Chapter II
Metric and Pseudometric Spaces

1. Introduction

By itself, a set doesn't have any "structure." For two arbitrary sets A and B, we can ask questions like "Is $A = B$?" or "Is A equivalent to a subset of B?" but not much more. If we add additional structure to a set, it becomes more interesting. For example, if we define a "multiplication operation" $a \cdot b$ in X that satisfies certain axioms (such as $a \cdot (b \cdot c) = (a \cdot b) \cdot c$)), then X becomes an algebraic structure called a group and a whole area of mathematics known as "group theory" begins.

We are not interested in making a set X into an algebraic system. For topology, we want additional structure on a set X for a different purpose: to talk about "nearness" in X. This is what we need to discuss topics like "convergence" and "continuity" – roughly, "f is continuous at a" means that "if x is near a, then $f(x)$ is near $f(a)$."

The simplest way to talk about "nearness" is to equip the set X with a "distance function" d to tell us "how far apart" two elements of X are.

Note: As we proceed we may use ideas taken from elementary analysis, such as the continuity of a function $f : \mathbb{R}^n \to \mathbb{R}^m$ as a source for motivation or examples, although these ideas will not be carefully defined until later in this chapter.

2. Metric and Pseudometric Spaces

Definition 2.1 Suppose $d : X \times X \to \mathbb{R}$ and that for all $x, y, z \in X$:

1) $d(x, y) \geq 0$
2) $d(x, x) = 0$
3) $d(x, y) = d(y, x)$ (symmetry)
4) $d(x, z) \leq d(x, y) + d(y, z)$ (the triangle inequality)

Such a "distance function" d is called a pseudometric on X. The pair (X, d) is called a pseudometric space. If d also satisfies

5) when $x \neq y$, then $d(x, y) > 0$

then d is called a metric on X and (X, d) is called a metric space. Of course, every metric space is automatically a pseudometric space.

If a pseudometric space (X, d) is not a metric space, it is because there are at least two points $x \neq y$ for which $d(x, y) = 0$. In most situations this doesn't happen; metrics come up in mathematics more often than pseudometrics. However pseudometrics do occasionally arise in a natural way. Moreover, many definitions and theorems actually only require using properties 1)-4). Therefore we will state our results in terms of pseudometrics rather than metrics in situations where 5) is irrelevant.

Example 2.2

1) The usual metric on $\mathbb{R}$ is $d(x,y) = |x - y|$. Clearly, properties 1) - 5) are true. In fact, 1)-5) are chosen so that a metric imitates the usual distance function.

2) The usual metric on $\mathbb{R}^n$ is defined as follows: if $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$ are in $\mathbb{R}^n$, then $d(x,y) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$. You should already know that d has properties 1) - 5). But the details for verifying the triangle inequality are a little tricky, so we will go through the steps. First, we prove another useful inequality.

Suppose $A = (a_1, a_2, ..., a_n)$ and $B = (b_1, b_2, ..., b_n)$ points in $\mathbb{R}^n$. Define

$$P(w) = \sum_{i=1}^{n}(a_i + wb_i)^2 = \sum_{i=1}^{n}a_i^2 + (2\sum_{i=1}^{n}a_i b_i)w + (\sum_{i=1}^{n}b_i^2)w^2$$

$P(w)$ is a quadratic function of w, and $P(w) \geq 0$ because $P(w)$ is a sum of squares. Therefore the equation $P(w) = 0$ has at most one real root, so it follows from the quadratic formula that

$$(2\sum_{i=1}^{n}a_i b_i)^2 - 4(\sum_{i=1}^{n}a_i^2)(\sum_{i=1}^{n}b_i^2) \leq 0, \qquad \text{which gives}$$

$$|\sum_{i=1}^{n}a_i b_i| \leq (\sum_{i=1}^{n}a_i^2)^{1/2}(\sum_{i=1}^{n}b_i^2)^{1/2}$$

This last inequality is called the Cauchy-Schwarz inequality. In vector notation it could be written in the form $|A \cdot B| \leq ||A|| \cdot ||B||$.

Then if $x = (x_1, x_2, ..., x_n)$, $y = (y_1, y_2, ..., y_n)$ and $z = (z_1, z_2, ..., z_n)$ are in $\mathbb{R}^n$, we can calculate

$$d(x,z)^2 = \sum_{i=1}^{n}(x_i - z_i)^2 = \sum_{i=1}^{n}((x_i - y_i) + (y_i - z_i))^2$$

$$= \sum_{i=1}^{n}(x_i - y_i)^2 + 2\sum_{i=1}^{n}(x_i - y_i)(y_i - z_i) + \sum_{i=1}^{n}(y_i - z_i)^2$$

$$\leq \sum_{i=1}^{n}(x_i - y_i)^2 + 2\sum_{i=1}^{n}|x_i - y_i||y_i - z_i| + \sum_{i=1}^{n}(y_i - z_i)^2$$

$$\leq \sum_{i=1}^{n}(x_i - y_i)^2 + 2(\sum_{i=1}^{n}(x_i - y_i)^2)^{1/2}(\sum_{i=1}^{n}(x_i - y_i)^2)^{1/2} + \sum_{i=1}^{n}(y_i - z_i)^2$$

$= (d(x,y) + d(y,z))^2$. Taking the square root of both sides gives

$d(x,z) \leq d(x,y) + d(y,z)$.

Example 2.3 We can also put other "unusual" metrics on the set $\mathbb{R}^n$.

1) Let d be the usual metric on $\mathbb{R}^n$ and define $d'(x,y) = 100d(x,y)$. Then d' is also a metric on $\mathbb{R}^n$. In $(\mathbb{R}^n, d')$, the "usual" distances are stretched by a factor of 100. It is as if we simply changed the units of measurement from meters to centimeters and that change shouldn't matter in any important way. In fact, it's easy to check that if d is any metric (or pseudometric) on a set X and $\alpha > 0$, then $d' = \alpha \cdot d$ is also a metric (or pseudometric) on X.

2) If $x = (x_1, x_2, ..., x_n)$, $y = (y_1, y_2, ..., y_n)$ are points in $\mathbb{R}^n$, define

$$d_t(x,y) = \sum_{i=1}^{n} |x_i - y_i|$$

It is easy to check that d_t satisfies properties 1) - 5) so $(\mathbb{R}^n, d_t)$ is a metric space. We call d_t the taxicab metric on $\mathbb{R}^n$. (*For $n = 2$, distances are measured as if you had to move along a rectangular grid of city streets from x to y – the taxicab cannot cut diagonally across a city block*).

3) If $x = (x_1, x_2, ..., x_n)$, $y = (y_1, y_2, ..., y_n)$ are points in $\mathbb{R}^n$, define

$$d^*(x,y) = \max\{|x_i - y_i| : i = 1, 2, ..., n\}$$

Then $(\mathbb{R}^n, d^*)$ is also a metric space. We will refer to d^* as the max metric on $\mathbb{R}^n$.

When $n = 1$, of course, d, d_t and d^* are exactly the same metric on $\mathbb{R}$.

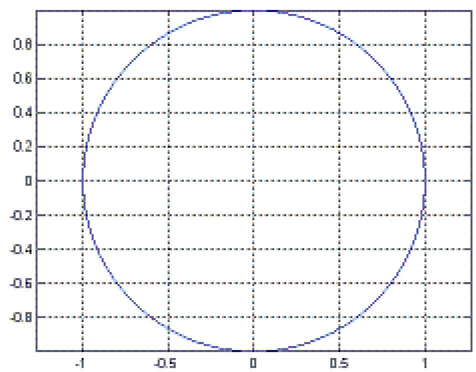
We will see later that d', d_t, d^* are all "equivalent" metrics on $\mathbb{R}^n$ for "topological" purposes. Roughly, this means that whichever of these metrics is used in $\mathbb{R}^n$, exactly the same functions are continuous and exactly the same sequences converge.

4) The “unit sphere” S^1 is the set of points in $\mathbb{R}^2$ that are at distance 1 from the origin. Sketch the unit sphere in $\mathbb{R}^2$ using the metrics $d,\ d_t,\ d^*$, and $d\,' = 100d$.

Since there are only two coordinates, we will write a point in $\mathbb{R}^2$ in the usual way as (x, y) rather than (x_1, x_2).

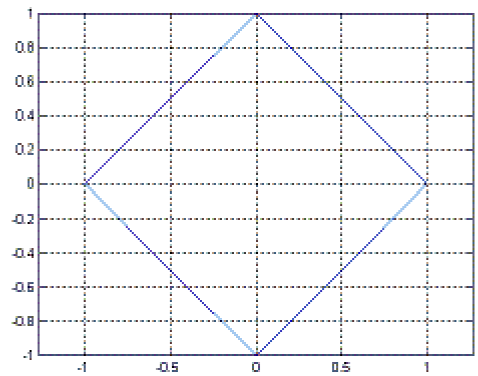
For d, we get

$$S^1 = \{(x,y) :\ d(\,(x,y),(0,0)) = 1\}$$
$$= \{(x,y) :\ x^2 + y^2 = 1\}$$



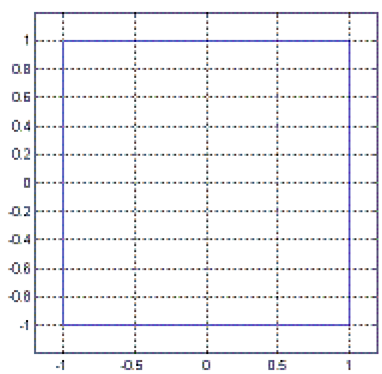
For d_t, we get

$$S^1 = \{(x,y) :\ d_t(\,(x,y),(0,0)) = 1\}$$
$$= \{(x,y) :\ |x| + |y| = 1\}$$



For d^*, we get

$$S^1 = \{(x,y) :\ d^*(\,(x,y),(0,0)) = 1\}$$
$$= \{(x,y) :\ \max\{|x|,\ |y|\} = 1\}$$



Of course for the metric $d\,' = 100d$, S^1 has the same shape as for the metric d, but the sphere is reduced in size by a scaling factor of 100.

Switching among the metrics $d,\ d\,',\ d_t,\ d^*$ produces unit spheres in $\mathbb{R}^n$ with different sizes and shapes. In other words, changing the metric on $\mathbb{R}^n$ may cause dramatic changes in the geometry of the space – for example, “areas” may change and “spheres” may no longer be “round.” Changing the metric can also affect smoothness features of the space (spheres may turn out to have sharp corners). But it turns out, as mentioned earlier, that $d,\ d\,',\ d_t$ and d^* are “equivalent” for “topological purposes.” For topology, “size,” “geometrical shape,” and “smoothness” don't matter.

When talking about $\mathbb{R}^n$, the usual metric d is the default – that is, we always assume that $\mathbb{R}^n$, or any subset of $\mathbb{R}^n$, has the usual metric d unless a different metric is explicitly stated.

Example 2.4 In each part, you should verify that d satisfies the properties of a pseudometric or metric.

1) For a set X, define $d(x, y) = 0$ for all $x, y \in X$. We call d the trivial pseudometric on X: all distances are 0. (*Under what circumstances is this* d *a metric?*)

2) For a set X, define $d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$. We call d the discrete unit metric on X.
To verify the triangle inequality: for points $x, y, z \in X$, $d(x, z) \leq d(x, y) + d(y, z)$ certainly is true if $x = z$; and if $x \neq z$, then $d(x, z) = 1$ and $d(x, y) + d(y, z) \geq 1$.

Definition 2.5 Suppose (X, d) is a pseudometric space, that $x_0 \in X$ and $\epsilon > 0$. Then $B_\epsilon(x_0) = \{x \in X : d(x, x_0) < \epsilon\}$ is called the ball of radius ϵ with center at x_0.

If there exists an $\epsilon > 0$ such that $B_\epsilon(x_0) = \{x_0\}$, then we say that x_0 is an isolated point in (X, d).

Example 2.6

1) In $\mathbb{R}$, $B_\epsilon(x_0) = (x_0 - \epsilon, x_0 + \epsilon)$. More generally, $B_\epsilon(x_0)$ in $\mathbb{R}^n$ is just the usual spherical ball with radius ϵ and center at x_0 (not including the boundary surface). If the metric d_t is used in $\mathbb{R}^n$, then $B_\epsilon(x_0)$ is the interior of a "diamond-shaped" region centered at x_0. (*See the earlier sketches of* S^1*: in* $(\mathbb{R}^2, d_t)$*,* $B_1((0,0))$ *is the region "inside" the diamond-shaped* S^1*.*)

In $X = [0, 1]$ with the usual metric d, then $B_{\frac{1}{2}}(0) = [0, \frac{1}{2})$, $B_1(0) = [0, 1)$, $B_2(0) = [0, 1]$.

2) If d is the trivial pseudometric on X and $x_0 \in X$, then $B_\epsilon(x_0) = X$ for every $\epsilon > 0$.

3) If d is the discrete unit metric on X, then $B_\epsilon(x_0) = \begin{cases} \{x_0\} & \text{if } \epsilon \leq 1 \\ X & \text{if } \epsilon > 1 \end{cases}$. Therefore every point x_0 in (X, d) is isolated. The same is true if we rescale and replace d by the metric αd (where $\alpha > 0$).

4) Let $C([0, 1]) = \{f \in \mathbb{R}^{[0,1]} : f \text{ is continuous}\}$. For $f, g \in C([0, 1])$, define

$$d(f, g) = \int_0^1 |f(x) - g(x)|\, dx \quad (*)$$

It is easy to check that d is a pseudometric on $C([0, 1])$. In fact d is a metric: if $f \neq g$, then there must be a point $x_0 \in [0, 1]$ where $|f(x_0) - g(x_0)| > 0$. By continuity, $|f(x) - g(x)| > 0$ for x's near x_0, that is, $|f(x) - g(x)| > 0$ on some interval $[a, b] \subseteq [0, 1]$, where $x_0 \in [a, b]$. (*carefully explain why!*). Let $m = \min_{x \in [a,b]} |f(x) - g(x)|$ (*why does* m *exist?*). Then $m > 0$, so

$$d(f, g) = \int_0^1 |f(x) - g(x)|\, dx \geq \int_a^b |f(x) - g(x)|\, dx \geq \int_a^b m\, dx = m(b - a) > 0.$$

Therefore, d is a metric on $C([0, 1])$.

$C([0,1])$ is a subset of the larger set $Y = \{f \in \mathbb{R}^{[0,1]} : f$ is integrable$\}$. We can define a distance function d on Y the same formula (*). In this case, d is a pseudometric on Y but not a metric. If

$$f(x) = 0 \text{ for all } x, \text{and } g(x) = \begin{cases} 0 & \text{if } x \neq \frac{1}{2} \\ 1 & \text{if } x = \frac{1}{2} \end{cases}, \text{ then } f \neq g \text{ but}$$
$$d(f,g) = \int_0^1 |f(x) - g(x)|\, dx = 0$$

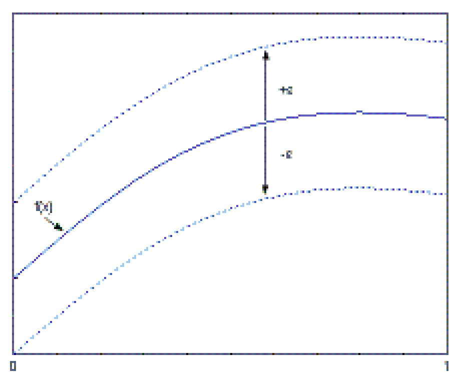
This example shows how a pseudometric that is not a metric can arise naturally in analysis.

