

Ordered Pairs, Products and Relations

An ordered pair (a, b) is built from two objects, a and b . As the name suggests, the “order” matters: (a, b) and (b, a) are two different ordered pairs (unless $a = b$). For two ordered pairs, $(a, b) = (c, d)$ iff $a = c$ and $b = d$.

If we are allowed to choose a from the set A and b from the set B , then the set of all possible ordered pairs that we can create is called the product set $A \times B$ (usually read as “ A cross B ”).

Definition If A and B are sets, then $A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$. If $A = B$, we sometimes write A^2 for the set $A \times A$.

For example, the plane $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x, y) : x \in \mathbb{R} \text{ and } y \in \mathbb{R}\}$ and we can think of $\{0\} \times \mathbb{R} = \{(0, y) : y \in \mathbb{R}\}$ as being the y -axis, a subset of \mathbb{R}^2 .

Since (a, b) is not the same as (b, a) , the sets $A \times B$ and $B \times A$ are usually not the same. For example, the x -axis in $\mathbb{R}^2 = \mathbb{R} \times \{0\} \neq \{0\} \times \mathbb{R} =$ the y -axis. (*If you think about it, you should be to state and prove a theorem that says: $A \times B = B \times A$ iff*)

Before we go on: we are taking the point of view that we should be able to use set theory as a foundation for all of mathematics – that it should be possible to think of everything as some sort of set. For example, we saw how to think of whole numbers as sets and how the system of whole numbers (including $+$, \cdot , \leq) and be defined in terms of sets.

From that point of view, we should also be able to think of an ordered pair as some set. For sets, we know that $\{a, b\} = \{b, a\}$, so the ordered pair (a, b) is not the same as the set $\{a, b\}$.

Just as with whole numbers, philosophical questions about what an ordered pair “really is” are irrelevant to mathematicians: all that matters to us is how ordered pairs behave. Because order matters, the crucial thing about the behavior of ordered pairs is that $(a, b) = (c, d)$ iff $a = c$ and $b = d$. So if, given a and b , we can define a set that behaves in this way, we can agree to let that set be the official definition for the ordered pair (a, b) .

There are different ways one can do this, and for mathematical purposes one way is as good as another, as long as we all agree. The following definition is the one most mathematicians use; it was first written down, I think, by a Polish mathematician named Kazimierz Kuratowski (1896-1980).

Definition $(a, b) = \{ \{a\}, \{a, b\} \}$

You should stop here and prove now that $\{ \{a\}, \{a, b\} \} = \{ \{c\}, \{c, d\} \}$ iff $a = c$ and $b = d$. (*When doing this, there are two cases: you do have to consider the possibility that $a = b$, which is a second case in the argument.*)

When you've done that, then you have shown that the set $\{\{a\}, \{a, b\}\}$ “acts just like” the intuitive ordered pair (a, b) , so we can all agree that we might as well accept this as the “official” definition of an ordered pair (a, b) .

For most purposes, we can then relax and continue to use the notation (a, b) just the way we always have – knowing that “officially” it's really a set.

We also can use ordered n -tuples: we say that $(a_1, a_2, \dots, a_n) = (b_1, b_2, \dots, b_n)$ iff $a_i = b_i$ for each $i = 1, 2, \dots, n$. We can also make an official definition of an n -tuple (a_1, a_2, \dots, a_n) as a set, but we won't bother going into details about that here.

We then define the product set $A_1 \times A_2 \times \dots \times A_n = \{(a_1, a_2, \dots, a_n) : a_i \in A_i \text{ for each } i = 1, \dots, n\}$.

As the book points out: strictly speaking, the sets $A_1 \times A_2 \times A_3$, $(A_1 \times A_2) \times A_3$ and $A_1 \times (A_2 \times A_3)$ are all different. Members of $A_1 \times A_2 \times A_3$ are “ordered triples” (a_1, a_2, a_3) . But a member of $(A_1 \times A_2) \times A_3$ is an ordered pair whose first element is itself an ordered pair: something that looks like $((a_1, a_2), a_3)$. A typical member of $A_1 \times (A_2 \times A_3)$ is different from either: $(a_1, (a_2, a_3))$. It turns out (as the text also states) that the fact that these three products are literally different really doesn't much matter: it's customary to ignore the difference and (usually) to view the elements in the product of the three sets as ordered triples (a_1, a_2, a_3) .