Math 417, Fall 2013
Some Problems on Connectedness

You do not need to hand in these problems.

1. Let (X, d) and (Y, s) be two unbounded connected metric spaces. Give $X \times Y$ the product topology. For $(a, b) \in X \times Y$ and $k > 0$, let $K = \{ (x, y) \in X \times Y\colon d(x, a) \leq k$ and $s(y, b) \leq k\}$. Prove that the complement of K in $X \times Y$ is connected.

2. Suppose (X, d) is a connected metric space with $|X| > 1$. Prove that $|X| \geq c$.

3. A metric space (X, d) satisfies the <u>ϵ-chain</u> <u>condition</u> if, for all $x, y \in X$ and for all $\epsilon > 0$, there exists a finite set of points $x = x_1, x_2, \ldots, x_n = y$ such that for all $i = 1, \ldots, n-1$, $d(x_i, x_{i+1}) < \epsilon$.

a) Prove that if (X, d) is connected, then (X, d) satisfies the ϵ-chain condition. (*Hint: Let $x \in X$ and $\epsilon > 0$, consider the set of all points y that can be "chained" to x. Is this set open? ...*)

b) Give an example of a metric space (X, d) which satisfies the ϵ-chain condition but which is not connected.

c) Prove that if (X, d) is a compact metric space that satisfies the ϵ-chain condition, then (X, d) is connected.

4. Prove or disprove: there is a continuous function $f\colon \mathbb{R} \to \mathbb{R}$ such that $f[\mathbb{P}] \subseteq \mathbb{Q}$ and $f[\mathbb{Q}] \subseteq \mathbb{P}$.
(*Hint: what you can say about the range of f.*)

5. a) Find the cardinality of the set of all compact connected subsets of the plane.

b) Find the cardinality of the set of all connected subsets of the plane.