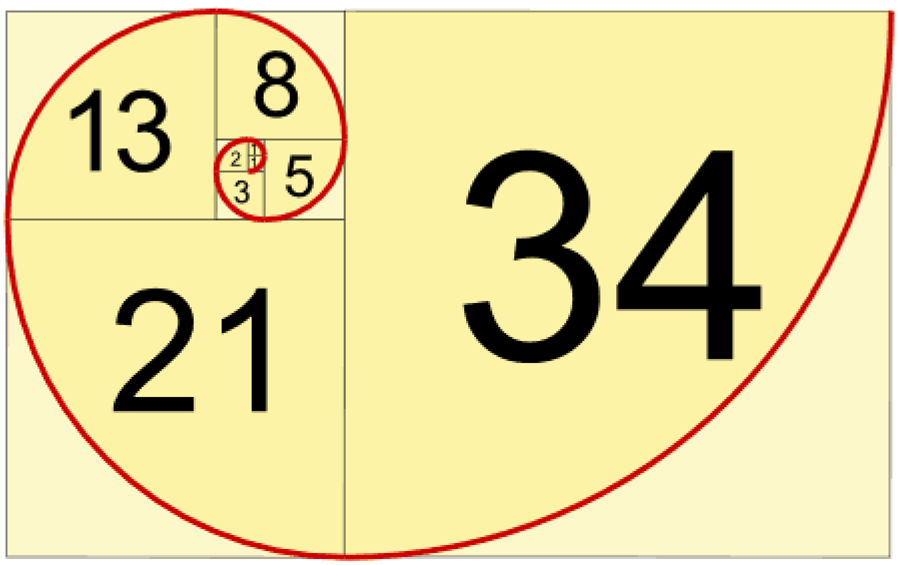
Diagonalization and Fibonacci Numbers

The famous Fibonacci sequence begins: $a_0,\ a_1,\ a_2,\ a_3,\ a_4,\ a_5,\ a_6,\ a_7,\ a_8,\ \ldots$
$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ \ldots$

After the first two 1's, each term of the sequence is the sum of the two preceding numbers. This is a recursive definition of the Fibonacci numbers: each number (after the first two) is defined in terms of numbers that have been previously defined. Precisely,

$$\begin{cases} a_0 = a_1 = 1 \\ a_{k+2} = a_{k+1} + a_k \end{cases}$$

The Fibonacci numbers can be pictured in a spiral of squares that fit neatly together:



Fibonacci numbers have many interesting properties, and they frequently occur in patterns found in the natural world. A Google search for a phrase like "Fibonacci numbers in nature" will produce a lot of hits. For example, there's a wealth of information at

http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/fib.html

We will look at a few interesting mathematical properties of Fibonacci numbers that are related to this course. In particular, using eigenvalues, eigenvectors, and a bit of algebra, we can find an explicit formula for each a_k – that is, a formula that doesn't depend on the previously defined Fibonacci numbers. We can also determine the value of $\lim_{k\to\infty} \frac{a_{k+1}}{a_k}$.

We begin by defining vectors $\boldsymbol{x}$ in $\mathbb{R}^2$ that have two successive Fibonacci numbers as entries:

$$\boldsymbol{x_0} = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ \boldsymbol{x_1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix},\ \boldsymbol{x_2} = \begin{bmatrix} 2 \\ 3 \end{bmatrix},\ \boldsymbol{x_3} = \begin{bmatrix} 3 \\ 5 \end{bmatrix},.\ \boldsymbol{x_4} = \begin{bmatrix} 5 \\ 8 \end{bmatrix},\ \ldots$$

In general, $\boldsymbol{x_k} = \begin{bmatrix} a_k \\ a_{k+1} \end{bmatrix}$. Notice that multiplication by the matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ gets us

from one $\boldsymbol{x_k}$ to the next:

$$A\boldsymbol{x_k} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a_k \\ a_{k+1} \end{bmatrix} = \begin{bmatrix} a_{k+1} \\ a_k + a_{k+1} \end{bmatrix} = \begin{bmatrix} a_{k+1} \\ a_{k+2} \end{bmatrix} = \boldsymbol{x_{k+1}}$$

We can find a basis for $\mathbb{R}^2$ that consists of eigenvectors of A (*that is, A is diagonalizable*).