5) On $C([0,1])$ we can also define another metric d^* by

$$\begin{aligned} d^*(f,g) &= \sup \ \{|f(x) - g(x)| : x \in [0,1]\} \\ &= \max \{|f(x) - g(x)| : x \in [0,1]\} \end{aligned}$$

(*Replacing "sup" with "max" makes sense because a theorem from analysis says that the continuous function* $|f - g|$ *has a maximum value on the closed interval* $[0,1]$.)

Then $d^*(f,g) < \epsilon$ if and only if $|f(x) - g(x)| < \epsilon$ at every point $x \in [0,1]$, so we can picture $B_\epsilon(f)$ in $(C([0,1]), d^*)$ as the set of all functions $g \in C([0,1])$ whose graph lies entirely inside a "tube of width ϵ" containing the graph of f – that is, $g \in B_\epsilon(f)$ iff g is "uniformly within ϵ of f on $[0,1]$." See the following figure.

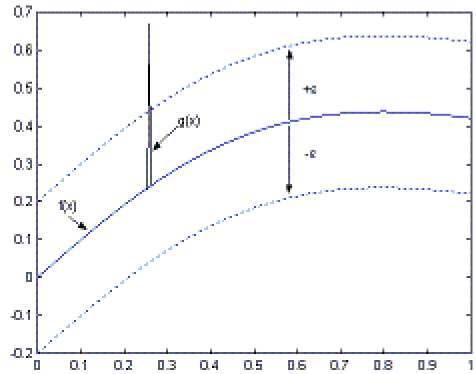


How are the metrics d and d^* from Examples 4) and 5) related? Notice that for $f, g \in C([0,1])$:

$$d(f,g) = \int_0^1 |f(x) - g(x)|\, dx \leq \int_0^1 \max_{x \in [0,1]} |f(x) - g(x)|\, dx = \int_0^1 d^*(f,g)\, dx = d^*(f,g)\,.$$

We abbreviate this observation by writing $d \leq d^*$. It follows that $B_\epsilon^{d^*}(f) \subseteq B_\epsilon^{d}(f)$: so, for a given $\epsilon > 0$, the larger metric produces the smaller ball. (*Note: the superscript notation on the balls indicates which metric is being used in each case.*)

The following below shows a function f and the graph of a function $g \in B_\epsilon^d(f) - B_\epsilon^{d^*}(f$. The graph of g coincides with the graph of f, except for a tall spike: the spike takes the graph of g outside the "ϵ-tube" around the graph of f, but the spike is so thin that the $d(f,g) = \int_0^1 |f(x) - g(x)|\, dx$ $=$ "the total area between the graphs of f and g" $< \epsilon$.



6) Let $\ell_2 = \{f \in \mathbb{R}^{\mathbb{N}} : \sum_{k=1}^{\infty} f^2(k) \text{ converges}\}$. If we write $f(k) = x_k$ and use the more informal sequence notation, then $\ell_2 = \{(x_k) : x_k \in \mathbb{R} \text{ and } \sum_{k=1}^{\infty} x_k^2 \text{ converges}\}$. Thus, ℓ_2 is the set of all "square-summable" sequences of real numbers.

Suppose $x = (x_k)$ and $y = (y_k)$ are in ℓ_2 and that $a, b \in \mathbb{R}$. We claim that the sequence $ax + by = (ax_k + by_k)$ is also in ℓ_2. To see this, look at partial sums:

$$\sum_{k=1}^{n}(ax_k + by_k)^2 = a^2\sum_{k=1}^{n}x_k^2 + 2ab\sum_{k=1}^{n}x_k y_k + b^2\sum_{k=1}^{n}y_k^2 \le a^2\sum_{k=1}^{n}x_k^2 + |2ab\sum_{k=1}^{n}x_k y_k| + b^2\sum_{k=1}^{n}y_k$$

$$\le a^2\sum_{k=1}^{n}x_k^2 + 2|a||b|(\sum_{k=1}^{n}x_k^2)^{1/2}(\sum_{k=1}^{n}y_k^2)^{1/2} + b^2\sum_{k=1}^{n}y_k$$ (*by the Cauchy-Schwarz inequality*)

$$\le a^2\sum_{k=1}^{\infty}x_k^2 + 2|a||b|(\sum_{k=1}^{\infty}x_k^2)^{1/2}(\sum_{k=1}^{\infty}y_k^2)^{1/2} + b^2\sum_{k=1}^{\infty}y_k = M \in \mathbb{R}$$ (*all the series converge because* $x, y \in \ell_2$.)

Therefore the nonnegative series $\sum_{k=1}^{\infty}(ax_k + by_k)^2$ converges because it has bounded partial sums. This means that $ax + by \in \ell_2$.

In particular, if $x, y \in \ell_2$, we now know that $x - y \in \ell_2$ so $\sum_{k=1}^{\infty}(x_k - y_k)^2$ converges. Therefore it makes sense to define $d(x, y) = (\sum_{k=1}^{\infty}(x_k - y_k)^2)^{1/2}$. You should check that d is a metric on ℓ_2.
(*For the triangle inequality, notice that* $(\sum_{k=1}^{n}(x_k - z_k)^2)^{1/2} \le (\sum_{k=1}^{n}(x_k - y_k)^2)^{1/2} + (\sum_{k=1}^{n}(y_k - z_k)^2)^{1/2}$ *by the triangle inequality in* $\mathbb{R}^n$. *Letting* $n \to \infty$ *gives the triangle inequality for* ℓ_2.)

7) Suppose (X_i, d_i) are pseudometric spaces $(i = 1, ..., n)$, and that $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$ are points in the product $X = X_1 \times ... \times X_n$. Then each of the following is a pseudometric on X :

$$d(x,y) = (\sum_{i=1}^{n} d_i^2(x_i, y_i))^{1/2} \qquad\qquad d_t(x,y) = \sum_{i=1}^{n} d_i(x_i, y_i)$$

$$d^*(x,y) = \max\{d_i(x_i, y_i) : i = 1, ..., n\}$$

If each d_i is a metric, then so are d, d_t, and d^*. Notice that if each $X_i = \mathbb{R}$ and each d_i is the usual metric on $\mathbb{R}$, then d, d_t, and d^* are just the usual metric, the taxicab metric, and the max metric on $\mathbb{R}^n$. As we shall see, it turns out that these metrics on X are all equivalent for "topological purposes."

Definition 2.7 Suppose (X, d) is a pseudometric space and $O \subseteq X$. We say that $\underline{O}$ <u>is open in</u> $\underline{(X, d)}$ if for each $x \in O$ there is an $\epsilon > 0$ such that $B_\epsilon(x) \subseteq O$. (*Of course, ϵ may depend on x.*)

For example,

i) The sets $\emptyset$ and X are open in any space (X, d).

ii) The intervals (a, b), $(-\infty, a)$, (b, ∞), and $(-\infty, \infty) = \mathbb{R}$ are open in $\mathbb{R}$.
(*Fortunately this terminology is consistent with the fact that these intervals were called "open intervals" in calculus books.*)

But notice that the interval (a, b), when viewed as a subset of the x-axis in $\mathbb{R}^2$, is <u>not</u> open in $\mathbb{R}^2$. Similarly, $\mathbb{R}$ is an open set in $\mathbb{R}$, but $\mathbb{R}$ (viewed as the x-axis) is <u>not</u> open in $\mathbb{R}^2$.

iii) The intervals $[a, b]$, $[a, b)$ and $(a, b]$ are not open in $\mathbb{R}$. But the sets $[a, b)$ and $[a, b]$ are open in the metric space $([a, b], d)$.

Examples ii) and iii) illustrate that "open" is not an a property that depends just on the set A: whether or not a set A is open depends on the larger space in which it is contained – that is, "open" is a relative term.

The next theorem tells us that the balls in (X, d) are the "building blocks" from which all open sets can be constructed.

Theorem 2.8 A set $O \subseteq X$ is open in (X, d) if and only if O is a union of a collection of balls.

Proof If O is open, then for each $x \in O$ there is an $\epsilon_x > 0$ such that $B_{\epsilon_x}(x) \subseteq O$ and $O = \bigcup_{x \in O} B_{\epsilon_x}(x)$.

Conversely, suppose $O = \bigcup_{x \in C} B_{\epsilon_x}(x)$ for some indexing set $C \subseteq O$. We must show that if $y \in O$, then $B_\epsilon(y) \subseteq O$ for some $\epsilon > 0$. Since $y \in O$, we know that $y \in B_{\epsilon_{x_0}}(x_0)$ for some $x_0 \in C$. Then $d(x_0, y) = \delta < \epsilon_{x_0}$. Let $\epsilon = \frac{1}{2}(\epsilon_{x_0} - \delta) > 0$ and consider $B_\epsilon(y)$. If $z \in B_\epsilon(y)$, then
$d(z, x_0) \le d(z, y) + d(y, x_0) < \epsilon + \delta = \frac{1}{2}(\epsilon_{x_0} - \delta) + \delta = \frac{1}{2}\epsilon_{x_0} + \frac{1}{2}\delta < \frac{1}{2}\epsilon_{x_0} + \frac{1}{2}\epsilon_{x_0} = \epsilon_{x_0}$,
so $z \in B_{\epsilon_{x_0}}(x_0)$. Therefore $B_\epsilon(y) \subseteq B_{\epsilon_{x_0}}(x_0) \subseteq O$. ∙

Corollary 2.9 a) Each ball $B_\epsilon(x)$ is open in (X, d).
b) A point x_0 in a pseudometric space (X, d) is isolated iff $\{x_0\}$ is an open set.

Definition 2.10 Suppose (X, d) is a pseudometric space. The topology $\mathcal{T}_d$ generated by d is the collection of all open sets in (X, d). In other words, $\mathcal{T}_d = \{O : O$ is open in $(X, d)\} = \{O : O$ is a union of balls$\}$.

Theorem 2.11 Let $\mathcal{T}_d$ be the topology in (X, d). Then

i) $\emptyset, X \in \mathcal{T}_d$
ii) if $O_\alpha \in \mathcal{T}_d$ for each $\alpha \in A$, then $\bigcup_{a \in A} O_a \in \mathcal{T}_d$
iii) if $O_1, ..., O_n \in \mathcal{T}_d$, then $O_1 \cap ... \cap O_n \in \mathcal{T}_d$.

(Conditions ii) and iii) say that the collection $\mathcal{T}_d$ is "closed under unions" and "closed under finite intersections.")

Proof $\emptyset$ is the union of the empty collection of open balls, and $X = \bigcup_{x \in X} B_1(x)$, so $\emptyset, X \in \mathcal{T}_d$.

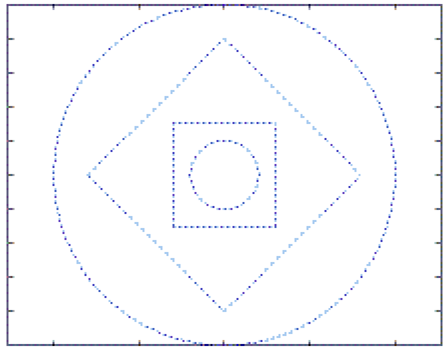
Suppose $x \in O = \bigcup_{\alpha \in A} O_\alpha$ where each $O_\alpha \in \mathcal{T}_d$. Then x is in one of these open sets, say O_{α_0}. So for some $\epsilon > 0$, $x \in B_\epsilon(x) \subseteq O_{\alpha_0} \subseteq O$. Therefore O is open, that is, $O \in \mathcal{T}_d$.

To verify iii), suppose $O_1, O_2, ..., O_n \in \mathcal{T}_d$ and that $x \in O_1 \cap O_2 \cap ... \cap O_n$. For each $i = 1, ..., n$, there is an $\epsilon_i > 0$ such that $x \in B_{\epsilon_i}(x) \subseteq O_i$. Let $\epsilon = \min\{\epsilon_1, \epsilon_2, ..., \epsilon_n\} > 0$. Then $B_\epsilon(x) \subseteq O_1 \cap O_2 \cap ... \cap O_n$. Therefore $O_1 \cap O_2 \cap ... \cap O_n \in \mathcal{T}_d$. •

Example 2.12 The set $O_n = (-\frac{1}{n}, \frac{1}{n})$ is open in $\mathbb{R}$ for every $n \in \mathbb{N}$. However, $\bigcap_{n=1}^{\infty} O_n = \{0\}$ is not open in $\mathbb{R}$: so an intersection of infinitely many open sets might not be open. (*Where does the proof for part iii) in Theorem 2.11 break down if we intersect infinitely many open sets?*)

Notice that different pseudometrics can produce the same topology on a set X. For example, if d is a metric on X and we set $d' = 2d$, then d and d' produce the same collection of balls (with radii measured differently): for each $\epsilon > 0$, the ball $B_\epsilon^d(x)$ is the same set as the ball $B_{2\epsilon}^{d'}(x)$. If we get the same balls from each metric, then we must also get the same open sets: $\mathcal{T}_d = \mathcal{T}_{d'}$ (*see Theorem 2.8*).

We can see a less trivial example in $\mathbb{R}^2$. Let d, d_t, and d^* be the usual metric, the taxicab metric, and the max metric on $\mathbb{R}^2$. Clearly any set which is a union of d-balls (or d^* balls) can also be written as a union of d_t-balls, and vice-versa. (*Explain why! See the following picture for* $\mathbb{R}^2$.)



Therefore all three metrics produce the same topology: $\mathcal{T}_d = \mathcal{T}_{d_t} = \mathcal{T}_{d^*}$ even though the balls are different for each metric. It turns out that the open sets in (X, d) are the most important objects from a "topological" point of view, so in that sense these metrics are all equivalent. (As mentioned above, these metrics do change the "shape" and "smoothness" of the balls and therefore these metrics are not equivalent "for geometric purposes.")

Definition 2.13 Suppose d and d' are two pseudometrics (or metrics) on a set X. We say that d and d' are equivalent (written $d \sim d'$) if $\mathcal{T}_d = \mathcal{T}_{d'}$, that is, if d and d' generate the same collection of open sets.

Example 2.14

1) If d is the discrete unit metric on X, then each singleton set $\{x\} = B_1(x)$ is a ball, so each $\{x\}$ is open – equivalently, every point x is isolated in (X, d). If $A \subseteq X$, then $A = \bigcup_{a \in A} \{a\}$ is open because A is a union of balls. Therefore $\mathcal{T}_d = \mathcal{P}(X)$, called the discrete topology on X

If $d'(x, y) = \begin{cases} 17 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$ on a set X, then $d' \sim d$, where d is the discrete unit metric.

More generally, $\alpha d \sim d$ for any $\alpha > 0.$, and all of these metrics generate the discrete topology.

2) Let $X = \{ \frac{1}{n} : n \in \mathbb{N} \}$. Let d be the usual metric on X and let d' be the discrete unit metric on X. For each n, $B^d_\epsilon(\frac{1}{n}) = \{\frac{1}{n}\}$ if we choose a sufficiently small ϵ. Therefore, just as in part 1), every subset of X is open in (X, d). But every subset in (X, d') also is open, so $d \sim d'$ (even though d and d' are not constant multiples of each other).

3) If d is the trivial pseudometric on a set X, are there any other pseudometrics d' on X for which $d' \sim d$?

