A Memorable Proof of the Cantor Schroeder Bernstein Theorem
Perhaps due to Irving Kaplansky: I've not found anyone who is sure

Lemma Let X be a set. Suppose $\psi : \mathcal{P}(X) \to \mathcal{P}(X)$ is monotone: that is, if $A \subseteq B$ then $\psi(A) \subseteq \psi(B)$. Then there is a subset $F \subseteq X$ for which $\psi(F) = F$.

Note: ψ is an arbitrary function mapping subsets of X to subsets of X, but not necessarily one induced by some mapping $f : X \to X$, so monotonicity is not automatic.

Proof Let $\mathfrak{B} = \{B \subseteq X : B \subseteq \psi(B)\}$ and define $F = \bigcup \mathfrak{B}$.

For each $B \in \mathfrak{B}$ we have $B \subseteq \psi(B) \subseteq \psi(\bigcup \mathcal{B}) = \psi(F)$. Therefore $\bigcup \mathfrak{B} = F \subseteq \psi(F)$.

Since $F \subseteq \psi(F)$, monotonicity gives $\psi(F) \subseteq \psi(\psi(F))$, so $\psi(F) \in \mathfrak{B}$. Therefore $\psi(F) \subseteq \bigcup \mathfrak{B} = F$.

So $\psi(F) = F$. •

Theorem (Cantor-Schroeder-Bernstein) Suppose X and Y are sets and that both $f : X \to Y$ and $g : Y \to X$ are one-to-one. Then there is a bijection $h : X \to Y$.

Proof Define $\psi : \mathcal{P}(X) \to \mathcal{P}(X)$ by $\psi(A) = X - g[\,Y - f[A]\,]$. Because functions preserve inclusion and complements reverse inclusion, we see that if $A \subseteq B$, then $\psi(A) \subseteq \psi(B)$.

By the Lemma, there is a set F for which $\psi(F) = F$. (*This means that f takes F to a set whose complement is mapped by g to the complement of F*).

Define $h : X \to Y$ by

$$h(x) = \begin{cases} f(x) & \text{if } x \in F \\ g^{-1}(x) & \text{if } x \in X - F. \end{cases}$$

Then h is a bijection. •