The Leontief Open Economy Production Model

We will look at the idea for Leontief's model of an "open economy" $-$ a term which we will explain below. It starts by looking very similar to an example of a "closed economy" that we saw near the beginning of this course. Here, however, we start with a matrix whose columns represent "consumption" (rather than "production" as in the earlier example).

In the case of simple examples, we can solve the equations that we will set up. However, proofs of the deeper and more interesting theorems about "what happens in general" are beyond the scope of this course.

Assume that an economy has n sectors $S_1, \ldots, S_n$. Amounts consumed or produced by each sector are measured in \$ (value of goods consmbed or produced).

We begin with a consumption matrix C: the j^{th} column lists the amount of goods (in \$) consumed (or "demanded") by sector S_j from sectors $S_1, S_2, \ldots, S_n$ to produce one unit (\$) of its product..

Columns of C are
consumption vectors by sectors
$S_1 \cdots S_j \cdots S_n$
to produce 1 unit of product (measured in \$)

$$\textbf{Consumed }\underline{\textbf{from}}\left\{\begin{array}{l} S_1 \\ \vdots \\ S_i \\ \vdots \\ S_n \end{array}\right. \begin{array}{c} \longrightarrow \\ \\ \longrightarrow \\ \\ \longrightarrow \end{array} \begin{array}{c} \quad\downarrow \qquad\quad \downarrow \qquad\quad \downarrow \\ \begin{bmatrix} c_{11} \cdots & c_{1j} & \cdots & c_{1n} \\ & \vdots & & \\ c_{i1} \cdots & c_{ij} & \cdots & c_{in} \\ & \vdots & & \\ c_{n1} \cdots & c_{nj} & \cdots & c_{nn} \end{bmatrix} \end{array} = C$$

Of course, each $c_{ij} \geq 0$.

For example, Column 1 $= \begin{bmatrix} c_{11} \\ \vdots \\ c_{i1} \\ \vdots \\ c_{n1} \end{bmatrix}$ lists the consumption (demands) of Sector 1 from

Sectors S_1 ,..., S_n to produce one unit (\$) of product. Of course, we could also think of this as a list of costs (\$) for Sector 1 to produce one unit of product, and the sum of the entries gives the total cost of all products consumed (demanded) by Sector 1 to produce one unit of its product.

The same is true for each column: the entries in Column j represent the consumption (demands, or costs) of sector S_j from sectors $S_1, S_2, ..., S_n$ in order for S_j to produce one unit ($) of product.

Each column sum represents the costs for that sector to produce one unit. If a column sum is < 1, we say that sector is profitable.)

If we weight Column j with a scalar $x_j \geq 0$, then the vector $x_j \begin{bmatrix} c_{1j} \\ \vdots \\ c_{ij} \\ \vdots \\ c_{nj} \end{bmatrix} = \begin{bmatrix} x_j c_{1j} \\ \vdots \\ x_j c_{ij} \\ \vdots \\ x_j c_{nj} \end{bmatrix}$ lists

the consumption (demand, cost) by sector S_j from other sectors to produce x_j units of its own product.

Suppose $x_1, x_2, ..., x_n$ (≥ 0) are amounts to be produced by sectors $S_1, S_2, ..., S_n$. We

call $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is called a production vector. Then what does the vector $C\boldsymbol{x}$ mean?

It is a linear combination of the columns of C with the x_i's as weights:

$$C\boldsymbol{x} = x_1 \begin{bmatrix} c_{11} \\ \vdots \\ c_{i1} \\ \vdots \\ c_{n1} \end{bmatrix} + x_2 \begin{bmatrix} c_{12} \\ \vdots \\ c_{i2} \\ \vdots \\ c_{n2} \end{bmatrix} + ... + x_n \begin{bmatrix} c_{1n} \\ \vdots \\ c_{\boldsymbol{in}} \\ \vdots \\ c_{nn} \end{bmatrix} = \begin{bmatrix} c_{11}x_1 + \ ... + c_{1n}x_n \\ \vdots \\ c_{i1}x_1 + ... + c_{in}x_n \\ \vdots \\ c_{n1}x_1 + ... + c_{nn}x_n \end{bmatrix}$$