3. The topology of $\mathbb{R}$

What do the open sets in $\mathbb{R}$ look like? Since an ϵ-ball in $\mathbb{R}$ is an interval of the form $(a - \epsilon, a + \epsilon)$, the open sets are precisely the sets which are unions of open intervals. But we can say more to make the situation even clearer. We begin by making precise the definition of "interval."

Definition 3.1 A subset I of $\mathbb{R}$ is convex if whenever $x \le y \le z$ and $x, z \in I$, then $y \in I$. A convex subset of $\mathbb{R}$ is called an interval.

It is easy to give examples of intervals in $\mathbb{R}$. The following theorem states that the obvious examples are the only examples.

Theorem 3.2 $I \subseteq \mathbb{R}$ is an interval iff I has one of the following forms (where $a < b$):

$$(-\infty, \infty),\ (-\infty, a),\ (-\infty, a],\ [a,\infty),\ (a,\infty),\ (a,b),\ [a, b),\ (a, b],\ [a, b],\ \{a\},\ \emptyset \qquad (*)$$

Proof It is clear that each of the sets in the list is convex and therefore is an interval.

Conversely, we need to show that every interval I has one of these forms. Clearly, if $|I| \le 1$, then $I = \emptyset$ or $I = \{a\}$. If $|I| \ge 2$ then the definition of interval implies that I must be infinite.

The remainder of the proof uses the completeness property (= "least upper bound property") of $\mathbb{R}$, and the argument falls into several cases:

Case I: I is bounded both above and below. Then I has a least upper bound and a greatest lower bound: let $a = \inf I$ and $b = \sup I$. Of course a and b may or may not be in I.

a) if $a, b \in I$, we claim $I = [a, b]$
b) if $a \in I$ but $b \notin I$, we claim $I = [a, b)$
c) if $a \notin I$ but $b \in I$, we claim $I = (a, b]$
d) if $a, b \notin I$, we claim $I = (a, b)$

Case II: I is bounded below but not above. In this case, let $a = \inf I$.
a) if $a \in I$, we claim $I = [a, \infty)$
b) if $a \notin I$, we claim $I = (a, \infty)$

Case III: I is bounded above but not below. In this case, let $b = \sup I$.
a) if $b \in I$, we claim $I = (-\infty, b]$
b) if $b \notin I$, we claim $I = (-\infty, b)$

Case IV: I is not bounded above or below. In this case, we claim $I = (-\infty, \infty)$.

The proof is similar in each case, using properties of sups and infs. To illustrate, We prove case Ic):

If $x \in I$, then $x \le \sup I = b$. Also, $x \ge \inf I = a$, and because $a \notin I$, we get $x > a$. So $I \subseteq (a, b]$.

We need to show that $I \subseteq (a, b]$, so suppose $x \in (a, b]$. Then $x > a = \inf I$, so x is not a lower bound for I. This means that there is a point $z \in I$ such that $z < x$. Then $z < x \le b$ where $z, b \in I$, and I is convex, so $x \in I$. Therefore $(a, b] \subseteq I$, so $I = (a, b]$. ∙

Note: We used the completeness property to prove Theorem 3.2. In fact, Theorem 3.2 is equivalent to the completeness property. To see this :

> *Assume Theorem 3.2 is true and that A is a nonempty subset of $\mathbb{R}$ that has an upper bound. Let $I = \{x \in \mathbb{R} : x \leq c$ for some $c \in A\}$. Then I is an interval (suppose $x \leq y \leq z$ where $x, z \in I$. Then $z \leq c$ for some $c \in A$; therefore $y \leq c$ so, by definition of I, $y \in I$.).*
>
> *Since $A \neq \emptyset$, I must be infinite. An upper bound for A must also be an upper bound for I. Since I is an interval with an upper bound, I must have one of the forms $(-\infty, b]$, $(-\infty, b)$, (a,b), $[a, b)$, $(a, b]$, $[a, b]$, Then it's not hard to check that A has a least upper bound, namely sup $A = b$.*

This is an observation I owe to Professor Robert McDowell.

It is clear that an intersection of intervals in $\mathbb{R}$ is an interval (*why*?). Also, a union of intervals need not be an interval: for example $[0,1] \cup [2,3]$. But if every pair of intervals in a collection "overlap," then the union is an interval. The following theorem makes this precise.

Theorem 3.3 Suppose $\mathcal{I}$ is a collection of intervals in $\mathbb{R}$. If $I \cap J \neq \emptyset$ for all $I, J \in \mathcal{I}$, then $\bigcup \mathcal{I}$ is an interval. In particular, if $\bigcap \mathcal{I} \neq \emptyset$, then $\bigcup \mathcal{I}$ is an interval.

Proof Let $a, b \in \bigcup \mathcal{I}$. Then there are intervals I, J in $\mathcal{I}$ with $a \in I$ and $b \in J$.
Suppose $a \leq x \leq b$.

Pick a point $z \in I \cap J$. If $x = z$, then $x \in \bigcup \mathcal{I}$ and we are done. Otherwise, either $z < x \leq b$ or $a \leq x < z$. Therefore either x is either between two points of J and so $x \in J$; or x is between two points of I, so $x \in I$. Either way, we conclude that
$x \in \bigcup \mathcal{I}$. •

We can now give a more careful description of the open sets in $\mathbb{R}$.

Theorem 3.4 Suppose $O \subseteq \mathbb{R}$. O is open in $\mathbb{R}$ if and only if O is the union of a countable collection of pairwise disjoint open intervals.

Proof ($\Leftarrow$) Open intervals in $\mathbb{R}$ are open sets, and a union of any collection of open sets is open.

($\Rightarrow$) Suppose O is open in $\mathbb{R}$. For each $x \in O$, there is an open interval (ball) I such that $x \in I \subseteq O$. Let $G_x = \bigcup \{I : I$ is an open interval and $x \in I \subseteq O\}$. Then $x \in G_x \subseteq O$.
By Theorem 3.3, G_x is also an open interval (*in fact, G_x is the largest open interval containing x and inside O; why?*). It is easy to see that there can be distinct points $x, y \in O$ for which $G_x = G_y$. In fact, we claim that if $x, y \in O$, then either $G_x = G_y$ or $G_x \cap G_y = \emptyset$.

> If $G_x \cap G_y \neq \emptyset$, then there is a point $z \in G_x \cap G_y$. By Theorem 3.3, $G_x \cup G_y$ is an open interval, a subset of O, and containing both x and y. Therefore $G_x \cup G_y$ is a set in the collection whose union is G_x. Therefore $G_x \cup G_y \subseteq G_x$. Similarly, $G_x \cup G_y \subseteq G_y$, so $G_x = G_y$.

Removing any repetitions, we let $\mathcal{D}$ be the collection of the distinct intervals G_x that arise in this way. Clearly, $O = \cup\,\mathcal{D}$ and $\mathcal{D}$ is countable because the members of $\mathcal{D}$ are pairwise disjoint: for each $I \in \mathcal{D}$, we can pick a rational number $q_I \in I$, and these q_I's are distinct. (*More formally, the function* $f : \mathcal{D} \to \mathbb{Q}$ *given by* $f(I) = q_I$ *is one-to-one.*) •

We are now able to find the number of open sets in $\mathbb{R}$.

Corollary 3.5 There are exactly *c* open sets in $\mathbb{R}$.

Proof Let $\mathcal{T}_d$ be the usual topology on $\mathbb{R}$. We want to prove that $|\mathcal{T}_d| = c$.

For each $r \in \mathbb{R}$, the interval $(-\infty, r) \in \mathcal{T}_d$, so $|\,\mathcal{T}_d\,| \geq c$.

Let $\mathcal{I}$ be the set of all open intervals in $\mathbb{R}$. Then $|\,\mathcal{I}\,| = c$ (*why?*). For each $O \in \mathcal{T}_d$, pick a sequence $I_1,\ I_2, ...I_n, ... \in \mathcal{I}$ for which $O = \bigcup_{n=1}^{\infty} I_n$. (*We could also choose the* I_n*'s to be pairwise disjoint, but that is unnecessary in this argument – the important thing here is that there are only countably many* I_n*'s.*) Then we have a function $f : \mathcal{T}_d \to \mathcal{I}^{\mathbb{N}}$ given by $f(O) = (I_1,\ I_2, ..., I_n, ...)$. The function f is clearly one-to-one, so $|\,\mathcal{T}\,| \leq |\,\mathcal{I}^{\mathbb{N}}\,| = c^{\aleph_0} = c$. •

Exercises

E1. The following statements refer to a metric space (X, d). Prove the true statements and give counterexamples for the false ones. (*The statements illustrate the danger of assuming that familiar features of* $\mathbb{R}^n$ *necessarily carry over to arbitrary pseudometric spaces.*)

a) $B_\epsilon(x) = B_\epsilon(y)$ implies $x = y$ (i.e., "a ball can't have two centers")
b) The diameter of $B_\epsilon(x)$ must be bigger than ϵ. (*The diameter of a set A in a metric space is defined to be sup* $\{d(x,y) : x, y \in A\} \leq \infty$.)

E2. The "taxicab" metric on $\mathbb{R}^2$ is defined by $d_t((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$. Draw the set of points in $(\mathbb{R}^2, d_t)$ that are equidistant from $(0, 0)$ and $(3, 4)$.

E3. Suppose (X, d) is a metric space.

$a)$ Define $d^*(x, y) = \min\{1, d(x, y)\}$. Prove that d^* is also a metric on X, and that $\mathcal{T}_d = \mathcal{T}_{d^*}$.
b) Define $d^{**}(x, y) = \frac{d(x,y)}{1 + d(x,y)}$. Prove that d^{**} is also a metric on X and that $\mathcal{T}_d = \mathcal{T}_{d^{**}}$

Hint: Let d *be a metric on* X *and suppose* f *is a function from the nonnegative real numbers to the nonnegative real numbers for which:* $f(0) = 0$, $x \leq y \Rightarrow f(x) \leq f(y)$, *and* $f(x + y) \leq f(x) + f(y)$ *for all nonnegative* x, y. *Prove that* $d'(x, y) = f(d(x, y))$ *is also a metric on* X. *Then consider the particular function* $f(x) = \frac{x}{1+x}$.

Note: For all x, y *in* X, $d^*(x, y) \leq 1$ *and* $d^{**}(x, y) \leq 1$ *– that is,* d^* *and* d^{**} *are* *__bounded__* *metrics on* X. *Thus any metric d on* X *can be replaced by an equivalent bounded metric – that is, a bounded metric generating the same topology. So "boundedness" is a property determined by the particular metric, not the topology.*

E4. Suppose $f : \mathbb{R} \to \mathbb{R}$. Let d be the usual metric on $\mathbb{R}$, and d' the usual metric on $\mathbb{R}^2$. Define a new distance function d'' on $\mathbb{R}$ by $d''(x, y) = d'((x, f(x)), (y, f(y)))$. Prove that d'' is a metric on $\mathbb{R}$.

Must d'' be equivalent to d ? If not, can you describe conditions which will guarantee that $d'' \sim d$?

E5. Suppose a function $d : X \times X \to \mathbb{R}$ satisfies the conditions 1), 2), 3), and 5) in the definition of a metric, but that instead of the triangle inequality, we have that for all $x, y, z \in X$

$$d(x, z) \geq d(x, y) + d(y, z).$$

Prove that $|X| \leq 1$.

E6. Suppose A is a finite open set in a metric space (X, d). Prove that every point of A is isolated in (X, d).

E7. Suppose (X, d) is a metric space and $x \in X$. Prove that the following two statements are __equivalent__:

i) x is not an isolated point of X
ii) every open set containing x contains an infinite number of points.

E8. The definition of an open set in (X,d) reads: O is open if for all $x \in O$, there is an $\epsilon > 0$ such that $B_\epsilon(x) \subseteq O$. In this definition, ϵ may depend on x.

Suppose we define O to be uniformly open if there is an $\epsilon > 0$ such that for all $x \in O$, $B_\epsilon(x) \subseteq O$ – that is, the same ϵ works for every $x \in O$. (*"Uniformly open" is not a standard term.*)

a) What are the uniformly open subsets of $\mathbb{R}^n$?
b) What are the uniformly open sets in (X,d) if d is the trivial pseudometric?
c) What are the uniformly open sets in (X,d) if d is the discrete unit metric?

E9. Let *p* be a fixed prime number. We define the p-adic absolute value $|\;|_p$ (sometimes called the p-adic norm) on the set of rational numbers $\mathbb{Q}$ as follows:

If $0 \neq x \in \mathbb{Q}$, write $x = \frac{p^k m}{n}$ for integers *k*,*m*,*n*, where *p* does not divide *m* or *n*, and define $|x|_p = p^{-k} = \frac{1}{p^k}$. (*Of course, k may be negative.*). Also, define $|0|_p = 0$.

Prove that $|\;|_p$ "behaves the way an absolute value (norm) should" – that is, for all $x, y \in \mathbb{Q}$

a) $|x|_p \geq 0$ and $|x|_p = 0$ iff $x = 0$
b) $|xy|_p = |x|_p \cdot |y|_p$
c) $|x+y|_p \leq |x|_p + |y|_p$

$|\;|_p$ actually satisfies a stronger inequality than the inequality in part c). Prove that

d) $|x+y|_p \leq \max\{|x|_p, |y|_p\} \leq |x|_p + |y|_p$

Whenever we have an absolute value (norm), we can use it to define a distance function:

for $x, y \in \mathbb{Q}$, let $d_p(x,y) = |x-y|_p$

e) Prove that d_p is a metric on $\mathbb{Q}$, and show that d_p actually satisfies an inequality stronger than the usual triangle inequality, namely:

for all $x, y, z \in \mathbb{Q}$, $d_p(x,z) \leq \max\{d_p(x,y), d_p(y,z)\}$

f) Give a specific example for x, y, z, p for which

$d_p(x,z) < \max\{d_p(x,y), d_p(y,z)\}$

(*Hint: It might be convenient to be able to refer to the exponent "k" associated with a particular x. If $x = \frac{p^k m}{n}$, then k roughly refers to the "number of p's that can be factored out of x" – so we can call $k = \nu(x)$. Prove that $\nu(a-b) \geq \min\{\nu(a), \nu(b)\}$ whenever a,b $\in \mathbb{Q}$ with a,b $\neq 0$ and a $\neq$ b. Note that strict inequality can occur here: for example, when p = 3, $\nu(8) = \nu(2) = 0$, but $\nu(8-2) = \nu(6) = 1$.*)

g) Suppose $p = 2$. Calculate $d_2(2^n, 0)$. What are $\lim_{n\to\infty} d_2(2^n, 0)$ and $\lim_{n\to\infty} d_2(4^n, 0)$?

4. Closed Sets and Operators on Sets

Definition 4.1 Suppose (X,d) is a pseudometric space and that $F \subseteq X$. We say that F is closed in (X,d) if $X - F$ is open in (X,d).

From the definitions:

F is closed in (X,d)
iff $X - F$ is open in (X,d)
iff for all $x \in X - F$, there is an $\epsilon > 0$ for which $B_\epsilon(x) \subseteq X - F$
iff for all $x \in X - F$, there is an $\epsilon > 0$ for which $B_\epsilon(x) \cap F = \emptyset$.

The closed sets are completely determined by the open sets and vice-versa so that in any space (X,d) the collection of closed sets and the collection of open sets, $\mathcal{T}_d$, contain exactly "the same information."

The close connection between the closed sets and the open sets is illustrated in the following theorem.

