

Introduction

These notes are an introduction to set theory and topology. They are the result of teaching a two-semester course sequence on these topics for many years at Washington University in St. Louis. Typically the students were advanced undergraduate mathematics majors, some beginning mathematics graduate students, and some graduate students from other areas such as economics and engineering.

Over time my lecture notes gradually evolved into written outlines for students, then written versions of the more involved proofs. The full set of notes was a project completed during the years 2003-2007 and since slightly revised.

An introductory undergraduate analysis course is the usual background for the material, primarily because it provides a solid introduction to Euclidean space \mathbb{R}^n and practice with rigorous arguments (in particular, about continuity). Strictly speaking, however, I can say that the material is mostly self-contained. Although examples are taken now and then from analysis, they are not crucial to the logical progression. The only real prerequisite is the level of mathematical interest, maturity and patience needed to handle abstract ideas and to read and write careful proofs. A few talented students have taken these courses before introductory analysis (rarely, even as university freshmen) and invariably they have commented later on how material eased their way into analysis.

The set-theoretic material is not done axiomatically. However, there is an attempt to provide some informal insights into why an axiomatization of the subject might be important and what some of the most important results are. A student with a good grasp of the set-theoretic material – scattered throughout the notes, but heavily concentrated in Chapters I and VIII – will know all the informal set theory most mathematicians ever need, and will be in a good position to continue into a study of axiomatic set theory.

The topological material is all within the area traditionally labeled “general topology.” No topics from algebraic topology are included. This was a conscious choice that reflects my own training and tastes, as well as a conviction that students are rushed too quickly nowadays through the basics of topology in order to get to “where the action is.” It is certainly true that general topology has not been a hotbed of research for several decades and that much of what does go on currently is closely related to set theory and mathematical logic. Nevertheless, general topology contains a set of tools that most mathematicians need, whether for work in analysis or other parts of topology.

Many of those basic tools (such as “compactness” and the “product topology”) seem very abstract when a student first meets them. It takes time to develop an ownership of these tools. This includes a sense of their significance, an appropriate “feel” for how they behave, and good technique – in short, all the things necessary to make, say, using a compactness argument into a completely routine tool. I believe this process is often short-circuited in the rush to move students along to algebraic topology. The result then can be an introduction to algebraic topology where many tedious details are (appropriately) omitted and the student is ill-equipped to fill them in – or even to feel confident that the omissions are genuinely routine. In that case, a student can begin to feel that the subject has a vague, hand-waving quality about it.

These notes are designed to give the student the necessary practice and build up intuition. They begin with the more concrete (metric spaces) and move outward to the more general ideas. The basic notions about topological spaces are introduced in the middle of the study of metric spaces to illustrate the idea of increasing abstraction and to highlight the most important properties of metric spaces against a background where these properties fail. The result is an exposition that is not as efficient as it could be if the more abstract definitions were stated in the first place. In particular, many of the basic ideas about metric spaces (Chapter II) are rehashed in the introductory chapter on topological spaces (Chapter III).

Just as in any mathematics course, solving problems is essential. There are many exercises in the notes, particularly in the early chapters. They vary in difficulty but it is fair to say that a majority of the problems require some thought. Few, if any, could be genuinely called “trivial.” For example, in Chapter I (Sets) there are no problems of the sort:

$$\text{Prove that } A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

It is assumed that students are sufficiently sophisticated not to need that sort of drill.

There are “Chapter Reviews” at the end of each chapter. Each review consists of a list of statements and asks for either an explanation or a counterexample. Presented with statements whose truth is uncertain, students can develop confidence and intuition, learn to make thoughtful connections and guesses, and build a tool chest of examples and counterexamples. Nearly every review statement requires only an insight, a use of an earlier result in a new situation, or the application of a more abstract result in a concrete situation. For almost every true statement, an appropriate explanation consists of at most a few sentences.

These notes were a “labor of love” over many years and have been intended as an aid for students, not as a work for publication. Such originality as there is lies in the selection of material and its organization. Many proofs and exercises have been refashioned but others are more-or-less standard fare drawn from sources some of which are now forgotten. Readers familiar with the material will probably recognize overtones of my predecessors and contemporaries such as Arthur H. Stone, Leonard Gillman, Robert McDowell and Stephen Willard. My thanks to them for all their insights and contributions, and to a few hundred students who have worked with various parts of these notes over the years. Nevertheless, any errors are my own.

The notes are organized into ten chapters (I, II, ..., X) and each chapter is divided into sections (1, 2, ..., n). Definitions, theorems, and examples are numbered consecutively within each of these sections – for example, Definition 4.1, Theorem 4.2, Theorem 4.3, Example 4.4. A reference to Theorem 6.4 refers to the 4th numbered item in Section 6 of the current chapter. A reference to an item outside the current chapter would include the chapter number: for example, Theorem III.6.4 means the 4th item in Section 6 of Chapter III.

Exercises are numbered consecutively within each chapter: E1, E2, A reference to an exercise outside the current chapter would include the chapter number – for example, Exercise III.E8.

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