First, <u>find the eigenvalues</u> of A :

$$\det(A - \lambda I) = \det \begin{bmatrix} -\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix} = \lambda^2 - \lambda - 1 = 0, \text{ so } \lambda = \tfrac{1 \pm \sqrt{1+4}}{2}.$$

The eigenvalues are $\lambda_1 = \frac{1-\sqrt{5}}{2}$ and $\lambda_2 = \frac{1+\sqrt{5}}{2}$

Next, <u>find an eigenvector for each eigenvalue</u>. To illustrate, solve $(A - \lambda_1 I)\boldsymbol{x} = \mathbf{0}$ to find the eigenvectors for $\lambda_1 = \frac{1-\sqrt{5}}{2}$:

$$\begin{bmatrix} 0-\lambda_1 & 1 & 0 \\ 1 & 1-\lambda_1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1-\lambda_1 & 0 \\ -\lambda_1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1-\lambda_1 & 0 \\ 0 & -\lambda_1^2 + \lambda_1 + 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1-\lambda_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\downarrow$

because $-\lambda_1^2 + \lambda_1 + 1 = 0$ *(just look at the characteristic equation)*

So the eigenspace for λ_1 is $\left\{ t \begin{bmatrix} \lambda_1 - 1 \\ 1 \end{bmatrix} : t \text{ real} \right\}$ (*using t for the free variable*). Setting $t = 1$ gives an eigenvector $\begin{bmatrix} \lambda_1 - 1 \\ 1 \end{bmatrix}$. However, we make find a "better looking" eigenvector if instead we choose $t = \lambda_1$ to get

$$\lambda_1 \begin{bmatrix} \lambda_1 - 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \lambda_1^2 - \lambda_1 \\ \lambda_1 \end{bmatrix} = \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix}$$

$\downarrow$

(*again, just look at the characteristic equation)*

For the eigenvalue $\lambda_2 = \frac{1+\sqrt{5}}{2}$, perfectly similar calculations give an eigenvector $\begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}$.

Note: we could have substituted the numeric values for λ_1 and λ_2 at the beginning and used them throughout the calculations, but the work would actually look messier! Specific numeric values are not always your best friends.

Now we have an eigenvector basis for $\mathbb{R}^2$: $\mathcal{B} = \{\boldsymbol{b_1}, \boldsymbol{b_2}\} = \left\{ \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix}, \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix} \right\}$, and we can write $A = PDP^{-1}$ where $P = \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix}$ and $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$. We will not actually need this factorization. However, we can make use of P^{-1}, so we compute it:

$$P^{-1} = \tfrac{1}{\lambda_2 - \lambda_1} \begin{bmatrix} \lambda_2 & -1 \\ -\lambda_1 & 1 \end{bmatrix} = \tfrac{1}{\sqrt{5}} \begin{bmatrix} \lambda_2 & -1 \\ -\lambda_1 & 1 \end{bmatrix}$$

↓

because $\lambda_2 - \lambda_1 = \sqrt{5}$ (check!)

Now, we return to our examination of the Fibonacci sequence. We begin by writing $\boldsymbol{x_0}$ as a linear combination of the eigenvectors: we want

$$\boldsymbol{x_0} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = c_1\boldsymbol{b_1} + c_2\boldsymbol{b_2} = P\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}, \text{ so}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = P^{-1}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \tfrac{1}{\sqrt{5}} \begin{bmatrix} \lambda_2 & -1 \\ -\lambda_1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \tfrac{1}{\sqrt{5}} \begin{bmatrix} \lambda_2 - 1 \\ -\lambda_1 + 1 \end{bmatrix}$$

$$= \tfrac{1}{\sqrt{5}} \begin{bmatrix} \lambda_2 - 1 \\ -\lambda_1 + 1 \end{bmatrix} = \tfrac{1}{\sqrt{5}} \begin{bmatrix} -\lambda_1 \\ \lambda_2 \end{bmatrix}$$

↓

because $\lambda_2 - 1 = \frac{1+\sqrt{5}}{2} - 1 = \frac{1+\sqrt{5}-2}{2} = -\lambda_1$, *and* ,,,

Thus $\boldsymbol{x_0} = \frac{-\lambda_1}{\sqrt{5}}\boldsymbol{b_1} + \frac{\lambda_2}{\sqrt{5}}\boldsymbol{b_2}$

Since $\boldsymbol{b_1}$ and $\boldsymbol{b_2}$ are eigenvectors with eigenvalues λ_1 and λ_2, we get

$$\boldsymbol{x_1} = A\boldsymbol{x_0} = \tfrac{-\lambda_1}{\sqrt{5}}A\boldsymbol{b_1} + \tfrac{-\lambda_2}{\sqrt{5}}A\boldsymbol{b_2} = \tfrac{-\lambda_1}{\sqrt{5}}\lambda_1\boldsymbol{b_1} + \tfrac{\lambda_2}{\sqrt{5}}\lambda_2\boldsymbol{b_2}$$