Theorem 4.2 For any pseudometric space (X,d),

i) $\emptyset$ and X are closed
ii) if F_α is closed for each $\alpha \in A$, then $\bigcap_{\alpha \in A} F_\alpha$ is closed
iii) if $F_1, ..., F_n$ are closed, then $\bigcup_{i=i}^{n} F_i$ is closed.

(Conditions ii) and iii) say that the collection of closed sets is closed under intersections and finite unions.)

Proof If we take complements, then these statements follow from the corresponding properties of open sets. Since $\emptyset$ and X are open, the complements $X - \emptyset = X$ and $X - X = \emptyset$ are closed.

Suppose F_α is closed for each $\alpha \in A$. Then $X - F_\alpha$ is open for each $\alpha \in A$, so $\bigcup_{\alpha \in A}(X - F_\alpha)$ is open, and therefore the complement $X - (\bigcup_{\alpha \in A}(X - F_\alpha))$ $= \bigcap_{\alpha \in A} X - (X - F_\alpha) = \bigcap_{\alpha \in A} F_\alpha$ is closed.

The proof of iii) uses the fact that a finite intersection of open sets is open. ∙

Exercise: Give an example to show that an infinite union of closed sets need not be closed.

Example 4.3

1) In $\mathbb{R}$, the interval $[0,1]$ is closed since its complement $\mathbb{R} - [0,1] = (-\infty, 0) \cup (1, \infty)$ is open. Equivalently, we can say that $[0,1]$ is closed because for each $x \notin [0,1]$, there is an $\epsilon > 0$ for which $B_\epsilon(x) \cap [0,1] = (x - \epsilon, x + \epsilon) \cap [0,1] = \emptyset$.

2) A set can be neither open nor closed: for example, consider the following subset of $\mathbb{R}$: $[0,1)$, $\mathbb{Q}$ and $\mathbb{P}$.

3) A set can be both open and closed – these terms are not mutually exclusive. Such sets in (X, d) are called clopen sets. For example, $\emptyset$ and X are clopen in every pseudometric space (X, d). Sometimes there are other clopen sets and sometimes not.

In the space $X = [0,1] \cup [3,4]$ with the usual metric d, the set $[0,1]$ is clopen. But in $\mathbb{R}$, for example, $\emptyset$ and $\mathbb{R}$ are the only clopen subsets. (*This fact is not too hard to prove but it is also not obvious – the proof depends on the completeness property* $($ = *"least upper bound property"* $)$ *in* $\mathbb{R}$*. We will prove this fact later when we need it in Chapter V.*)

4) In any pseudometric space (X, d), the set $\{x \in X : d(a, x) \leq \epsilon\} = F$ is a closed set. To see this, suppose $y \notin F$, Then $d(a, y) = \delta > \epsilon$. Let $\epsilon_1 = \delta - \epsilon > 0$. Then $B_{\epsilon_1}(y) \cap F = \emptyset$. (If $z \in B_{\epsilon_1}(y) \cap F$, then we would have $d(a, y) \leq d(a, z) + d(z, y) < \epsilon + \epsilon_1 = \epsilon + (\delta - \epsilon) = \delta$, which is false.)

$\{x \in X : d(a, x) \leq \epsilon\}$ is called the closed ball centered at x with radius ϵ.

For example, $[0,1] = \{x \in \mathbb{R} : d(\frac{1}{2}, x) = |x - \frac{1}{2}| \leq \frac{1}{2}\}$ is the "closed ball" centered at $\frac{1}{2}$.

5) Let $X = [0,1] \cup [2,5)$ with the usual metric d.

$[0,1]$ and $[2,5)$ are both clopen in (X, d).
$[3,4)$ is neither open nor closed in (X, d).

Notice again that "open" and "closed" are not absolute terms: whether a set A is open (or closed) depends on what larger space X that A "lives in."

6) Let d be the discrete unit metric, so that $\mathcal{T}_d$ is the discrete topology: every subset of X is open. Then every subset of X is clopen.

7) Let d be the trivial pseudometric on X. Then $B_\epsilon(x) = X$ for every $x \in X$ and every $\epsilon > 0$. Therefore a union of balls must be either X or $\emptyset$ (in the case of the union of an empty collection of balls), so $\mathcal{T}_d = \{\emptyset, X\}$, which is called the trivial topology on X. Since $\emptyset$ and X must be open in any space (X, d), the trivial topology is the smallest possible topology on X.

In (X, d), the only closed sets are $\emptyset$ and X.

Using the open and closed sets in (X, d), we can define some useful "operators" on subsets of X. An "operator" creates a new subset of X from an old one.

Definition 4.4 Suppose (X, d) is a pseudometric space and $A \subseteq X$.

The interior of A in X $= \text{int}_X\, A = \bigcup\{O : O \text{ is open and } O \subseteq A\}$
The closure of A in X $= \text{cl}_X A = \bigcap\{F : F \text{ is closed and } F \supseteq A\}$
The frontier (or boundary) of A in X $= \text{Fr}_X A = \text{cl}_X A \cap \text{cl}_X(X - A)$

We will omit the subscript "X" when the context makes clear the space X in which the operations are being performed. Sometimes int A and cl A are denoted A° and $\overline{A}$ respectively. Some books use the notation ∂A for Fr A, but the symbol ∂A has a different meaning in algebraic topology so we will avoid using it here.

Theorem 4.5 Suppose (X, d) is a pseudometric space and that $A \subseteq X$. Then

1) a) int A is the largest open subset of A (that is, if O is open and $O \subseteq A$, then $O \subseteq$ int A).
b) A is open iff $A =$ int A (since int $A \subseteq A$, the equality is equivalent to $A \subseteq$ int A).
c) $x \in$ int A iff there is an open set O such that $x \in O \subseteq A$
iff $\exists\, \epsilon > 0$ such that $B_\epsilon(x) \subseteq A$.
Informally, we can think of 1c) as saying that the interior of A consist of those points "comfortably inside A surrounded by a small cushion," $-$ that is, points not "on the edge of A."

2) a) cl A is the smallest closed set containing A (that is, if F is closed and $F \supseteq A$, then $F \supseteq$ cl A)
b) A is closed iff $A =$ cl A (since $A \subseteq$ cl A, the equality is equivalent to cl $A \subseteq A$).
c) $x \in$ cl A iff for every open set O containing x, $O \cap A \neq \emptyset$
iff for every $\epsilon > 0$, $B_\epsilon(x) \cap A \neq \emptyset$.
Informally, 2c) states that clA consists of the points in X that can be approximated arbitrarily closely by points from within the set A.

3) a) Fr A is closed and Fr $A =$ Fr $(X - A)$.
b) A is clopen iff Fr $A = \emptyset$.
c) $x \in$ Fr A iff for every open set O containing x, $O \cap A \neq \emptyset$ and $O \cap (X - A) \neq \emptyset$
iff for every $\epsilon > 0$ $B_\epsilon(x) \cap A \neq \emptyset$ and $B_\epsilon(x) \cap (X - A) \neq \emptyset$.
Informally, 3c) states that Fr A consists of those points in X that can be approximated arbitrarily closely both by points from within A and by points from outside A.

Proof

1) a) Clearly int $A \subseteq A$. If O open and $O \subseteq A$, then O is one of the sets whose union is int A, so $O \subseteq$ int A.

b) int A is open (it is a union of open sets), so if int $A = A$, then A is open. Conversely, if A is open, then A is the largest open subset of A, so $A =$ int A.

c) Since int A is a union of open sets, it is clear that $x \in$ int A iff $x \in O \subseteq A$ for some open set O. Since a open set is open iff it is a union of ϵ-balls, the remainder of the assertion is obviously true.

2) Exercise

3) a) Fr A is closed because it is an intersection of two closed sets, and

$$\text{Fr}\,(X - A) = \text{cl}(X - A) \cap \text{cl}(X - (X - A)) = \text{cl}(X - A) \cap \text{cl}(A) = \text{Fr}\,A.$$

b) If A is clopen, then so is $X - A$. Therefore $\text{Fr}(A) = \text{cl}\,(A) \cap \text{cl}(X - A)$ $= A \cap (X - A) = \emptyset$. Conversely, if $\text{cl}\,(A) \cap \text{cl}(X - A) = \emptyset$, then $\text{cl}\,A \subseteq X - \text{cl}(X - A) \subseteq X - (X - A) = A$, so A is closed. Similarly we show that $X - A$ is closed, so A is clopen.

c) $x \in \text{Fr}\,A$ iff x in both cl A and cl $(X - A)$. By 2c), this is true iff each open set containing x intersects both A and $X - A$.

Since an open set is a union of ϵ-balls, the remainder of the assertion is clearly true. •

Notice that section c) in each part of Theorem 4.5, there is a criterion that lets you decide whether x is in one of the sets int A, cl A, or Fr A by using only the open sets, and not mentioning ϵ-balls. *This is important!* It means that if we change the metric d to an equivalent metric d', then int A, cl A, and Fr A do not change, since d and d' produce the same open sets. In other words, we can say that int, cl, and Fr are topological operators: they depend on only the topology, and not on the particular metric that produced the topology. For example if $A \subseteq \mathbb{R}^n$, then A will have the same interior, same closure, and same frontier whether we measure distances using the usual metric d, the taxicab metric d_t, or the max-metric d^*.

Example 4.6 (*Be sure you understand each statement!*)

1) In $\mathbb{R}$:

$\text{int}\,\mathbb{R} = \mathbb{R}$	$\text{cl}\,\mathbb{R} = \mathbb{R}$	$\text{Fr}\,\mathbb{R} = \emptyset$
$\text{int}\,\mathbb{Q} = \emptyset$	$\text{cl}\,\mathbb{Q} = \mathbb{R}$	$\text{Fr}\,\mathbb{Q} = \mathbb{R}$

$\text{int}\,[0,1) = (0,1)$ $\qquad$ $\text{cl}\,[0,1) = [0,1]$

$\text{Fr}\,[0,1) = \{0,1\}$ $\qquad$ $\text{Fr}\,(\text{Fr}\,\mathbb{Q}) = \text{Fr}\,(\mathbb{R}) = \emptyset$.

In any space (X, d), it is obviously true that $\text{int}(\text{int}\,A) = \text{int}\,A$ and $\text{cl}(\text{cl}\,A) = \text{cl}\,A$. But this need not be true for Fr, as the last example shows. (*It is true that Fr(Fr(Fr A)) = Fr(Fr A) in any space X. But this is not a useful fact, and it is also not interesting to prove.*)

2) $X = [0,2)$ (with the usual metric)

$\text{cl}_X\,[0,1) = [0,1]$ $\qquad$ $\text{int}_X[0,1) = [0,1)$

$\text{cl}_X[1,2) = [1,2)$ $\qquad$ $\text{int}_X[1,2) = (1,2)$

$\text{Fr}_X[0,1) = [0,1] \cap [1,2) = \{1\}$

$0 \notin Fr_X[0,1)$ *because* 0 *cannot be "approximated arbitrarily closely" by points from* $X - [0,1)$.

3) Suppose d is the discrete unit metric on X. If $A \subseteq X$, then A is clopen so we get $\text{cl}\,A = A$, $\text{int}\,A = A$, and $\text{Fr}\,A = \emptyset$.

Suppose d is the trivial pseudometric on X. If A is any nonempty, proper subset of X, then $\text{cl}\,A = X$, $\text{int}\,A = \emptyset$, and $\text{Fr}\,A = X$.

4) In (ℓ_2, d), let A be the set of sequences with all terms rational:

$$A = \{x = (x_i) \in \ell_2 : \forall i,\ x_i \in \mathbb{Q}\} = \mathbb{Q}^{\mathbb{N}} \cap \ell_2.$$

We claim that cl $A = \ell_2$ $-$ in other words, that any point $y = (y_i) \in \ell_2$ can be approximated arbitrarily closely by a point from A. So let $\epsilon > 0$. We must show that $B_\epsilon(y) \cap A \neq \emptyset$.

Since $\sum_{i=1}^{\infty} y_i^2$ cinverges, we can pick an N such that $\sum_{i=N+1}^{\infty} y_i^2 < \frac{\epsilon^2}{2}$. Using this value of N, choose rational numbers a_i so that $|a_i - y_i| < \frac{\epsilon}{\sqrt{2N}}$ for $i = 1, ..., N$. Define $a = (a_1, ..., a_N, 0, 0, 0, ...)$. Then $a \in A$ and

$$d(a, y) = \sqrt{\sum_{i=1}^{\infty}(a_i - y_i)^2} = \sqrt{\sum_{i=1}^{N}(a_i - y_i)^2 + \sum_{i=N+1}^{\infty}(a_i - y_i)^2}$$

$$= \sqrt{\sum_{i=1}^{N}(a_i - y_i)^2 + \sum_{i=N+1}^{\infty} y_i^2} < \sqrt{N \cdot \frac{\epsilon^2}{2N} + \frac{\epsilon^2}{2}} = \sqrt{\epsilon^2} = \epsilon.$$

Therefore $a \in B_\epsilon(y) \cap A$.

We also claim that int $A = \emptyset$. To prove this, we need to show that if $x = (x_i) \in A$, then no ball centered at x is a subset of A. To see this, pick any $\epsilon > 0$ and choose an irrational y_1 such that $|y_1 - x_1| < \epsilon$. Modify x by changing x_1: define $y = (y_1, x_2, ..., x_n, ...)$. Then $d(x, y) = |x_1 - y_1| < \epsilon$ and $y \notin A$, so $B_\epsilon(x) \not\subseteq A$.

What is Fr A ?

5) In any pseudometric space (X, d), $B_\epsilon(a)$ is a subset of the closed set $\{x \in X : d(a, x) \leq \epsilon\}$. Therefore cl $B_\epsilon(a) \subseteq \{x \in X : d(a, x) \leq \epsilon\}$.
But these two sets are not necessarily equal: sometimes the closed ball is larger than the closure of the open ball $B_\epsilon(a)$! For example, suppose d is the discrete unit metric on a set X where $|X| > 1$. Then

$$\{a\} = B_1(a) = \text{cl}\, B_1(a) \underset{\neq}{\subseteq} X = \{x \in X : d(a, x) \leq 1\}$$

Definition 4.7 Let (X, d) be a pseudometric space and suppose that $D \subseteq X$. We say that D is dense in (X, d) if cl $D = X$. The space (X, d) is called separable if it possible to find a countable dense set D in X. (*Note the spelling: "separable," not "seperable."*)

Since "separable" is defined in terms of the closure and the closure operator depends only on the topology, not the particular metric that generates the topology. Therefore separability is a topological property: if (X, d) is separable and $d' \sim d$, then (X, d') is also separable.

More informally, "D is dense in X" means that each point $x \in X$ can be approximated arbitrarily closely by a point from D. A countable dense set in X (if one exists) is a "small" set which can be used to approximate any point in X arbitrarily closely.

Example 4.8

1) $\mathbb{R}^n$ is separable because $\mathbb{Q}^n$ is a countable dense set in $\mathbb{R}^n$; in particular, $\mathbb{Q}$ is a countable dense set in $\mathbb{R}$, so $\mathbb{R}$ is separable. $\mathbb{P}$ is an example of an <u>uncountable</u> dense subset of $\mathbb{R}$.

2) Any countable space (X, d) is separable, because X is dense in X.

3) Suppose $\mathcal{T}_d$ is the discrete topology on X. Then (X, d) is separable iff X is countable (because any proper subset of X is closed and therefore not dense).

