous system:

$$(A - 1I)\boldsymbol{x} = \boldsymbol{0}\,\boldsymbol{x} = \boldsymbol{t}\begin{bmatrix} 5 \\ -5 \\ 1 \end{bmatrix}$$

$$(A - 3I)\boldsymbol{x} = \boldsymbol{0}\,\boldsymbol{x} = \boldsymbol{t}\begin{bmatrix} 0 \\ -3 \\ 5 \end{bmatrix}$$

$$(A - 6I)\boldsymbol{x} = \boldsymbol{0}\,\boldsymbol{x} = \boldsymbol{t}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

For each equation, the solutions form the eigenspace corresponding to the eigenvalue.
Each eigenspace here is one dinemsional (a line through $\mathbf{0}$ in $\mathbb{R}^3$)

We can choose (from these eigenspaces) eigenvectors $\begin{bmatrix} 5 \\ -5 \\ 1 \end{bmatrix}. \begin{bmatrix} 0 \\ -3 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ which are then an eigenvector basis for $\mathbb{R}^3$.

We can then diagonalize A :

$$A = PDP^{-1} = \begin{bmatrix} 5 & 0 & 0 \\ -5 & -3 & 0 \\ 1 & 5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ -5 & -3 & 0 \\ 1 & 5 & 1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 5 & 0 & 0 \\ -5 & -3 & 0 \\ 1 & 5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} \frac{1}{5} & 0 & 0 \\ -\frac{1}{3} & -\frac{1}{3} & 0 \\ \frac{22}{15} & \frac{5}{3} & 1 \end{bmatrix}$$

Question: Since the eigenvalues of a square upper triangular matrix are the entries on its diagonal, can we row reduce a square matrix A to an echelon form (which is upper triangular) and then just read off the eigenvalues?

Answer: NO, EROs can change the eigenvalues of a matrix as the following example shows

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} = B \qquad \text{(only one ERO used)}$$

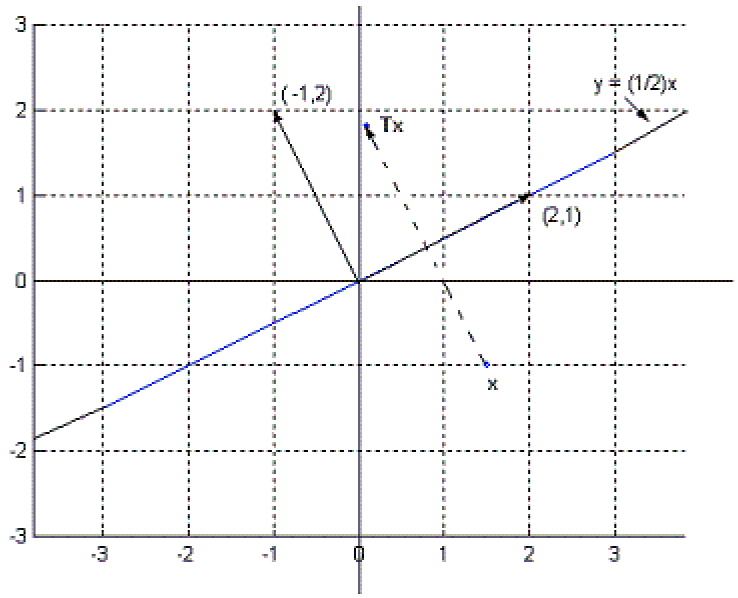
A is upper triangular, and its eigenvalues are $\lambda = 1, 2$

For B : $\det(B - \lambda I) = \det \begin{bmatrix} 1-\lambda & 1 \\ 1 & 3-\lambda \end{bmatrix} = \lambda^2 - 4\lambda + 2 = 0$

gives that the eigenvalues of B are $\frac{4 \pm \sqrt{16-8}}{2} = 2 \pm \sqrt{2}$

Example

The linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ "relect across the line $y = \frac{1}{2}x$"
What is the matrix A for which $T(\boldsymbol{x}) = A\boldsymbol{x}$?



One way to find A is to use a little geometry to figure out $T(\boldsymbol{e_1}) = \begin{bmatrix} ? \\ ? \end{bmatrix}$ and $T(\boldsymbol{e_2}) = \begin{bmatrix} ? \\ ? \end{bmatrix}$.
Then we will know the columns of A and can write down immediately: $A = [\, T(\boldsymbol{e_1}) \;\; T(\boldsymbol{e_2})]$

A different insight, using eigenvectors and eigenvalues, gives us some insight and also leads A by a more roundabout computation.