$$= \begin{bmatrix} \textit{total demanded } \underline{\textit{from}} \textit{ sector } S_1 \textit{ by sectors } S_1, ... , S \textit{ to produce } \boldsymbol{x} \\ \vdots \\ \textit{total demanded } \underline{\textit{from}} \textit{ sector } S_2 \textit{ by sectors } S_1, ... , S_n \textit{ to produce } \boldsymbol{x} \\ \vdots \\ \textit{total demanded } \underline{\textit{from}} \textit{ sector } S_n \textit{ by sectors } S_1, ... , S_n \textit{ to produce } \boldsymbol{x} \end{bmatrix}$$

In other words, $C\boldsymbol{x}$ lists the demands that must be supplied from each sector to the others to deliver the production vector $\boldsymbol{x}$.

The equation $C\boldsymbol{x} = \boldsymbol{x}$ asks for an "equilibrium value" – an production vector $\boldsymbol{x}$ for which everything is "in balance," where the list $\boldsymbol{x}$ of products (in $) from each sector $=$ the list $C\boldsymbol{x}$ of total demands ($) from the sectors. For equilibrium, the production from each sector must be not too large, not too small, but "just right." This is a closed economy – everything is just exchanged among the sectors.

Now we add another feature to the economic model: an open sector. It is nonproductive, that is, it produces nothing that sectors $S_1, \ldots, S_n$ use. The open sector simply "demands" the prodcuts of sectors $S_1, \ldots, S_n$. (For example, these could be demands of the government, demands for foreign export, for individual consumption, for charitable groups, ...)

Supppose $\boldsymbol{d} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix}$ tells how many units (in $) the open sector demands from each sector $S, \ldots, S_n$. The vector $\boldsymbol{d}$ is called the final demand vector

We want to know if it is possible to set a level of production $\boldsymbol{x}$ so that both the productive and open sectors are satisfied, with nothing left over. That is, we want to find an $\boldsymbol{x}$ so that

Leontief Open Economy Production Model

$$\boldsymbol{x} = C\boldsymbol{x} + \boldsymbol{d} \qquad (***)$$

$\uparrow$		$\uparrow$		$\uparrow$
Total production	$=$	Demand from productive sectors to produce $\boldsymbol{x}$ (*called "intermediate demand"*)	+	Demand from open sector (*called "final demand"*)

If I is the identity matrix, we can rewrite this equation as

:

$$I\boldsymbol{x} = C\boldsymbol{x} + \boldsymbol{d}$$
$$I\boldsymbol{x} - C\boldsymbol{x} = \boldsymbol{d} \quad \text{or}$$

$$(I - C)\,\boldsymbol{x} = \boldsymbol{d}$$

To take a simple example, suppose the economy has 4 productive sectors $S_1, \ldots S_4$ and the consumption matrix is

Consumption vectors for

$$\begin{array}{cc} & \begin{array}{cccc} S_1 & S_2 & S_3 & S_4 \\ \downarrow & \downarrow & \downarrow & \downarrow \end{array} \\ C = & \begin{bmatrix} .10 & .05 & .30 & .20 \\ .15 & .25 & .05 & .10 \\ .30 & .10 & .10 & .25 \\ .15 & .20 & .10 & .20 \end{bmatrix} \end{array}$$

(*The column sums in C are all < 1, so each sector is profitable.*)

Then $$(I - C) = \begin{bmatrix} .90 & -.05 & -.30 & -.20 \\ -.15 & .75 & -.05 & -.10 \\ -.30 & -.10 & .90 & -.25 \\ -.15 & -.20 & -.10 & .80 \end{bmatrix}$$

If the demand from the open sector is $\boldsymbol{d} = \begin{bmatrix} 25000 \\ 10000 \\ 30000 \\ 50000 \end{bmatrix}$, then we want to solve

$$(I - C)\boldsymbol{x} = \begin{bmatrix} .90 & -.05 & -.30 & -.20 \\ -.15 & .75 & -.05 & -.10 \\ -.30 & -.10 & .90 & -.25 \\ -.15 & -.20 & -.10 & .80 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 25000 \\ 10000 \\ 30000 \\ 50000 \end{bmatrix} = \boldsymbol{d}$$