If $\mathcal{T}_d$ is the trivial topology and $X \neq \emptyset$, then every nonempty subset D is dense (*why?*). Therefore (X, d) is separable because, for example, each singleton set $\{x\}$ is dense.

4) The set $A = \mathbb{Q}^{\mathbb{N}} \cap \ell_2$ is dense in ℓ_2 (*see Example 4.6(4)*). This set A is an <u>uncountable</u> dense set because every sequence of rationals (x_i) with $|x_i| \leq \frac{1}{i}$ is in A (*why?*), and there are c such sequences (*why?*). However, (ℓ_2, d) <u>is</u> separable. Can you find a <u>countable</u> dense set D? (*The computation in Example 4.6.4 might give you an idea.*)

Definition 4.9 For nonempty subsets A, B in a pseudometric space (X, d), we define the distance between them by

$$\text{dist}(A, B) = \inf \{d(a, b) : a \in A \text{ and } b \in B\}.$$

Although it is an abuse of notation "d" we usually abbreviate dist(A, B) by $d(A, B)$. Going one step further, we also write $d(a, B)$ as an abbreviation for $d(\{a\}, B)$.

Of course, if $a \in B$, then $d(a, B) = 0$. But the converse may not be true. For example, let $B = \{1, \frac{1}{2}, ..., \frac{1}{n}, ...\} \subseteq \mathbb{R}$. Then $d(0, B) = 0$ even though $0 \notin B$. Similarly, if A is the y-axis in $\mathbb{R}^2$ and $B = \{(x, y) \in \mathbb{R}^2 : y = \frac{1}{x}\}$, then A and B are disjoint, <u>closed</u> sets but $d(A, B) = 0$.

The "distance from a point to a set" can be used to describe the closure of a set.

Theorem 4.10 Suppose $A \subseteq X$, where (X, d) is a pseudometric space. Then $x \in \text{cl}\, A$ iff $d(x, A) = 0$.

Proof $x \in \text{cl}\, A$ iff for every $\epsilon > 0$, $B_\epsilon(x) \cap A \neq \emptyset$
iff for every $\epsilon > 0$ there is a $y \in A$ with $d(x, y) < \epsilon$
iff $d(x, A) = 0$. •

Exercises

E10. Let (X, d) be a pseudometric space. Prove or disprove each statement:

a) $B_\epsilon(x)$ is never a closed set.
b) if $A \subseteq X$, then Fr $A =$ cl $A -$ int A.
c) for any $A \subseteq X$, diam $(A) =$ diam (cl A).
(*The diameter of a set A in a metric space is defined to be sup* $\{d(x, y) : x, y \in A\} \leq \infty$.)
d) for any $A \subseteq X$, diam $(A) =$ diam (int A).
e) for any $A \subseteq X$, int$(A \cup B) =$ int A $\cup$ int B.
f) for every $x \in X$ and $\epsilon > 0$, cl $(B_\epsilon(x)) = \{y \in X : \ d(x, y) \leq \epsilon\}$.

E11. a) Give an example of a metric space (X, d) with a proper nonempty clopen subset.
b) Give an example of a metric space (X, d) and a subset that is neither open nor closed.
c) Give an example of a metric space (X, d) and a subset A for which every point in A is a limit point of A. (*Note: a point x is called a* <u>*limit point of a set*</u> A *if, for every open set* O *containing* x, $O \cap (A - \{x\}) \neq \emptyset$.)
d) Give an example of a metric space (X, d) and a nonempty subset *A* such that every point is a limit point of *A* but int$(A) = \emptyset$. Can you also arrange that *A* is closed in *X*?
e) For each of the following subsets of $\mathbb{R}$, find the interior, closure and frontier ("boundary") in $\mathbb{R}$. Which points of the set are isolated in $\mathbb{R}$? which points of the set are isolated in the set ?

i) $A = \{m + n\pi : m, n \in \mathbb{N}\}$
ii) $B = \{\frac{1}{m} + \frac{1}{n} : m, n \in \mathbb{Z}\}$

E12. An infinite union of closed sets need not be closed. However if infinitely many closed sets are "spread out enough" from each other, their union is closed. Parts a) and b) illustrate this.

a) Suppose that for each $n \in \mathbb{N}$, F_n is a closed set in $\mathbb{R}$ and that $F_n \subseteq (n, n+1)$. Prove that $\bigcup_{n=1}^{\infty} F_n$ is closed in $\mathbb{R}$.

b) More generally: suppose for $\alpha \in A$, each F_α is a closed set in (X, d) and that for each point $x \in X$ there is an $\epsilon > 0$ such that $B_\epsilon(x)$ has nonempty intersection with at most finitely many F_α's. Prove that $\bigcup_{\alpha \in A} F_\alpha$ is closed in (X, d). (*Notice that b $\Rightarrow$ a. Why?*)

E13. a) Give an example of 2^c subsets of $\mathbb{R}$ all of which have the same closure. Do the same in $\mathbb{R}^2$.

b) Prove or disprove: there exist 2^c subsets of $\mathbb{R}^2$ such that any two have different closures. Can you do the same in $\mathbb{R}$?

E14. The <u>Hilbert cube</u>, *H*, is a certain subset of ℓ_2: $H = \{x \in \ell_2 : \ |x_i| \leq \frac{1}{i}\}$. Prove that *H* is closed in ℓ_2. Is *H* also open? (Prove or disprove)

E15. A subset A of a space X is called a G_δ *set* if A can be written as a countable intersection of open sets; A is called an F_σ set if it can be written as a countable union of closed sets.

> *Note: Open sets are often denoted letters like O, U, or V (from open, and from the French ouvert), and sometimes by the letter G —from older literature where the German word is "Gebiet". Closed sets often are denoted by the letter F—from the French "ferme." Of course using these letters is just a common tradition – but many topologists follow it and would usually wince to read something like "let F be an open set."*
>
> *The names G_δ and F_σ go back to the classic book Mengenlehre of the German mathematician Felix Hausdorff. The σ and the δ in the notation represent abbreviations for the German words used for union and intersection: Summe and Durchschnitt.*

a) Prove that in a pseudometric space (X, d) every closed set is a G_δ set and every open set is an F_σ set.

b) Find the error in the following argument which "proves" that every subset of $\mathbb{R}$ is a G_δ set:

> *Let $A \subseteq \mathbb{R}$. For $x \in A$, let $J_n = \bigcup\{B_{\frac{1}{n}}(x) : x \in A\}$. J_n is open for each $n \in \mathbb{N}$. Since $\{x\} = \bigcap_{n=1}^{\infty} B_{\frac{1}{n}}(x)$, it follows that $A = \bigcap_{n=1}^{\infty} J_n$, so A is a countable intersection of open sets, that is, A is a G_δ set.*

c) In the argument in part b), the truth is that $\bigcap_{n=1}^{\infty} J_n =$?

d) Suppose we enumerate the members of $\mathbb{Q}$: $x_1, x_2, ..., x_n, ...$. For each n, consider the interval $B_{\frac{1}{n}}(x_n) = (x_n - \frac{1}{n}, x_n + \frac{1}{n}) \subseteq \mathbb{R}$ and let $J_n = \bigcup_{n=1}^{\infty} B_{\frac{1}{n}}(x_n)$. Is $J_n = \mathbb{R}$?

E16. Let (X, d) be a pseudometric space. Suppose that for every $\epsilon > 0$, there exists a countable subset D_ϵ of X with the following property: $\forall x \in X,\ \exists y \in D_\epsilon$ such that $d(x, y) < \epsilon$. Prove that (X, d) is separable.

E17. Suppose that X is an uncountable set and d is any metric on X which produces the discrete topology. (*Such a metric d does not have to be a constant multiple of the discrete unit metric: see Example 2.14.2*). Show that for some $\epsilon > 0$ there is an uncountable subset A of X such that $d(x, y) \geq \epsilon$ for all $x \neq y \in A$

E18. Let (X, d) be an infinite metric space. Prove that there exists an open set U such that both U and $X - U$ are infinite.

(*Hint: Consider a non-isolated point, if one exists.)*

E19. A metric space (X, d) is called extremally disconnected if the closure of every open set is open. (*Note: "extremally" is the correct spelling; the word is not the same as the everyday word "extremely."*)

Prove that if (X, d) is extremally disconnected, then the topology $\mathcal{T}_d$ is the discrete topology.

E20. Since "dist" provides a measure of "distance" between nonempty subsets of (X, d), one might ask whether $(\mathcal{P}(X) - \{\emptyset\}, \text{dist})$ is a metric (or pseudometric) space. Is it?

E21. Suppose (x_n) is a sequence in (X, d). We say that x_0 is a cluster point of (x_n) if for every open set O containing x_0 and for all $n \in \mathbb{N}$, $\exists k > n$ such that $x_k \in O$. (*This is clearly equivalent to saying that* $\forall \epsilon > 0$ *and* $\forall n \in \mathbb{N}$, $\exists k > n$ *such that* $x_k \in B_\epsilon(x_0)$.) Informally, x_0 is a cluster point of (x_n) if the sequence is "frequently in every open set containing x_0."

a) Show that there is a sequence in $\mathbb{R}$ for which every $r \in \mathbb{R}$ is a cluster point.

b) A neurotic mathematician is walking along $\mathbb{R}$ from 0 toward 1. Halfway to 1, she (or he) remembers that she forgot something at 0 and starts back. Halfway back to 0, she decides to go to 1 anyway and turns around, only to change her mind again after traveling half the remaining distance to 1. She continues in this back-and-forth fashion forever. Find the cluster point(s) of the sequence (x_n), where x_n is the point where she reverses direction for the n^{th} time.

5. Continuity

Suppose $a \in A \subseteq \mathbb{R}$ and that $f : A \to \mathbb{R}$. In elementary calculus, the set A is usually and interval, and the idea of continuity at a point a in A is introduced very informally. Roughly, it means that "if x is a point near a in the domain, then $f(x)$ is near $f(a)$." In advanced calculus or analysis, the idea of "continuity of f at a" is defined carefully. The intuitive version of continuity – stated in terms of "nearness" – is made precise by measuring distances:

f is continuous at a means: $\forall\, \epsilon > 0\ \exists\, \delta > 0$ such that

$$\text{if } x \in A \text{ and } |x - a| < \delta, \text{ then } |f(x) - f(a)| < \epsilon.$$

We say that "f is continuous" if f is continuous at every point a in its domain A.

An important thing to notice is that the definition is made using the distance function in $\mathbb{R}$: $d(x, a) = |x - a|$ and $d(f(x), f(a)) = |f(x) - f(a)|$. Since we have a way to measure distances in pseudometric spaces, we can make an a completely similar definition of continuity for functions from one pseudometric space to another.

Definition 5.1 Suppose $f : X \to Y$ where (X, d) and (Y, s) are pseudometric spaces. If $a \in X$ and $f : X \to Y$, we say f is continuous at a if $\forall \epsilon > 0\quad \exists \delta > 0$ such that: if $d(a, x) < \delta$, then $s(f(x), f(a)) < \epsilon$.

Notice that the sets X and Y may have two completely unrelated metrics d and s: d measures distances in dom(f) and s measures distances in Y. But the idea is exactly the same as in calculus: "continuity of f at a" means, roughly, that "points x near a in the domain X have images $f(x)$ near $f(a)$ in Y."

Theorem 5.2 Suppose (X, d) and (Y, s) are pseudometric spaces. If $a \in X$ and $f : X \to Y$, then the following statements are equivalent:

1) f is continuous at a
2) $\forall \epsilon > 0\, \exists \delta > 0$ such that $f\,[B_\delta\,(a)] \subseteq B_\epsilon(f(a))$
3) $\forall \epsilon > 0\, \exists \delta > 0$ $B_\delta\,(a) \subseteq f^{-1}[B_\epsilon\,(f(a))]$
4) $\forall N \subseteq Y$: if $f\,(a) \in \text{int}\, N$, then $a \in \text{int}\ f^{-1}[N]$.

Proof It is clear that conditions 1) - 2) - 3) are just equivalent restatements the definition of continuity at a phrased in terms of images and inverse images of balls. Condition 4), however, seems a bit strange. We will show that 3) and 4) are equivalent.

3) $\Rightarrow$ 4) Suppose $f(a) \in \text{int}\, N$. By definition of interior, there is an $\epsilon > 0$ such that $B_\epsilon(f(a)) \subseteq N$, so that $f^{-1}[B_\epsilon(f(a))] \subseteq f^{-1}[N]$. By 3), we can pick $\delta > 0$ so that $a \in B_\delta\,(a) \subseteq f^{-1}[B_\epsilon\,(f(a))] \subseteq f^{-1}[N]$. Since the δ-ball at a is an open subset of $f^{-1}[N]$, we get that $a \in \text{int}\, f^{-1}[N]$, as desired.

4) $\Rightarrow$ 3) Suppose $\epsilon > 0$ is given. Let $N = B_\epsilon(f(a))$. Then N is open and $f(a) \in N = \text{int}\, N$. We conclude from 4) that $a \in \text{int}\, f^{-1}[N] = \text{int}\, f^{-1}[B_\epsilon(f(a))]$. Therefore for some $\delta > 0$, $B_\delta\,(a) \subseteq \text{int}\, f^{-1}[B_\epsilon(f(a))] \subseteq f^{-1}[B_\epsilon(f(a))]$, so 3) holds. •

Definition 5.3 If $N \subseteq X$ and $x \in \text{int}\, N$, then N is called a neighborhood of x in (X, d).
Thus, N is a neighborhood of x if there is an open set O such that $x \in O \subseteq N$.

Notice that:

1) The term "neighborhood" goes together with a point $x \in X$. We might say "O is an open set in X," but we would never say "N is a neighborhood in X" – but rather "N is a neighborhood of x in X."

2) A neighborhood N of x need not be an open set. Be aware, however, that in some texts "a neighborhood of x" means "an open set containing x." It's not really important which way one makes the definition of neighborhood (each version has its own technical advantages), but it is important that we all agree in these notes. So, in $\mathbb{R}^n$ for example, we say that the closed ball $F = \{(a, x) : d(a, x) \le \epsilon\}$ is a neighborhood of a; in fact the closed ball is a neighborhood of any point x in its interior. A point x for which $d(a, x) = \epsilon$ is in F, but F is not a neighborhood of such a point x.

The following observation is almost trivial but it is important enough to state and remember.

Theorem 5.4 A subset N in (X, d) is open iff N is a neighborhood of each of its points.

Proof Suppose N is open and $x \in N$. Then $x \in \text{int}\, N = N$. Since $\text{int}\, N$ is open, N is a neighborhood of x.

Conversely, if N is a neighborhood of each of its points, then for every $x \in N$, we have $x \in \text{int}\, N$. Therefore $N \subseteq \text{int}\, N$, so $N = \text{int}\, N$ and N is open. •

With this new terminology, we can restate the equivalence of 1) and 4) in Theorem 5.2 as:

f is continuous at a iff
whenever N is a neighborhood of $f(a)$ (*in* Y), then $f^{-1}[N]$ is a neighborhood of a (*in* X).

This tells us something very important. Interiors (and therefore neighborhoods of points) are defined in terms of the open sets (without needing to mention the distance function), so the neighborhoods of a point x depend only on the topology, not on the specific metric that generates the topology. Therefore whether or not f is continuous at a does not actually depend on the specific metrics but only on the topologies in the domain and range. In other words, "continuity at a" is a topological property.

