Transcendental Numbers: An Extended Example

The relatively simple facts about countable and uncountable sets that we know are enough to prove an interesting fact about the real numbers.

Definition A real number r is called algebraic if there is a polynomial

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \ ... \ + a_1 x + a_0$$

where the coefficients $a_0 . a_1, ..., a_n$ are all integers and for which $P(r) = 0$.

Loosely paraphrasing, an algebraic real number is one that's a root of some polynomial equation with integer coefficients. [*What's slightly "loose" about the paraphrase"?*]

It is clearly equivalent to say "rational" instead of "integers" in the definition of an algebraic number. If r is a root of a polynomial equation with rational coefficients (*for example,* $\frac{3}{4}x^2 + \frac{1}{7}x + \frac{1}{2} = 0$), *then r is also the root of a polynomial equation with integer coefficients* (*for example,* $21x^2 + 4x + 14 = 0$). *Multiplying an equation by a nonzero constant (to clear the fractions) doesn't change the roots.*

Complex roots $a + bi$ of the equation $P(x) = 0$ are also called (complex) algebraic numbers. Although everything that follows is about real algebraic numbers, the arguments work just as well if we allow complex algebraic numbers.

A real number is rational iff it is the root of a first degree polynomial equation with integer coefficients : the rational $\frac{p}{q}$ is a root of the equation $qx - p = 0$. Therefore the algebraic numbers are a natural generalization of rational numbers : the algebraic numbers include all numbers that are roots of higher degree polynomial equations with integer coefficients. For example, $\sqrt{2}$ is algebraic, because it is a root of $x^2 - 2 = 0$. A reasonable question would be: are there any nonalgebraic real numbers?

Theorem The set $\mathbb{A} = \{r \in \mathbb{R} : \ r \text{ is algebraic}\}$ is countable.

Proof For each $n \geq 1$, let $\mathcal{P}_n = \{a_n x^n + ... + a_1 x + a_0 : \ a_n, ..., a_1, a_0 \in \mathbb{Z}\}$ = the set of all polynomials of degree $\leq n$ with integer coefficients (*the degree will be $< n$ if the coefficient a_n happens to be* 0).

Each polynomial in $\mathcal{P}_n$ is completely identified by a list of its coefficients. Such a list is an $(n+1)$-tuple of integers : $(a_n, a_{n-1}, \ ... \ , a_0) \in \mathbb{Z} \times ... \times \mathbb{Z} = \mathbb{Z}^{n+1}$, so

$$a_n x^n + ... + a_1 x + a_0 \qquad \leftrightarrow \qquad (a_n, a_{n-1}, \ ... \ , a_0)$$

is a one-to-one correspondence between $\mathcal{P}_n$ and $\mathbb{Z}^{n+1}$. $\mathbb{Z}^{n+1}$ is a finite product of countable sets, so $\mathbb{Z}^{n+1}$ is countable and therefore $\mathcal{P}_n$ is countable – which means there exists a bijection $f : \mathbb{N} \to \mathcal{P}_n$.

Thus the set $\mathcal{P}_n$ can be listed as a sequence: $\mathcal{P}_n = \{P_{n,1}(x), P_{n,2}(x), ..., P_{n,k}(x), ... \ \}$
(*Here,* $P_{n,k} = f(k)$ = *the* k^{th} *term of the sequence* f.)

Each polynomial $P_{n,k}$ in $\mathcal{P}_n$ has at most n roots (since $P_{n,k}(x)$ has degree $\leq n$), so

$R_{n,k} = \{r \in \mathbb{R} : P_{n,k}(r) = 0\} = \{r : r$ is a root of $P_{n,k}(x) = 0\}$ is finite (so, countable).

Now we put all this together using our earlier theorems about countable sets.

$$A_n = \bigcup_{k=1}^{\infty} R_{n,k} = R_{n,1} \cup R_{n,2} \cup \ldots \cup R_{n,k} \cup \ldots$$
$$= \{r : r \text{ is a root of a polynomial of degree } \leq n \text{ with integer coefficients}\}$$

is a countable union of countable sets; so A_n is countable.

Then $\bigcup_{n=1}^{\infty} \mathbb{A}_n$ is countable, since it is a countable union of countable sets. But $\bigcup_{n=1}^{\infty} \mathbb{A}_n = \mathbb{A}$ $=$ the set of all algebraic numbers. ●

Definition A real number which is not algebraic is called transcendental. (*Euler called these numbers "transcendental" because they "transcend the power of algebraic methods." To be more politically correct, we might call them "polynomially challenged."*)

Corollary Transcendental numbers exist.

Proof Let $\mathbb{T}$ be the set of (real) transcendental numbers. Then $\mathbb{R} = \mathbb{A} \cup \mathbb{T}$. Since $\mathbb{A}$ is countable and $\mathbb{R}$ is uncountable, $\mathbb{T}$ cannot be empty. ●

In fact, this two-line argument actually proves much more: not only is $\mathbb{T}$ nonempty, but $\mathbb{T}$ must be uncountable! In the sense of one-to-one correspondence, there are "more" transcendental numbers than algebraic numbers on the real line $\mathbb{R}$.

This is an example of a "pure existence" proof – it does not exhibit any particular transcendental number, nor give any computational hints about how to find one. To do that is harder. Transcendental numbers were first shown to exist by Liouville in 1844. (*Liouville used other (more difficult) methods. The ideas about countable and uncountable sets were not developed until the early 1870's by Georg Cantor. See the biography on the course website.*)

Two famous examples of transcendental numbers are e (proven to be transcendental by Hermite in 1873) and π (Lindemann, 1882).

One method for producing many transcendental numbers is contained in a theorem of the Russian mathematician Gelfand (1934). It implies, for example, that $\sqrt{2}^{\sqrt{2}}$ is transcendental.

> **Gelfand's Theorem** If α is an algebraic number, $\alpha \neq 0$ or 1, and β is algebraic and not rational, then α^{β} is transcendental.

The number e^{π} is also transcendental. This follows from Gelfand's Theorem (which allows complex algebraic numbers) if you know something about the arithmetic of complex numbers:

$e^{\pi} = e^{-i^2\pi} = (e^{i\pi})^{-i}$ and $e^{i\pi} = \cos\pi + i\sin\pi = -1$.
So $e^{\pi} = (-1)^{-i}$, which is transcendental by Gelfand's Theorem.