What are the eigenspaces and eigenvalues (we need to argue <u>geometrically</u> since we don't know the entries in the matrix A) – which vectors go into multiples of themselves? what multiples?

i) for each vector $\boldsymbol{x}$ that is <u>on</u> the line $y = \frac{1}{2}x$, reflecting across that line does nothing. Each such $\boldsymbol{x}$ is an eigenvector with eigenvalue $\lambda = 1$ because $A\boldsymbol{x} = 1 \cdot \boldsymbol{x}$. The line $y = \frac{1}{2}x$ is the eigenspace for $\lambda = 1$. The eigenspace can be described, for example, as $\{t \begin{bmatrix} 2 \\ 1 \end{bmatrix} : t \text{ real}\}$.

ii) each vector $\boldsymbol{x}$ on the perpendicular line $y = -2x$ is also an eigenvector, but with eigenvalue $\lambda = -1$. For example, $A \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} = (-1) \begin{bmatrix} -1 \\ 2 \end{bmatrix}$. The line $y = -2x$ is the eigenspace for $\lambda = -1$. The eigenspace can be described, for example, as $\{t \begin{bmatrix} -1 \\ 2 \end{bmatrix} : t$ real$\}$.

Pick one eigenvector from each of these eigenspaces: say $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$. These eigenvectors are linearly independent (*as we proved earlier they must be, since they have different eigenvalues*) and form a basis for $\mathbb{R}^2$. By our diagonalization theorem, <u>A is diagonalizable</u> and

$$A = PDP^{-1} = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}^{-1}$$

Since $\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}^{-1} = \frac{1}{5}\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$, we can multiply out and find

$$A = PDP^{-1} = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \frac{1}{5}\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{bmatrix} = \begin{bmatrix} .6 & .8 \\ .8 & -.6 \end{bmatrix}$$

(So, in retrospect, we know $T(\boldsymbol{e_1}) = \begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix}$ and $T(\boldsymbol{e_2}) = \begin{bmatrix} \frac{4}{5} \\ -\frac{3}{5} \end{bmatrix}$ – as we could have figured out in the first place, with some geometry, as a way to get the matrix A)

Note: $\begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix}$ *= the first column of A satisfies* $(\frac{3}{5})^2 + (\frac{\backslash 4}{5})^2 = 1$, *so* $\begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ *for some angle* θ. *The matrix A resembles a rotation matrix; but for a rotation the second column would then need to be* $\begin{bmatrix} -\frac{4}{5} \\ \frac{3}{5} \end{bmatrix} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$; *A is <u>not</u> a rotation matrix (and geometrically the reflection* $\boldsymbol{x} \mapsto A\boldsymbol{x}$ *is obviously not a rotation).*

A 2×2 *rotation matrix* $R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *satisfies*

$a^2 + c^2 = 1 = b^2 + d^2 = 1$ *and det* $R = 1$.

This reflection matrix A satisfies the first comdition, but det $A = -1$

Example (continued)

We have $A = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{bmatrix} = PDP^{-1} = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}^{-1}$

Suppose we pick $\boldsymbol{x_0} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, then let

$$
\begin{aligned}
&\boldsymbol{x_1} = A\boldsymbol{x_0}, \\
&\boldsymbol{x_2} = A\boldsymbol{x_1} = A^2\boldsymbol{x_0}, \\
&\vdots \\
&x_{k+1} = A\boldsymbol{x_k} = A^{k+1}\boldsymbol{x_0} \\
&\vdots
\end{aligned}
$$

We would like to know how the $\boldsymbol{x}_k$'s behave as $k \to \infty$

Computing a nice formula for A^{k+1} could be hard, but the diagonalization of A, makes our work easy.

Notice that if $A = PDP^{-1}$, then
$$
\begin{aligned}
A^2 &= (PDP^{-1})(PDP^{-1}) = PD^2P^{-1} \\
A^3 &= A \cdot A^2 = PDP^{-1}(PD^2P^{-1}) = PD^3P^{-1} \\
&\vdots \\
A^{k+1} &= A \cdot A^k = PDP^{-1}(PD^kP^{-1}) = PD^{k+1}P^{-1}
\end{aligned}
$$

So $\boldsymbol{x}_{k+1}$
$$
\begin{aligned}
&= A^{k+1}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = PD^{k+1}P^{-1}\begin{bmatrix} 1 \\ 3 \end{bmatrix} \\
&= \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^{k+1}\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 3 \end{bmatrix} \\
&= \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^{k+1}\frac{1}{5}\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} \\
&= \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1^{k+1} & 0 \\ 0 & (-1)^{k+1} \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \\
&= \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & (-1)^{k+1} \end{bmatrix} = \begin{cases} \begin{bmatrix} 1 \\ 3 \end{bmatrix} & \text{if } k \text{ is odd} \\ \begin{bmatrix} 3 \\ -1 \end{bmatrix} & \text{if } k \text{ is even} \end{cases}
\end{aligned}
$$

Thus, $\boldsymbol{x_1} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$, $\boldsymbol{x_2} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $\boldsymbol{x_3} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$, ..., $\boldsymbol{x_{99}} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$, $\boldsymbol{x_{100}} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, ...

Multiplying by A over and over just reflects the initial vector $\boldsymbol{x_0}$ back and forth across the line $y = \frac{1}{2}x$.

In this example, it should have been clear (geometrically) that this is exactly what a reflection across $y = \frac{1}{2}x$ would do. The point of the calculation is just to illustrate how, perhaps in a more interesting situation (see the material about "Spotted Owls" later in Chapter 5, the diagonalization of a matrix A could be useful in looking for thelong term behavior of the $\boldsymbol{x_k}$'s.