For example, the function $\sin : \mathbb{R} \to \mathbb{R}$ is continuous at each point a in $\mathbb{R}$, and this is remains true if we measure distances in the domain with, say, the taxicab metric d_t and distance in the range with the max-metric d^* – since these are both equivalent to the usual metric d on $\mathbb{R}$.

We now define "f is a continuous function" in the usual way.

Definition 5.5 Suppose (X, d) and (Y, s) are pseudometric spaces. We say that $f : X \to Y$ is continuous if f is continuous at each point of X.

Theorem 5.6 Suppose $f : X \to Y$ where (X, d) and (Y, s) are pseudometric spaces.
The following are equivalent:

1) f is continuous
2) if O is open in Y, then $f^{-1}[O]$ is open in X
3) if F is closed in Y, then $f^{-1}[F]$ is closed in X.

Proof 1) $\Rightarrow$ 2) Suppose O is open in Y and that $x \in f^{-1}[O]$. Since O is a neighborhood of $f(x)$ and f is continuous at x, we know from Theorem 5.2 that $f^{-1}[O]$ is a neighborhood of x. Therefore $f^{-1}[O]$ is a neighborhood of each of its points, so $f^{-1}[O]$ is open.

2) $\Rightarrow$ 3) If F is closed in Y, then $Y - F$ is open. By 2), $f^{-1}[Y - F] = X - f^{-1}[F]$ is open in X, so $X - (X - f^{-1}[F]) = f^{-1}[F]$ is closed in X.

3) $\Rightarrow$ 2) Exercise

2) $\Rightarrow$ 1) Suppose $a \in X$ and that N is a neighborhood of $f(a)$ in Y, so that $f(a) \in \text{int}\, N \subseteq N$. By 2), $f^{-1}[\text{int}\, N]$ is open in X, and $a \in f^{-1}[\text{int}\, N] \subseteq f^{-1}[N]$. Therefore $f^{-1}[N]$ is a neighborhood of a. Therefore f is continuous at a. Since a was an arbitrary point in X, f is continuous. ●

Notice again that Theorem 5.6 shows that continuity is completely described in terms of the open sets (or equivalently, the closed sets), and the proof of the theorem is phrased entirely in terms of open (closed) sets, without any explicit mention of the pseudometrics on X and Y. Replacing d and s with equivalent pseudometrics would not affect the continuity of f.

Theorem 5.7 Suppose (X, d), (Y, s) and (Z, t) are pseudometric spaces and that $f : X \to Y$ and $g : Y \to Z$. If f is continuous at $a \in X$ and g is continuous at $f(a) \in Y$, then $g \circ f$ is continuous at a. (Therefore, if f and g are continuous, so is $g \circ f$.)

Proof If N is a neighborhood of $g(f(a))$, then $g^{-1}[N]$ is a neighborhood of $f(a)$, because g is continuous at $f(a)$. Since f is continuous at a, $f^{-1}[g^{-1}[N]]$ is a neighborhood of a. But $f^{-1}[g^{-1}[N]] = (g \circ f)^{-1}[N[$, so $g \circ f$ is continuous at a. ●

Example 5.8

1) Suppose $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{if } x \neq 0 \end{cases}$. Then $N = (\frac{1}{2}, \frac{3}{2})$ is a neighborhood of $f(0) = 1$, but $f^{-1}[N] = \{0\}$ is not a neighborhood of 0. Therefore f is not continuous at 0. To see the same thing using slightly different language: f is not continuous at 0 because, choosing $\epsilon = \frac{1}{2}$, there is no choice of $\delta > 0$ such that $f[B_\delta(0)] \subseteq B_\epsilon(f(0)) = B_\epsilon(1)$.

2) If $f : (X, d) \to (Y, s)$ is a constant function, then f is continuous. To see this, suppose $f(x) = c$ for every x. If O is open in Y, then $f^{-1}[O] = \begin{cases} \emptyset & \text{if } c \notin O \\ X & \text{if } c \in O \end{cases}$. In both cases, $f^{-1}[O]$ is open.

3) Suppose $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{P} \end{cases}$. Then f is not continuous at any point $a \in \mathbb{R}$ (why?). However $f|\mathbb{Q} = g : \mathbb{Q} \to \mathbb{R}$ is continuous at every point of $\mathbb{Q}$, because g is a constant function.

> *There is a curious old result called* <u>*Blumberg's Theorem*</u> *which states:*
> *For any $f : \mathbb{R} \to \mathbb{R}$, there exists a dense set $D \subseteq \mathbb{R}$ such that $f|D = g : D \to \mathbb{R}$ is continuous.*
> *Blumberg's Theorem is rather difficult to prove, and not very useful.*

4) Suppose $f, g : X \to \mathbb{R}$ where (X, d) is a pseudometric space. Since these functions are real-valued, it makes sense to define functions $f + g$, $f - g$, $f \cdot g$ and $\frac{f}{g}$ in the obvious way. For example, $(f + g)(x) = f(x) + g(x)$, where the " $+$ " on the right is ordinary addition in $\mathbb{R}$.

If f and g are continuous at a point $a \in X$, then the functions $f + g$, $f - g$, and $f \cdot g$ are also continuous at a; and $\frac{f}{g}$ is continuous at a if $g(a) \neq 0$. The proofs are just like those given in calculus (where $X = \mathbb{R}$)

For example, consider $f + g$: given $\epsilon > 0$, then (because f, g are continuous at a), we can find $\delta_1 > 0$ and $\delta_2 > 0$ so that

if $d(x, a) < \delta_1$, then $|f(x) - f(a)| < \frac{\epsilon}{2}$ and
if $d(x, a) < \delta_2$, then $|g(x) - g(a)| < \frac{\epsilon}{2}$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then if $d(x, a) < \delta$, we have

$$|(f + g)(x) - (f + g)(a)| = |f(x) - f(a) + g(x) - g(a)|$$
$$\leq |f(x) - f(a)| + |g(x) - g(a)| < \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon$$

You can find at the other proofs in an analysis textbook.

5) For $x = (x_1, x_2, ..., x_n, ...) \in \ell_2$, define $f : \ell_2 \to \mathbb{R}$ by $f(x) = x_1$. (f *is a "projection" of* ℓ_2 *onto* $\mathbb{R}$.) Then f is continuous at x. To see this, suppose $\epsilon > 0$, and let $\delta = \epsilon$. If $y = (y_1, y_2, ..., y_n, ...) \in B_\delta(x)$, then

$$|f(x) - f(y)| = |x_1 - y_1| \leq d(x, y) = \left(\sum_{i=1}^{\infty} (x_i - y_i)^2\right)^{1/2} < \delta = \epsilon,$$

so $f[B_\delta(x)] \subseteq B_\epsilon(f(x))$.

A similar argument shows that the projection $g(x) = x_n$ is also continuous, and an argument only slightly more complicated would show, for example, that the projection function $h : \ell_2 \to \mathbb{R}^3$ given by $h(x) = (x_3, x_9, x_{11})$ is continuous.

6) If a is an isolated point in (X,d), then every function $f:(X,d)\to(Y,s)$ is continuous at a. To see this, suppose N is a neighborhood of $f(a)$. Then $a\in\{a\}\subseteq f^{-1}[N]$. But $\{a\}$ is open in X, so $f^{-1}[N]$ is a neighborhood of a.

If $\mathcal{T}_d$ happens to be the discrete topology, then every point in X is isolated so f is continuous. (*In this case, we could argue instead that whenever O is open in Y then $f^{-1}[O]$ must be open in X – because every subset of X is open.*)

7) A function $f:(X,d)\to(Y,s)$ is called an <u>isometry of X into Y</u> if it preserves distances, that is, if $d(a,b)=s(f(a),f(b))$ for all $a,b\in X$. An isometry is clearly continuous (given $\epsilon>0$, choose $\delta=\epsilon$).

Note that if d is a <u>metric</u>, then f one-to-one (*Why?*). If f happens to be a bijection, we say that (X,d) and (Y,s) are <u>isometric</u> to each other. In that case, it is clear that the inverse function f^{-1} is also an isometry, so f^{-1} isalso continuous.

Theorem 5.9 If (X,d) is a <u>metric</u> space and $x\neq y\in X$, then there exist open sets U and V such that $x\in U$, $y\in V$ and $U\cap V=\emptyset$. (*More informally: distinct points in a metric space can be separated by disjoint open sets.*)

Proof Since $x\neq y$, $d(x,y)=\delta>0$. Let $U=B_{\frac{\delta}{2}}(x)$ and $V=B_{\frac{\delta}{2}}(y)$. These sets are open, and if there were a point $z\in U\cap V$, we would have a contradiction: $d(x,y)\leq d(x,z)+d(z,y)$ $<\frac{\delta}{2}+\frac{\delta}{2}=\delta$. •

Theorem 5.9 may not be true if d is not a metric. For example, if d is the trivial pseudometric on X, then the only open sets containing x and y are $U=V=X$.)

Example 5.10

1) Suppose $f:(X,d)\to(Y,s)$, where d is the trivial pseudometric on X and Y is any <u>metric</u> space. We already know that if f is constant, then f is continuous.

If f is not constant, then there are points $a,b\in X$ for which $f(a)\neq f(b)$. Since s is a metric, we can pick disjoint open sets $U$and V in Y with $f(a)\in U$ and $f(b)\in V$. Then $f^{-1}[U]\neq\begin{cases}\emptyset & (a\in f^{-1}[U])\\ X & (b\notin f^{-1}[U])\end{cases}$

Since $\mathcal{T}_d=\{\emptyset,X\}$, we see that $f^{-1}[U]$ is not open so f is not continuous.

So, in this situation: f is continuous iff f is constant.

2) Suppose d is the usual metric and s is the discrete unit metric on $\mathbb{R}$. Let $i:(\mathbb{R},d)\to(\mathbb{R},s)$ is the identity map $i(x)=x$. For every open set O in $(\mathbb{R},d)$, the image set $i[O]$ is open in $(\mathbb{R},s)$, but this is not the criterion for continuity: in fact, f is <u>not continuous at any point</u>. The criterion for continuity is that the <u>inverse</u> image of every open set must be open.

Example 5.10.2 leads us to a definition.

Definition 5.11 A function $f:(X,d)\to(Y,s)$ is called an open function or open mapping if: whenever O is open in X, then the image set $f[O]$ is open in Y. Similarly, we say f a closed mapping if whenever F is closed in X, then the image set $f[F]$ is closed in Y.

The identity mapping i in Example 5.10.2 is both open and closed but i is not continuous. You can convince yourself fairly easily that the projection function $\pi_x:\mathbb{R}^2\to\mathbb{R}$ given by $\pi_x(x,y)=x$ is open and continuous, but it is not closed – for example, the set $F=\{(x,y):\ y=\frac{1}{x}\}$ is a closed set in $\mathbb{R}^2$ but $\pi_x[F]=(0,\infty)$ is not closed in $\mathbb{R}$. The general observation is that for a function $f:(X,d)\to(Y,s)$, the properties "open", "closed", and "continuous" are completely independent. You should provide other examples: for instance, a function that is continuous but not open or closed.

With just the basic ideas about continuous functions, we can already prove some rather interesting results.

Theorem 5.12 Suppose $f,g:(X,d)\to(Y,s)$, where d is a pseudometric and s is a metric. Let D be a dense subset of X. If f and g are continuous and $f|D=g|D$, then $f=g$. (*More informally: if two continuous functions with values in a metric space agree on a dense set, then they agree everywhere.*)

Proof Suppose $f\neq g$. Then $f(a)\neq g(a)$ for some point $a\in X$. Since s is a metric, we can find disjoint open sets U and V in Y with $f(a)\in U$ and $g(a)\in V$. Since f and g are continuous at a, there are open sets U_1 and V_1 in X which contain a and satisfy $f[U_1]\subseteq U$ and $g[V_1]\subseteq V$.

We know $a\in \operatorname{cl} D$. Since $U_1\cap V_1$ is an open set containing a, there must be a point $d\in(U_1\cap V_1)\cap D$. Then $f(d)\neq g(d)$ because $f(d)\in U$, $g(d)\in V$ and $U\cap V=\emptyset$. Therefore $f|D\neq g|D$. •

Example 5.13 If $f,g\in C(\mathbb{R})$ and $f|\mathbb{Q}=g|\mathbb{Q}$, then $f=g$ by Theorem 5.12. In other words, the mapping $\Phi:C(\mathbb{R})\to C(\mathbb{Q})$ given by $\Phi(f)=f|\mathbb{Q}\in C(\mathbb{Q})$ is one-to-one. Therefore $|C(\mathbb{R})|\le|C(\mathbb{Q})|\le c^{\aleph_0}=c$.

On the other hand, each constant function $f(x)=r$ is in $C(\mathbb{R})$, so $|C(\mathbb{R})|\ge c$. It follows that $|C(\mathbb{R})|=c$. In other words, there are exactly c continuous functions from $\mathbb{R}$ to $\mathbb{R}$.

Example 5.14 Find all continuous functions $f:\mathbb{R}\to\mathbb{R}$ satisfying the functional equation

$$f(x+y)=f(x)+f(y) \text{ for all } x,y\in\mathbb{R} \qquad (*)$$

Simple induction shows that for $n\in\mathbb{N}$, $f(x_1+...+x_n)=f(x_1)+...+f(x_n)$.
By $(*)$,

$$f(0)=f(0+0)=f(0)+f(0), \text{ so } f(0)=0.$$

Let $f(1)=c$. Then

$$\begin{aligned} f(2) &= f(1+1)=f(1)+f(1)=c\cdot 2\\ f(3) &= f(2+1)=f(2)+f(1)=c\cdot 2+c=c\cdot 3\\ &\vdots \end{aligned}$$

Continuing, we see that $f(n)=cn$ for every $x\in\mathbb{N}$. Similarly, for each $m,n\in\mathbb{N}$, we have

$$f(1) = f(\underbrace{\tfrac{1}{n}+...+\tfrac{1}{n}}_{n \text{ terms}})=f(\tfrac{1}{n})+...f(\tfrac{1}{n})=nf(\tfrac{1}{n}),\ \text{so } f(\tfrac{1}{n})=c(\tfrac{1}{n})$$

$$f(\frac{m}{n}) = f(\underset{\substack{\uparrow \\ m \text{ terms}}}{\frac{1}{n} + \ldots + \frac{1}{n}}) = mf(\frac{1}{n}) = m \cdot c(\frac{1}{n}) = c(\frac{m}{n})$$

So far, we have shown that a function f satisfying $(*)$ must have the formula $f(x) = cx$ for every positive rational $x = \frac{m}{n}$.

Since $$0 = f(0) = f(\frac{m}{n} + (-\frac{m}{n})) = f(-\frac{m}{n}) + f(\frac{m}{n}), \text{ we get that}$$

$$f(-\frac{m}{n}) = -f(\frac{m}{n}) = -(\frac{m}{n})c = c(-\frac{m}{n}).$$

Therefore, $f(x) = cx$ for every $x \in \mathbb{Q}$.

So far, we have not used the hypothesis that f is continuous. Let $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = cx$. Since f and g are continuous and $f|\mathbb{Q} = g|\mathbb{Q}$, Theorem 5.12 tells us that $f = g$, that is, $f(x) = cx$ for <u>all</u> $x \in \mathbb{R}$.

The continuous functions satisfying the functional equation $*$) were first described by Cauchy in 1821. It turns out that there are also discontinuous functions f satisfying $(*)$; however they are not easy to find. In fact, they must satisfy a nasty condition called "nonmeasurability" (which makes them "extremely discontinuous").