Row reducing the augmented matrix gives

$$\begin{bmatrix} .90 & -.05 & -.30 & -.20 & 25000 \\ -.15 & .75 & -.05 & -.10 & 10000 \\ -.30 & -.10 & .90 & -.25 & 30000 \\ -.15 & -.20 & -.10 & .80 & 50000 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 85580 \\ 0 & 1 & 0 & 0 & 50620 \\ 0 & 0 & 1 & 0 & 96160 \\ 0 & 0 & 0 & 1 & 103220 \end{bmatrix}$$

These calculations were carried out to many decimal places, but the displayed numbers are rounded, at the end, to the nearest unit ($)

The equation (***) will be satisfied if $\boldsymbol{x} = \begin{bmatrix} 85580 \\ 50620 \\ 96160 \\ 103220 \end{bmatrix}$, that is,

if S_1 produces 85580 units ($) ,
S_2 produces 50620 units ($) , ... etc.

Notice: the row reduced echelon form shows that $I - C$ is invertible, so the solution for $\boldsymbol{x}$ is actually unique: $\boldsymbol{x} = (I - C)^{-1}\boldsymbol{d}$. (*We could have solved the matrix equation* $(I - C)\boldsymbol{x} = \boldsymbol{d}$ *in the first place by trying to find* $I - C)^{-1}$*, but that's even more work.*)

This theorem is too hard for us to prove, but we can give an intuitive argument for it.

Theorem (*using the same notation as above*) If C and $\boldsymbol{d}$ have nonnegative entries and the column sums in C are all < 1 (*that is, if every sector is profitable*), then $(I - C)$ must be invertible and therefore there is a unique production vector satisfying the equation $(I - C)\boldsymbol{x} = \boldsymbol{d}$, namely $\boldsymbol{x} = (I - C)^{-1}\boldsymbol{d}$. Moreover, this $\boldsymbol{x}$ is economically feasible in the sense that all its entries are ≥ 0.

Here is an informal, imprecise discussion that might give you some confidence that the theorem is actually true.

1) Suppose C is any $n \times n$ matrix with entries all ≥ 0 and all column sums < 1. Then $C^m \to 0$ as $m \to \infty$.

$\uparrow$
(the $n \times n$ zero matrix)

This statement just means that "each entry c in the matrix C^m can be made as close to 0 as we like by choosing m large enough."

To keep things simple, we will only look at the case when $C = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, a 2×2 matrix. But the idea is similar when C is $n \times n$.

Let α = the <u>larger</u> of the two column sums in C. By definition of α, $a + c \leq \alpha$ and $b + d \leq \alpha$ (*one of the two actually <u>equals</u>* α.) By our assumptions above about C, we see that $0 \leq \alpha < 1$.

An observation: suppose $x, y, z, w \geq 0$. What happens to column sums when a 2×2 matrix $\begin{bmatrix} x & y \\ z & w \end{bmatrix}$ is multiplied by C <u>on the left</u>?

$$C\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} ax+bz & ay+bw \\ cx+dz & cy+dw \end{bmatrix},$$

so the column sums satisfy:

$$\begin{aligned} \text{Sum Column 1} &= (a+c)x + (b+d)z \leq \alpha x + \alpha z = \alpha\,(x+z) \\ \text{Sum Column 2} &= (a+c)y + (b+d)w \leq \alpha y + \alpha w = \alpha\,(y+w) \end{aligned}$$

In other words: for each column in the product $C\begin{bmatrix} x & y \\ z & w \end{bmatrix}$,

<u>column sum in product</u> $\leq \alpha$ (<u>old</u> column sum in $\begin{bmatrix} x & y \\ z & w \end{bmatrix}$)

Use this fact, letting $\begin{bmatrix} x & y \\ z & w \end{bmatrix} = C = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then

$$C\begin{bmatrix} x & y \\ z & w \end{bmatrix} = C^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ and}$$

sum for column 1 of $C^2 \leq \alpha$ (old column 1 sum in C) $= \alpha(a+c) \leq \alpha \cdot \alpha = \alpha^2$
sum for column 2 of $C^2 \leq \alpha$ (old column 2 sum in C) $= \alpha(b+d) \leq \alpha \cdot \alpha = \alpha^2$
.

