EFFECT OF EROs ON DETERMINANT

(*)Theorem Let A be a square matrix:

1) if a multiple of one row of A is added to another to get a matrix B, then det $A =$ det B
(*no effect on determinant*)

2) If two rows of A are interchanged to get B,
then det $B = -$ det A
(*<u>each</u> interchange reverses sign on determinant*)

3) If one row of A is multiplied by k ($\neq 0$) to get B, then det $B = k$ det A
(*rescaling a row rescales the determinant too*)

Example: $\det \begin{bmatrix} 5a & 5b \\ c & d \end{bmatrix} = 5 \det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

(3) often used to "factor out" a number

Depends on theorem:

Theorem Suppose E and A are $n \times n$ and that E is an elementary matrix. Then

$$\det(EA) = \det(E)\det(A) \quad \text{and}$$

$$\det E = \begin{cases} 1 \text{ if } E \leftrightarrow \text{add multiple of one row to another} \\ k \text{ if } E \leftrightarrow \text{rescale a row by factor } k \neq 0 \\ -1 \text{ if } E \leftrightarrow \text{interchange two rows} \end{cases}$$

Why true for 2×2 ?

Suppose $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is 2×2 and that E is a 2×2 elementary matrix ($\leftrightarrow$ ERO)

$$E \downarrow$$

$$= \quad \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix} \cdot A \quad = \quad \begin{bmatrix} a & b \\ ka+c & kb+d \end{bmatrix},$$

$\text{det}(E) = 1$

$\text{det}(EA) = ad - bc = \text{det}(A)$

(*so this ERO doesn't change det*(A))

$= 1 \cdot \text{det}(A)$

$= \text{det}(E) \cdot \text{det}(A)$

$$E \downarrow$$

$$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix} \cdot A = \begin{bmatrix} ka & kb \\ c & d \end{bmatrix}$$

$\text{det}(E) = k$

$\text{det}(EA) = k(ad - bc) = k\,\text{det}(A)$

(*so this ERO multiplies det*(A) *by* k)

$= \text{det}(E) \cdot \text{det}(A)$

$$E \downarrow$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot A = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

$\text{det}(E) = -1$
$\text{det}(EA) = (bc - ad) = (-1)\text{det}(A)$

(*so this ERO changes sign of det* A)

$= \text{det}(E) \cdot \text{det}(A)$

Conclusion: <u>for size 2×2 matrices</u>

if E is an elementary matrix, then

$$\det(EA) = \det(E)\det(A)$$

and det E =

$$\begin{cases} 1 & \text{if } E \text{ is: "add a multiple of one row to another"} \\ -1 & \text{if } E \text{ is: a row interchange} \\ k & \text{if } E \text{ is: a row rescaling by a factor of } k \neq 0 \end{cases}$$

2×2's, and E elementary: $\det(EA) = \det(E)\det(A)$;
this forces the same equation to be true for 3×3;
this forces the same equation to be true for 4×4; etc

Illustrate for 3×3: <u>for example</u>: "add multiple of one row to another"

$\det(E) = 1$ ↙

$$\overset{E}{\begin{bmatrix} 1 & 0 & 0 \\ k & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \overset{A}{\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}} = \begin{bmatrix} a & b & c \\ d+ka & e+kb & f+kc \\ g & h & i \end{bmatrix}$$

calculate det(EA) going across an "uninvolved" row

↘

$$\det(EA) = g \det\begin{bmatrix} b & c \\ e+kb & f+kc \end{bmatrix} - h \det\begin{bmatrix} a & c \\ d+ka & f+kc \end{bmatrix} + i \det\begin{bmatrix} a & b \\ d+ka & e+kb \end{bmatrix}$$

why?

↓

$$= g \det\begin{bmatrix} b & c \\ e & f \end{bmatrix} - h \det\begin{bmatrix} a & c \\ d & f \end{bmatrix} + i \det\begin{bmatrix} a & b \\ d & e \end{bmatrix}$$

$$= \det(A) \qquad = 1 \cdot \det(A) \quad = \det(E) \cdot \det(A)$$