In calculus, another important is the idea of a convergent sequence (in $\mathbb{R}$ or $\mathbb{R}^n$). We can generalize the idea of a convergent sequence in $\mathbb{R}^n$ to any pseudometric space.

Definition 5.15 A <u>sequence</u> (x_n) <u>in (X, d)</u> <u>converges to x</u> $\in X$ if any one of the following (clearly equivalent) conditions holds:

1) $\forall \epsilon > 0\ \exists N \in \mathbb{N}$ such that if $n \geq N$, then $x_n \in B_\epsilon(x)$
(that is, if the sequence of numbers $(d(x_n, x)) \to 0$ in $\mathbb{R}$)

2) if O is open and $x \in O \subseteq X$, then $\exists N \in \mathbb{N}$ such that if $n \geq N$, then $x_n \in O$

3) if W is a neighborhood of x, then $\exists N \in \mathbb{N}$ such that if $n \geq N$, then $x_n \in W$.

If (x_n) converges to X, we write $(x_n) \to x$.

Condition$s$ 2) and 3) describe the convergence of sequences in terms of open sets (or neighborhoods) rather than directly using distances. Therefore replacing d with an equivalent metric d' does not affect which sequences converge to which points: sequential convergence is a topological property.

If a sequence (x_n) has a certain property P for all $n \geq$ some N, we say that "(x_n) <u>eventually</u> has property P." For example, the sequence (0,3,1,7,7,7,7,...) is <u>eventually constant</u>; the sequence $(-1, -3, 5, -2, 5, 6, 7, 8, \ldots, n, n+1, \ldots)$ is <u>eventually increasing</u>. Using this terminology, we can give a completely precise definition of convergence by saying: (x_n) converges to x if (x_n) is eventually in each neighborhood of x.

Example 5.16 1) In $\mathbb{R}$, $(\frac{1}{n}) \to 0$.

2) Suppose d is the discrete unit metric on X. Then each set $\{x\}$ is open so $(x_n) \to x$ iff (x_n) is eventually in each neighborhood of X iff (x_n) is eventually in $\{x\}$. In other words, $(x_n) \to x$ iff $x_n = x$ eventually. Every convergent sequence is eventually constant.

At the other extreme, suppose d is the trivial pseudometric on X. Then every sequence (x_n) converges to every point $x \in X$ (since the only neighborhood of x is X).

Example 5.16.2 shows that limits of sequences in a pseudometric space do not have to be unique: the same sequence can have many limits. However if d is a metric, then sequential limits in (X, d) must be unique, as the following theorem shows.

Theorem 5.17 A sequence (x_n) in a metric space (X, d) has at most one limit.

Proof Suppose $x \neq y \in X$ and let U, V be disjoint open sets with $x \in U$ and $y \in V$. If $(x_n) \to x$, then (x_n) must be eventually in U. Since U and V are disjoint, this means that (x_n) cannot be eventually in V (*in fact, the sequence is eventually outside* V), so $(x_n) \not\to y$. ●

In a pseudometric space, sequences can be used to describe the closure of a set.

Theorem 5.18 Suppose $A \subseteq X$, where (X, d) is a pseudometric space. Then $x \in \text{cl}\, A$ iff there is a sequence (a_n) in A for which $(a_n) \to x$.

Proof First, suppose there is a sequence (a_n) in A for which $(a_n) \to x$. If N is any neighborhood of x, then (a_n) is eventually in N. Therefore $N \cap A \neq \emptyset$, so $x \in \text{cl}\, A$.

Conversely, suppose $x \in \text{cl}\, A$. Then $B_{\frac{1}{n}}(x) \cap A \neq \emptyset$ for each $n \in \mathbb{N}$, so we can choose a point $a_n \in B_{\frac{1}{n}}(x) \cap A$. Then $(a_n) \to x$ (because $d(a_n, x) \to 0$). ●

Note: the sequence (a_n) is actually a function $f : \mathbb{N} \to X$ with the property that $a_n = f(n) \in B_{\frac{1}{n}}(x) \cap A$. Informally, the existence of such a function is completely clear. But to be precise, this argument actually depends on the Axiom of Choice. The proof as written doesn't describe how to pick specific a_n's: it depends on making "arbitrary choices." But using AC gives us a function f which "chooses" one point from each set in the collection $\{B_{\frac{1}{n}}(x) \cap A : n \in \mathbb{N}\}$.

Theorem 5.18 tells us something very important about the role of sequences in pseudometric spaces. The set A is closed iff $A = \text{cl}\, A$. But $A \subseteq \text{cl}\, A$ is always true, so we can say A is closed iff $\text{cl}\, A \subseteq A$. But that is true iff the limits of convergent sequences (a_n) from A must also be in A. Therefore a complete knowledge about what sequences converge to what points in (X, d) would let you determine which sets are closed (and therefore, by taking complements, which sets are open). In other words, all the information about "which sets in (X, d) are open or closed?" is revealed by the convergent sequences. We summarize this by saying that in a pseudometric space (X, d), sequences are sufficient to describe the topology.

Example 5.19 If d is a pseudometric on X, then d' defined by $d'(x, y) = \min\{1, d(x, y)\}$ is also a pseudometric on X. It is clear that $d'(x_n, x) \to 0$ iff $d(x_n, x) \to 0$. In other words, the metrics d and d' produce exactly the same convergent sequences and limits in (X, d). Since sequences are sufficient to determine the topology in pseudometric spaces, we conclude that d and d' are equivalent pseudometrics on X.

This example also shows that for any given (X, d) there is always an equivalent pseudometric d' on X for which all distances are ≤ 1: every pseudometric is equivalent to a bounded pseudometric.

Another modification of d that accomplishes the same thing is $d''(x, y) = \frac{d(x,y)}{1 + d(x,y)}$. This time, it is a little harder to verify that d'' is in fact a pseudometric $-$ the triangle inequality for d'' takes a bit of work. Clearly $d''(x, y) \leq 1$, and $d''(x_n, x) \to 0$ iff $d(x_n, x) \to 0$. So $d'' \sim d \sim d'$.

Definition 5.20 The diameter of a set A in (X, d) is defined by $\text{diam}(A) = \sup\{d(x, y) : x, y \in A\}$ (*we allow the possibility that diam*$(A) = \infty$). If A has finite diameter, we say that A is a bounded set.

The diameter of a set depends on the particular metric used. Since we can always replace d by an equivalent metric d' or d'' for which $\text{diam}(X) \leq 1$, boundedness is not a topological property.

It is an easy exercise to show that A is bounded iff $A \subseteq B_k(x_0)$ for some sufficiently large k (where x_0 can be any point in X).

The fact that the convergent sequences determine the topology in (X, d) gives us an upper bound on the size of certain metric spaces.

Theorem 5.21 If D is a dense set in a metric space (X, d), then $|X| \leq |D|^{\aleph_0}$. In particular, for a separable metric space (X, d), it must be true that $|X| \leq \aleph_0^{\aleph_0} = c$.

Proof For each $x \in X$, pick a sequence (d_n) in D such that $(d_n) \to x$. This sequence is actually a function $f_x \in D^{\mathbb{N}}$. Since a sequence in a metric space has at most one limit, the mapping $\Phi : X \to D^{\mathbb{N}}$ given by $\Phi(x) = f_x$ is one-to-one, so $|X| \leq |D^{\mathbb{N}}| = |D|^{\aleph_0}$. •

> *Note: If $|D| = m > \aleph_0$, do not jump to the (false) conclusion that $|X| \leq m^{\aleph_0} = m$. See the note of caution in Chapter I at the end of Example 14.8.*

Theorem 5.21 is not true if (X, d) is merely a pseudometric space. For example, let X be an uncountable set (with arbitrarily large cardinality) and let d be the trivial pseudometric on X. Then any singleton $\{x\}$ is dense, but $|X| > 1^{\aleph_0} = 1$. In this case, where does the proof of Theorem 5.21 fall apart?

Sequences are sufficient to determine the topology in a pseudometric space, and continuity is characterized in terms of open sets, so it should not be a surprise that sequences can be used to decide whether or not a function f between pseudometric spaces is continuous.

Theorem 5.22 Suppose (X, d) and (Y, s) are pseudometric spaces, and that $f : X \to Y$. Then f is continuous at $a \in X$ iff $(f(x_n)) \to f(a)$ for every sequence $(x_n) \to a$.

Proof Suppose f is continuous at a and consider any sequence $(x_n) \to a$. If W is a neighborhood of $f(a)$, then $f^{-1}[W]$ is a neighborhood of a, so (x_n) is eventually in $f^{-1}[W]$. This implies that $(f(x_n))$ is eventually in W, so $(f(x_n)) \to f(a)$.

Conversely, if f is not continuous at a, then $\sim (\forall \epsilon > 0\ \exists \delta > 0\ \ f[B_\delta(x)] \subseteq B_\epsilon(f(a)))$, that is, $\exists \epsilon > 0\ \forall \delta > 0\ \ f[B_\delta(x)] \nsubseteq B_\epsilon(f(a))$.

For this ϵ and $\delta = \frac{1}{n}$, we have that $f[B_{\frac{1}{n}}(x)] \nsubseteq B_\epsilon(f(a))$. So for each n we can choose a point $x_n \in B_{\frac{1}{n}}(x)$ for which $f(x_n) \notin B_\epsilon(f(x))$. Then $(x_n) \to x$ (because $d(x_n, x) < \frac{1}{n} \to 0$), but $(f(x_n)) \not\to f(a)$ (because $s(f(x_n), f(a)) \geq \epsilon$ for all n). Therefore f is not continuous at a. •

Note: the first half of the proof is phrased completely in terms of neighborhoods of x_0 and $f(x_0)$; that part of the proof is topological. However the second half makes explicit use of the metric. In fact, as we shall see later, the second half of the proof must involve a little more than just the open sets.

Notice also that the second half of the proof makes uses the Axiom of Choice (the function x "chooses" x_n for each n.

The following theorem and its corollaries are often technically useful. Moreover, they show us that a pseudometric space (X, d) has lots of "built-in" continuous functions – these functions can be defined using pseudometric d.

Theorem 5.23 In a pseudometric space (X, d):

$$\text{if } (x_n) \to x \text{ and } (y_n) \to y, \text{ then } d(x_n, y_n) \to d(x, y).$$

Proof Given $\epsilon > 0$, pick N large enough so that $n \geq N$ implies both $d(x_n, x) < \frac{\epsilon}{2}$ and $d(y_n, y) < \frac{\epsilon}{2}$. Since $d(x_n, y_n) \leq d(x_n, x) + d(x, y_n) \leq d(x_n, x) + d(x, y) + d(y, y_n)$, we get that

$$\text{if } n \geq N, \quad d(x_n, y_n) < \tfrac{\epsilon}{2} + d(x, y) + \tfrac{\epsilon}{2} = d(x, y) + \epsilon \quad (*)$$

Similarly, $d(x, y) \leq d(x, x_n) + d(x_n, y_n) + d(y_n, y)$, so that

$$\text{if } n \geq N, \quad d(x, y) < \tfrac{\epsilon}{2} + d(x_n, y_n) + \tfrac{\epsilon}{2} = d(x_n, y_n) + \epsilon \ \ (**)$$

Combining $(*)$ and $(**)$ gives that $|d(x_n, y_n) - d(x, y)| < \epsilon$ if $n \geq N$, so $d(x_n, y_n) \to d(x, y)$. •

Corollary 5.24 If a is a point in the pseudometric space (X, d) and if $(x_n) \to x$, then $d(x_n, a) \to d(x, a)$ (in $\mathbb{R}$).

Proof In Theorem 5.23, let (y_n) be the constant sequence where $y_n = a$ for each n. •

Corollary 5.25 Suppose $a \in (X, d)$. Define $f : X \to \mathbb{R}$ by $f(x) = d(x, a)$. Then f is continuous.

Proof Let x_0 be a point in X. If $(x_n) \to x_0$, then by Corollary 5.24, $(d(x_n, a)) \to d(x_0, a)$, that is $(f(x_n)) \to f(x_0)$. So f is continuous at x_0 (by Theorem 5.22). •

Recall that for a nonempty subset A of X, we defined $d(x, A) = \inf\{d(x, a) : a \in A\}$.
The following theorem is also useful.

Theorem 5.26 If A is a nonempty subset of the pseudometric space (X, d), then the function $f : X \to \mathbb{R}$ defined by $f(x) = d(x, A)$ is continuous.

Proof We show that f is continuous at each point $x_0 \in X$. Let $a \in A$. Then the following inequalities are true for any $x \in X$:

$$\begin{aligned} d(a, x_0) &\le d(a, x) + d(x, x_0) \\ d(a, x) &\le d(a, x_0) + d(x_0, x) \end{aligned}$$

Apply "$\inf_{a \in A}$" to each inequality to get

$$\begin{aligned} d(A, x_0) &\le d(A, x) + d(x, x_0) \text{ or } & d(A, x_0) - d(A, x) &\le d(x, x_0) \\ d(A, x) &\le d(A, x_0) + d(x_0, x) \text{ or } & d(A, x) - d(A, x_0) &\le d(x, x_0) \end{aligned}$$

so for all $x \in X$,

$|d(A, x) - d(A, x_0)| \le d(x, x_0)$. In other words,

for all $x \in X$, $|f(x) - f(x_0)| \le d(x, x_0)$. (*)

So for $\epsilon > 0$, we can choose $\delta = \epsilon$. Then if $d(x, x_0) < \delta$, we have $|f(x) - f(x_0)| < \epsilon$. Therefore f is continuous at x_0. •

Comments on the proof:

i) Given $\epsilon > 0$, the <u>same choice</u> $\delta = \epsilon$ can be used for every point x_0. A function f that satisfies this condition – stronger than mere continuity – is called <u>uniformly continuous</u>. We will discuss uniform continuity more in Chapter IV.)

ii) From the last inequality (*) we could have argued instead: for any sequence $(x_n) \to x_0$, we have $d(x_n, x_0) \to 0$, and this forces $(f(x_n)) \to f(x_0)$. Therefore f is continuous at x_0.
However this argument hides the observation about "uniform continuity" made in i).

Exercises

E22. Suppose $a \in A \subseteq \mathbb{R}$ and that $f : A \to \mathbb{R}$. We said that:

f is continuous at a if $\forall\, \epsilon > 0\ \exists\, \delta > 0$ such that if $x \in A$ and $|x - a| < \delta$,
then $|f(x) - f(a) < \epsilon$.

The order of the quantifiers is important. What functions are described by each of the following modifications of the definition:

a) $\forall\, \delta > 0\ \exists\, \epsilon > 0$ such that if $x \in A$ and $|x - a| < \delta$, then $|f(x) - f(a) < \epsilon$
b) $\forall\, \epsilon > 0\ \forall\, \delta > 0$ such that if $x \in A$ and $|x - a| < \delta$, then $|f(x) - f(a) < \epsilon$
c) $\exists\, \epsilon > 0\ \exists\, \delta > 0$ such that if $x \in A$ and $|x - a| < \delta$, then $|f(x) - f(a) < \epsilon$.

In each case, what happens if the restriction " > 0" is dropped on either ϵ or δ ?