Repeating the calculation again using $\begin{bmatrix} x & y \\ z & w \end{bmatrix} = C^2$, we see that in the matrix $C^3 = C \cdot C^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot C^2$, the sum of each column is $\leq \alpha^3$.

Continuing in this way, we get that each column sum in C^m is $\leq \alpha^m$. Since all the elements in C^m are nonnegative, this means that every individual entry c in the matrix C^m satisfies $0 \leq c \leq \alpha^m$. Since $\alpha^m \to 0$ as $m \to \infty$ (because $0 \leq \alpha < 1$), we get that each entry c in C^m approaches 0 as $m \to \infty$.

2) $\boldsymbol{I - C^{m+1} = (I - C)(I + C + C^2 + ... + C^m)}$

Why? Just multiply it out to verify:

$$\begin{aligned}
&(I - C)(I + C + C^2 + ... + C^m) \\
&= (I - C)\,I + (I - C)\,C + (I - C)\,C^2 + ... + (I - C)\,C^m \\
&= I^2 - CI + IC - C^2 + IC^2 - C^3 + ... + IC^m - C^{m+1} \\
&= I - C + C - C^2 + C^2 ... \qquad ... - C^m + C^m - C^{m+1} \\
&= I - C^{m+1}
\end{aligned}$$

3) Since $C^m \to 0$ as $m \to \infty$, it follows that for large m $C^{m+1} \approx 0$ and therefore

$$\boldsymbol{I \approx (I - C)(I + C + C^2 + ... + C^m)}$$

↑ so $(I + C + C^2 + ... + C^m)$ is approximately an inverse for $(I - C)$, that is

$$\boldsymbol{(I + C + C^2 + ... + C^m) \approx (I - C)^{-1}}$$

This approximation suggests that $I - C$ really is invertible, and, since $(I + C + C^2 + ... + C^m)$ has nonnegative entries, it also suggests that all the entries in the exact matrix $(I - C)^{-1}$ will be nonnegative.

4) **If $(I - C)$ is invertible and $(I - C)^{-1}$ has all nonnegative elements, then of course there is a unique solution $x = (I - C)^{-1}d$ and $(I - C)^{-1}d$ has all nonnegative entries.**

Having done all this, we can also write:

$$(I - C)^{-1} \approx\ (I + C + C^2 + ... + C^m),\ \text{so}$$

$$\boldsymbol{x} = (I - C)^{-1}\boldsymbol{d} \approx\ (I + C + C^2 + ... + C^m) \cdot \boldsymbol{d} \qquad \text{or}$$

$$\boldsymbol{x} \approx \boldsymbol{d} + C\boldsymbol{d} + C^2\boldsymbol{d} + ... + C^m\boldsymbol{d}$$

Does that make any sense? Think about it like this: we ask the sectors to meet the final demand $\boldsymbol{d}$, and they set out to do it. But producing the goods listed in $\boldsymbol{d}$ creates additional demands: each sector has demands from the others to be able to produce <u>anything</u> at all. In fact, the sectors must supply each other with the products listed in $C\boldsymbol{d}$ so that they can produce $\boldsymbol{d}$. But then, to produce $C\boldsymbol{d}$, the sectors require an additional $C(C\boldsymbol{d})$ in products – and so on and so on .... To make it all work, the needed production $\boldsymbol{x}$ is an infinite sum:

$$\boldsymbol{x} = \boldsymbol{d} + C\boldsymbol{d} + C^2\boldsymbol{d} + ... + C^m\boldsymbol{d} + ...$$

But because $C^m \to 0$ as $m \to \infty$, we approximate $\boldsymbol{x} \approx \boldsymbol{d} + C\boldsymbol{d} + C^2\boldsymbol{d} + ... + C^m\boldsymbol{d}$ for large m.

