(See http://www.math.wustl.edu/~freiwald/Math131/reciprocal.pdf

The derivative of the reciprocal of a function $\frac{1}{g}$:

Suppose $g(x)$ is differentiable. Then $(\frac{1}{g(x)})' = \frac{d}{dx}(\frac{1}{g(x)}) = \frac{-g'(x)}{(g(x))^2}$, provided $g'(x) \neq 0$

Proof: $(\frac{1}{g(x)})' = \lim_{h \to 0} \frac{\frac{1}{g(x+h)} - \frac{1}{g(x)}}{h} = \lim_{h \to 0} \frac{\frac{g(x) - g(x+h)}{g(x)\,g(x+h)}}{h} = \lim_{h \to 0} \frac{g(x) - g(x+h)}{h\,g(x)\,g(x+h)}$

$$= \lim_{h \to 0} -\frac{g(x+h) - g(x)}{h\,g(x)\,g(x+h)} = \lim_{h \to 0} \frac{-\frac{g(x+h) - g(x)}{h}}{g(x)\,g(x+h)} = \lim_{h \to 0} -\left(\frac{g(x+h) - g(x)}{h}\right) \cdot \frac{1}{g(x)\,g(x+h)}$$

Since g is differentiable, $\lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = g'(x)$.

Since $g'(x)$ exists, g must be continuous at x, so $\lim_{h \to 0} g(x+h) = g(x)$.

Therefore,

$$(\frac{1}{g(x)})' = \lim_{h \to 0} \frac{-\frac{g(x+h) - g(x)}{h}}{g(x)\,g(x+h)} = \frac{-g'(x)}{(g(x))^2}.$$

The derivative of a quotient $\frac{f}{g}$:

Suppose f and g are differentiable. Then $(\frac{f(x)}{g(x)})' = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$, provided $g(x) \neq 0$.

Proof: Most of the work is already done. We just make use of the product rule and the rule for the derivative of a reciprocal.

$$(\frac{f(x)}{g(x)})' = (f(x) \cdot \frac{1}{g(x)})'$$

(using the product rule) $= f'(x) \cdot \frac{1}{g(x)} + f(x) \cdot (\frac{1}{g(x)})'$

(using the reciprocal rule) $= f'(x) \cdot \frac{1}{g(x)} + f(x) \cdot (\frac{1}{g(x)})'$

$$= \frac{f'(x)}{g(x)} + f(x) \cdot (\frac{-g'(x)}{g(x)^2}) = \frac{f'(x)}{g(x)} + \frac{-f(x)\,g'(x)}{g(x)^2}$$

$$= \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$$