E23. Suppose $f : \mathbb{R} \to \mathbb{R}$. What does each of the following statements tell us about f?
(*In this exercise, "interval" means "bounded open interval* (a, b)*"; and "*f *takes* I *into* J*" means that* $f[I] \subseteq J$.)

a) For every interval I containing a and every interval J containing b, f takes I into J.
b) There exists an interval J containing b and there exists an interval I containing a such that f takes I into J.
c) There exists an interval J containing b such that for every interval I containing a, f takes I into J.
d) There exists an interval J containing b such for every interval I containing a, f does not take I into J.
e) For every interval I containing a, there exists an interval J containing b such that f takes I into J.
f) There exists an interval I containing a such that for every interval J containing b, f takes I into J.

E24. (The Pasting Lemma, easy version) *The two parts of the problem give conditions when a collection of continuous functions defined on subsets of* X *can be "united" (* $=$ *"pasted together") to form a new continuous function.* In (X, d), suppose the sets O_α $(\alpha \in A)$ are open and that $F_1, ..., F_n$ $(n \in \mathbb{N})$ are closed.

a) Suppose that functions $f_\alpha : O_\alpha \to (Y, s)$ are continuous and that, if $\alpha \neq \beta$, then $f_\alpha|(O_\alpha \cap O_\beta) = f_\beta|(O_\alpha \cap O_\beta)$ (that is, f_α and f_β agree where their domains overlap).
Then $\bigcup_{\alpha \in A} f_\alpha = f : \bigcup_{\alpha \in A} O_\alpha \to Y$ is continuous.

b) Suppose that for each $i = 1, ..., n$, $f_i : F_i \to (Y, s)$ is continuous and that, if $i \neq j$, then $f_i|(F_i \cap F_j) = f_j|(F_i \cap F_j)$ (that is, f_i and f_j agree where their domains overlap).
Then $f = \bigcup_{i=1}^n f_i : \bigcup_{i=1}^n F_i \to Y$ is a continuous function.

c) Give an example to show that b) may be false for an infinite collection of functions f_i $(i \in \mathbb{N})$ defined on closed subsets of X, even if the domains F_i are pairwise disjoint.

E25. A point x_0 in (X, d) is <u>a cluster point of the sequence</u> (x_n) if for every neighborhood N of x_0 and for all $n \in \mathbb{N}$, $\exists k > n$ such that $x_k \in N$. (*When this condition is true, we say that "(x_n) is frequently in every neighborhood of x_0."*)

Prove that if $f : (X, d) \to (Y, s)$ is continuous and x_0 is a cluster point of (x_n) in X, then $f(x_0)$ is a cluster point of the sequence $(f(x_n))$ in Y.

E26. For $A \subseteq \mathbb{R}$, its characteristic function is defined by $\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$.
For which sets A is $\chi_A : \mathbb{R} \to \mathbb{R}$ continuous?

E27. Show that a set O is open in (X, d) if and only if there is a continuous function $f : X \to \mathbb{R}$ and an open set W in $\mathbb{R}$ such that $O = f^{-1}[W]$.

E28. Suppose (x_n) is a sequence in X, that x is some point in X and that $(f(x_n)) \to f(x)$ for <u>every</u> $f \in C(X)$. Prove that $(x_n) \to x$ or give a counterexample to show the statement is false.

E29. Let d be the usual metric on $\mathbb{R}$. Find a metric d' on $\mathbb{R}$ such that $(x_n) \to 0$ with respect to d iff $(x_n) \to 0$ with respect to d', but d' is not equivalent to d.

E30. a) Suppose A is a closed set in the pseudometric space (X, d) and that $x_0 \notin A$. Prove that there is a continuous function $f : X \to [0, 1]$ such that $f|A = 0$ and $f(x_0) = 1$. (*Hint: Consider the function "distance to the set A."*)

b) Suppose A and B are disjoint closed sets in (X, d). Prove that there exists a continuous function $f : X \to \mathbb{R}$ such that $f|A = 0$ and $f|B = 1$. (*Hint: Consider* $\frac{d(x,A)}{d(x,A) + d(x,B)}$)

c) Using b) (*or by some other method*) prove that if A and B are disjoint closed sets in (X, d), then there exist open sets U and V such that $A \subseteq U$, $B \subseteq V$ and $U \cap V = \emptyset$. Can U and V always be chosen so that $\text{cl}\, U \cap \text{cl}\, V = \emptyset$?

E31. A function $f: (X, d) \to (Y, s)$ is called an <u>isometry between X and Y</u> if f is <u>onto</u> and, for all $x, y \in X$, $d(x, y) = s(f(x), f(y))$. If such an f exists, we say that (X, d) and (Y, s) are <u>isometric</u> to each other. If f is <u>not</u> onto, we say f is an isometry of X <u>into</u> Y, or that f is an <u>isometric embedding of (X, d) into (Y, s)</u>. Let $\mathbb{R}$ and $\mathbb{R}^2$ have their usual metrics.

a) Prove that there is no isometry between $\mathbb{R}$ and $\mathbb{R}^2$.

b) Let $a \in \mathbb{R}$. Prove that there are exactly two isometries from $\mathbb{R}$ onto $\mathbb{R}$ which hold the point a fixed (that is, for which $f(a) = a$).

c) Give an example of a metric space which is isometric to a proper subset of itself.

E32. Use convergent sequences to prove the theorem that two continuous functions f and g from (X, d) into a metric space are identical if they agree on a dense set in X.

E33. Suppose $f : \mathbb{R} \to \mathbb{R}$. Then we can define $F : \mathbb{R} \to \mathbb{R}^2$ by $F(x) = (x, f(x))$, so that ran(F) is the graph of f.

a) Prove that the following statements are equivalent:

i) f is continuous

ii) F is continuous

iii) The sets $\{(x, y) : y \geq f(x)\}$ and $\{(x, y) : y \leq f(x)\}$ are both closed in $\mathbb{R}^2$.

b) Prove that if f is continuous, then its graph is a closed set in $\mathbb{R}^2$. Give a proof or a counterexample for the converse.

E34. Suppose $X = \{x, y, z, w\}$ is a four point set.

a) Show that the information $s(x, y) = s(y, z) = s(z, x) = 2$ and $s(x, w) = s(y, w) = s(z, w) = 1$ determines a unique metric s on X, i.e., that there is one and only one metric s on X which satisfies the given conditions.

b) Show that (X, s) cannot be isometrically embedded into the plane $\mathbb{R}^2$ (with its usual metric).

c) Prove or disprove: (X, s) can be isometrically embedded in (ℓ_2, d), where d is the usual metric on ℓ_2.

E35. Suppose (X, d) is a metric space for which $|X| > 1$ and in which $\emptyset$ and X are the only clopen sets. Prove that $|X| \geq c$.

(*Hint: First prove that there must be a nonconstant continuous function $f : X \to \mathbb{R}$. What can you say about the range of f ?)*

E36. Let X be a finite set and let $C^*(X)$ be the set of all bounded continuous functions from X into $\mathbb{R}$. Let s denote the "uniform metric" on $C^*(X)$ given by $s(f, g) = \sup \{|f(x) - g(x)| : x \in X\}$.

a) Prove that $(C^*(X), s)$ is separable.

b) If $X = \mathbb{N}$, is part a) still true?

Chapter II Review

Explain why each statement is true, or provide a counterexample.

1. A finite set in a metric space must be closed.

2. For $m, n \in \mathbb{N}$, write $m - n = 2^k z$, where z is an integer not divisible by 2. Define $d(m, m) = 0$ and, for $m \neq n$, $d(m, n) = k$. Then d is a pseudometric on $\mathbb{N}$.

3. Consider the set $[1, \infty)$ with the metric $s(x, y) = \frac{3|x-y|}{3 + 3|x-y|}$. Let $\mathbb{N}$ have its usual metric d and define $f : [1, \infty) \to \mathbb{N}$ by $f(x) =$ "the largest integer $\leq x$". Then f is continuous.

4. If N_1 and N_2 are neighborhoods of x in (X, d), then $N_1 \cap N_2$ is also a neighborhood of x.

5. For any open subset O of a metric space (X, d), $\text{int}(\text{cl}(O)) = O$.

6. The metric $d(n, m) = |\frac{1}{n} - \frac{1}{m}|$ on $\mathbb{N}$ is equivalent to the usual metric on $\mathbb{N}$.

7. Define $f : \mathbb{N} \to \mathbb{R}$ by $f(n) =$ the n^{th} digit of the decimal expansion of π. Then f is continuous.

8. The set of all real numbers with a decimal expansion of the form $x = 0.x_1x_1x_3 ... x_n010101...$ is dense in $[0, 1]$.

9. There are exactly c countable dense subsets of $\mathbb{R}$.

10. Suppose we measure distances in $\mathbb{R}$ using the metric $d(x, y) = \frac{|x-y|}{1+|x-y|}$. Then the function cos : $\mathbb{R} \to \mathbb{R}$ is continuous at every point $a \in \mathbb{R}$.

11. In a pseudometric space (X, d), a subset A is dense if and only if int $(X - A) = \emptyset$.

12. Let d_t denote the "taxicab" metric $d_t(x, y) = \sum_{i=1}^{n} |x_i - y_i|$ on $\mathbb{R}^n$. $\mathbb{Q}^n$ is dense in $(\mathbb{R}^n, d_t)$.

13. If B is a countable subset of $\mathbb{R}$, then $\mathbb{R} - B$ is dense in $\mathbb{R}$.

14. If $A \subseteq [0, 1]$ and cl $A \neq [0, 1]$, then int $A \neq \emptyset$.

15. Let $A = \{\frac{1}{n} : n \in \mathbb{N}\}$. The discrete unit metric produces the same topology on A as the usual metric.

16. In a pseudometric space (X, d), cl $A =$ cl $(X - A)$ if and only if A is clopen.

17. If U is an open set in $\mathbb{R}$ and $U \supseteq \mathbb{Q}$, then $U = \mathbb{R}$.

18. There is a sequence of open sets O_n in $\mathbb{R}$ such that $\bigcap_{n=1}^{\infty} O_n = \mathbb{P}$.

19. If $A \subseteq (X, d)$, then int $A = X -$ cl $(X - A)$.

20. There are exactly c continuous functions $f : \mathbb{N} \to \mathbb{N}$.

21. Let $\mathbb{R}$ have the metric $d(x,y) = \frac{|x-y|}{1+|x-y|}$ and let $a_n = \frac{n^2}{n^2+1}$. In $(\mathbb{R}, d)$, $(a_n) \to 1$.

22. Suppose d is the usual metric on $\mathbb{R}$ and d_1 is another metric with the property that for every sequence (r_n) :

$(r_n) \to 5$ with respect to d_1 if and only if $(r_n) \to 5$ with respect to d.

Then $d_1 \sim d$.

23. Suppose $A \subseteq B \subseteq (X, d)$. If $\text{cl}\, A = \text{cl}\, B$, $\text{int}\, A = \text{int}\, B$, and $\text{Fr}\, A = \text{Fr}\, B$, then $A = B$.

24. Suppose $C([0,1])$ has the metric $d(f,g) = \int_0^1 |f-g|$ and define $\Phi : (C([0,1]), d) \to \mathbb{R}$ by $\Phi(f) = \int_0^1 f$. Then Φ is continuous.

25. Let d be the trivial pseudometric on $\mathbb{R}$. In $(\mathbb{R}, d)$, each real number r is the limit of a sequence of irrational numbers.

26. Suppose $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are continuous and that $f \neq g$. Then there must exist a point $p \in \mathbb{Q}$ where $f(p) \neq g(p)$ and a point $q \in \mathbb{P}$ where $f(q) \neq g(q)$.

27. In a pseudometric space (X, d), $\text{cl}\, A = \text{int}\, A$ if and only if A is clopen.

28. If (X, d) is a metric space in which every convergent sequence is eventually constant, then $\mathcal{T}_d = \mathcal{P}(X)$.

29. Let $x \in (X, d)$ Suppose (x_n) is a sequence such that $(f(x_n)) \to f(x)$ for every continuous $f : X \to \mathbb{R}$. Then $(x_n) \to x$.

30. Let $d^*(f,g) = \sup\{|f(x) - g(x)| : x \in [0,1]\}$ for $f, g \in C([0,1])$.

Let $f \in C([0,1])$ be the function given by $f(x) = x + 2$ and let (f_n) be a sequence such that $(f_n) \to f$ in $(C([0,1]), d^*)$

Then there is an $N \in \mathbb{N}$ such that $f_n(x) \geq x$ for all $x \in [0,1]$ and all $n \geq N$.

31. If $f : \mathbb{R} \to \mathbb{R}$ is continuous, then the graph of f is a closed subset of $\mathbb{R}^2$.

32. If the graph of a function $f : \mathbb{R} \to \mathbb{R}$ is closed subset of $\mathbb{R}^2$, then f is continuous.

33. The are exactly c different metrics d on $\mathbb{R}$ for which $\mathcal{T}_d =$ the usual topology on $\mathbb{R}$.

34. In $\mathbb{R}$, the interval $[-2, 1]$ can be written as a countable intersection of open sets.

35. For any $f \in \mathbb{N}^{\mathbb{N}}$, then there is a continuous function $g \in \mathbb{R}^{\mathbb{R}}$ such that $g|\mathbb{N} = f$.

36. Let d be the usual metric on $\mathbb{R}$ and d' the discrete unit metric. Suppose $f : (\mathbb{R}, d) \to (\mathbb{N}, d')$ is continuous. Then f is constant.

37. If $|X| > 1$, then there are infinitely many different metrics d on X for which $\mathcal{T}_d$ is the discrete topology.

38. Suppose (X, d) is a pseudometric space and $A \subseteq X$. A is dense in X if and only if $\operatorname{int}(X - A) = \emptyset$.

39. The space $\mathbb{P}$ of irrational numbers is separable.

40. Suppose (X, d) and (Y, d') are pseudometric spaces and that $f : X \to Y$ is continuous at a. If $d(a, b) = 0$, then f is also continuous at b.

41. If A and B are subsets of (X, d), then $\operatorname{int}(A \cap B) = \operatorname{int}(A) \cap \operatorname{int}(B)$.

42. Let d^* be the "max metric" on $\mathbb{R}$ and let d_t be the "taxicab metric" on $\mathbb{R}$. For $x \in \mathbb{R}$, let $f(x) = \cos(x^3)$. Then $f : (\mathbb{R}, d^*) \to (\mathbb{R}, d_t)$ is continuous.

43. Let d be a pseudometric on the set $X = \{0, 1\}$. Then either $\mathcal{T}_d = \{\emptyset, X\}$ or $\mathcal{T}_d = \mathcal{P}(X)$.

44. Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous and $f(p) = p - \sqrt{2}$ for each <u>irrational</u> p. Then $f(17) \in \mathbb{P}$.

45. A finite set in a pseudometric space must be closed.

46. If $f : \mathbb{R} \to \mathbb{N}$ is continuous and $f(1) = 1$, then there must exist an irrational number x for which $f(x) = 1$.

47. In a metric space (X, d), it cannot happen that $B_\epsilon(a) = X - B_\epsilon(b)$.

48. The discrete unit metric produces the same topology on $\mathbb{N}$ as the usual metric on $\mathbb{N}$.

49. If D is an uncountable dense subset of $\mathbb{R}$ and C is countable, then $D - C$ is dense in $\mathbb{R}$.

50. Suppose that $f : \ell_2 \to \mathbb{R}$ is continuous and that $f(x) = 0$ for whenever x is any sequence in ℓ_2 that is eventually 0. Then $f((1, \frac{1}{2}, \frac{1}{3}, ..., \frac{1}{n}, ...)) = 0$.

51. If $f : (X, d) \to (Y, d')$ is continuous and onto, and X is separable, then Y is separable.

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