The column vectors in $(I - C)^{-1}$ actually have an economic interpretation: see the remarks following the next example

Example Here is an example with a slightly more realistic set of data involving the Leontief Open Economy Production Model. The data is from #13, p. 137 in the textbook.

The consumption matrix C is based on input-output data for the U.S. economy in 1958, with data for 81 sectors grouped here (for manageability) into 7 larger sectors: (1) nonmetal household and personal products (2) final metal products (such as autos) (3) basic metal products and mining (4) basic nonmetal products and agriculture (5) energy (6) services and (7) entertainment and miscellaneous products. Units are in <u>millions of dollars</u>. (From Wassily W. Leontief, "The Structure of the US Economy", *Scientific American*, April 1965, pp. 30-32).

$$C = \begin{matrix} 0.1588 & 0.0064 & 0.0025 & 0.0304 & 0.0014 & 0.0083 & 0.1594 \\ 0.0057 & 0.2645 & 0.0436 & 0.0099 & 0.0083 & 0.0201 & 0.3413 \\ 0.0264 & 0.1506 & 0.3557 & 0.0139 & 0.0142 & 0.0070 & 0.0236 \\ 0.3299 & 0.0565 & 0.0495 & 0.3636 & 0.0204 & 0.0483 & 0.0649 \\ 0.0089 & 0.0081 & 0.0333 & 0.0295 & 0.3412 & 0.0237 & 0.0020 \\ 0.1190 & 0.0901 & 0.0996 & 0.1260 & 0.1722 & 0.2368 & 0.3369 \\ 0.0063 & 0.0126 & 0.0196 & 0.0098 & 0.0064 & 0.0132 & 0.0012 \end{matrix}$$

It turns out that $I - C$ is invertible (*as it should be, according to the Theorem stated)*

For a final demand vector $\boldsymbol{d} = \begin{bmatrix} 74000 \\ 56000 \\ 10500 \\ 25000 \\ 17500 \\ 196000 \\ 5000 \end{bmatrix}$, the solution to $\boldsymbol{x} = C\boldsymbol{x} + \boldsymbol{d}$ is easy with

computer software such as Matlab. Rounded to 4 places, Matlab gives

$$(I - C)^{-1} = \begin{matrix} 1.2212 & 0.0271 & 0.0226 & 0.0677 & 0.0135 & 0.0227 & 0.2167 \\ 0.0432 & 1.4046 & 0.1244 & 0.0466 & 0.0404 & 0.0516 & 0.5103 \\ 0.0806 & 0.3387 & 1.5927 & 0.0555 & 0.0508 & 0.0326 & 0.1810 \\ 0.6732 & 0.1905 & 0.1763 & 1.6448 & 0.0948 & 0.1266 & 0.3265 \\ 0.0636 & 0.0531 & 0.1010 & 0.0897 & 1.5393 & 0.0575 & 0.0590 \\ 0.3409 & 0.2711 & 0.2953 & 0.3253 & 0.3842 & 1.3674 & 0.6371 \\ 0.0213 & 0.0303 & 0.0392 & 0.0231 & 0.0175 & 0.0211 & 1.0246 \end{matrix}$$

so $\boldsymbol{x} = (I - C)^{-1}\boldsymbol{d} = (I - C)^{-1} \begin{bmatrix} 74000 \\ 56000 \\ 10500 \\ 25000 \\ 17500 \\ 196000 \\ 5000 \end{bmatrix} = \begin{bmatrix} 99580 \\ 97700 \\ 51230 \\ 13157 \\ 49490 \\ 32955 \\ 13840 \end{bmatrix}$

(*entries rounded to whole numbers*)

<u>An economic interpretation for the columns of</u> $(\boldsymbol{I} - \boldsymbol{C})^{-1}$