Continuing in the way, we get

$$\boldsymbol{x_2} = A\boldsymbol{x_1} = \tfrac{-\lambda_1}{\sqrt{5}}\lambda_1 A\boldsymbol{b_1} + \tfrac{\lambda_2}{\sqrt{5}}\lambda_2 A\boldsymbol{b_2} = \tfrac{-\lambda_1}{\sqrt{5}}\lambda_1^2\boldsymbol{b_1} + \tfrac{\lambda_2}{\sqrt{5}}\lambda_2^2\boldsymbol{b_2}$$

$$\vdots$$

$$\boldsymbol{x_k} = A\boldsymbol{x_{k-1}} = \tfrac{-\lambda_1}{\sqrt{5}}\lambda_1^k\boldsymbol{b_1} + \tfrac{\lambda_2}{\sqrt{5}}\lambda_2^k\boldsymbol{b_2}$$

$$= \tfrac{-1}{\sqrt{5}}\lambda_1^{k+1}\boldsymbol{b_1} + \tfrac{1}{\sqrt{5}}\lambda_2^{k+1}\boldsymbol{b_2}$$

$$= \tfrac{-1}{\sqrt{5}}\lambda_1^{k+1}\begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix} + \tfrac{1}{\sqrt{5}}\lambda_2^{k+1}\begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}$$

$$= -\frac{1}{\sqrt{5}}\begin{bmatrix}\lambda_1^{k+1}\\ \lambda_1^{k+2}\end{bmatrix} + \frac{1}{\sqrt{5}}\begin{bmatrix}\lambda_2^{k+1}\\ \lambda_2^{k+2}\end{bmatrix}$$

$$= \frac{1}{\sqrt{5}}\begin{bmatrix}\lambda_2^{k+1} - \lambda_1^{k+1}\\ \lambda_2^{k+2} - \lambda_1^{k+2}\end{bmatrix} = \begin{bmatrix}a_k\\ a_{k+1}\end{bmatrix}$$

Equating the first entries of the last two vectors gives

$$a_k = \frac{1}{\sqrt{5}}(\lambda_2^{k+1} - \lambda_1^{k+1}) = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}\right)$$

This is a explicit formula for the Fibonacci number a_k: it doesn't refer to any previous Fibonacci numbers, Interestingly, just to look at it, it's not even obvious that the expression on the right side of the equation is an integer!

Since $\begin{bmatrix}a_k\\ a_{k+1}\end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix}\lambda_2^{k+1} - \lambda_1^{k+1}\\ \lambda_2^{k+2} - \lambda_1^{k+2}\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{5}}(\lambda_2^{k+1} - \lambda_1^{k+1})\\ \frac{1}{\sqrt{5}}(\lambda_2^{k+2} - \lambda_1^{k+2})\end{bmatrix}$

we get that $\frac{a_{k+1}}{a_k} = \frac{\frac{1}{\sqrt{5}}(\lambda_2^{k+2}-\lambda_1^{k+2})}{\frac{1}{\sqrt{5}}(\lambda_2^{k+1}-\lambda_1^{k+1})} = \frac{(\lambda_2^{k+2}-\lambda_1^{k+2})}{(\lambda_2^{k+1}-\lambda_1^{k+1})}$.

But $|\lambda_1| = \left|\frac{1-\sqrt{5}}{2}\right| \approx 0.6180$, so $\lambda_1^k \to 0$ as $k \to \infty$ and therefore

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{(\lambda_2^{k+2}-\lambda_1^{k+2})}{(\lambda_2^{k+1}-\lambda_1^{k+1})} = \lim_{k\to\infty} \frac{\lambda_2^{k+2}}{\lambda_2^{k+1}} = \lambda_2 = \frac{1+\sqrt{5}}{2}.$$

The number $\frac{1+\sqrt{5}}{2}$ is called the Golden Ratio or Golden Section, often denoted by ϕ. Its decimal value is approximately 1.6180. For more information, look at

http://mathworld.wolfram.com/GoldenRatio.html.

Notice that the "classical" geometric definition for ϕ given there leads to the same quadratic equation that was the characteristic equation in the earlier discussion.

For some illustrations of the use of the "golden ratio" or "divine proportion" in art, look at
http://www.goldennumber.net/art-composition-design/