Suppose C is an $n \times n$ consumption matrix with entries ≥ 0 and column sums each less than 1. Let

$$(**) \begin{cases} \boldsymbol{x} & = \text{production vector that } \underline{\text{satisfies}} \text{ a final demand } \boldsymbol{d} \\ \Delta\boldsymbol{x} & = \text{production vector that } \underline{\text{satisfies}} \text{ a different final demand } \Delta\boldsymbol{d} \end{cases}$$

(*for the moment, just think of* $\Delta\boldsymbol{x}$ *and* $\Delta\boldsymbol{d}$ *as fancy names for two new production/final demand vectors having no connection to* $\boldsymbol{x}$, $\boldsymbol{d}$)

Then by (**)

$$\begin{aligned} \boldsymbol{x} &= C\boldsymbol{x} + \boldsymbol{d} \\ \Delta\boldsymbol{x} &= C\Delta\boldsymbol{x} + \Delta\boldsymbol{d} \end{aligned}$$

Adding the equations and rearranging, we get

$$\boldsymbol{x} + \Delta\boldsymbol{x} = C(\boldsymbol{x} + \Delta\boldsymbol{x}) + (\boldsymbol{d} + \Delta\boldsymbol{d})$$

So production level $(\boldsymbol{x} + \Delta\boldsymbol{x})$ fulfils the final demand $(\boldsymbol{d} + \Delta\boldsymbol{d})$

At this point you can think of $\Delta\boldsymbol{d}$ as we would in calculus – as a change in the final demand $\boldsymbol{d}$, and $\Delta\boldsymbol{x}$ is the corresponding change made necessary in the production vector $\boldsymbol{x}$ to satisfy the new final demand.

Suppose we let $\Delta\boldsymbol{d} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, that is, "increase the final demand from sector S_1 by 1 unit."

Since $\quad \Delta\boldsymbol{x} = C\Delta\boldsymbol{x} + \Delta\boldsymbol{d}$, we have $(I - C)\,\Delta\boldsymbol{x} = \Delta d$, or

$$\Delta\boldsymbol{x} = (I - C)^{-1}\Delta\boldsymbol{d} = (I - C)^{-1}\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$= 1 \cdot \begin{bmatrix} \text{col } 1 \\ \text{of} \\ (I - C)^{-1} \end{bmatrix} + 0 \cdot \begin{bmatrix} \text{col } 2 \\ \text{of} \\ (I - C)^{-1} \end{bmatrix} + \ldots + 0 \cdot \begin{bmatrix} \text{col } n \\ \text{of} \\ (I - C)^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} \text{col } 1 \\ \text{of} \\ (I - C)^{-1} \end{bmatrix}$$

So the $\Delta\boldsymbol{x}$ corresponding to $\Delta\boldsymbol{d} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ is the first column of $(I - C)^{-1}$.

In other words: a change of one unit in the final demand from sector S_1 requires a change $\Delta\boldsymbol{x}$ in the production vector required to satisfy the new demand, and $\Delta\boldsymbol{x} =$ the first column of $(I - C)^{-1}$.

Similarly, you can show that a change of one unit in the demand of sector S_2 causes a change in the production vector $\Delta x =$ the second column of $(I - C)^{-1}$; etc.

The column vectors in $(I-C)^{-1}$ are not just "accidental" vectors that pop up in computing the inverse; they have economic meaning.

So, in the preceding example: a change of $\boldsymbol{\Delta d} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, that is a change of $+1$ unit $(= +1$ million dollars$)$ in the demand for goods from sector S_1 forces the necessary production vector

$$\boldsymbol{x} = \begin{bmatrix} 99580 \\ 97700 \\ 51230 \\ 13157 \\ 49490 \\ 32955 \\ 13840 \end{bmatrix} \text{ to change to } \boldsymbol{x} + \Delta\boldsymbol{x},$$

where $$\Delta\boldsymbol{x} = 1^{st} \text{ column of } (I-C)^{-1} = \begin{bmatrix} 1.2212 \\ 0.0432 \\ 0.0806 \\ 0.6732 \\ 0.0636 \\ 0.3409 \\ 0.0213 \end{bmatrix}$$