Chapter III
Topological Spaces

1. Introduction

In Chapter I we looked at properties of sets, and in Chapter II we added some additional structure to a set – a distance function d – to create a pseudometric space. We then looked at some of the most basic definitions and properties of pseudometric spaces. There is much more, and some of the most useful and interesting properties of pseudometric spaces will be discussed in Chapter IV. But in Chapter III we look at an important generalization.

Early in Chapter II we observed that the idea of continuity (in calculus) depends on talking about "nearness," so we used a distance function d to make the idea of "nearness" precise. The result was that we could carry over the definition of continuity from calculus to pseudometric spaces. The distance function d also led us to the idea of an open set in a pseudometric space. From there we developed properties of closed sets, closures, interiors, frontiers, dense sets, continuity, and sequential convergence.

One important observation was that open (or closed) sets are all we need to work with many of these concepts; that is, we can often do what we need using the open sets without knowing what specific d that generated these open sets: the topology $\mathcal{T}_d$ is what really matters. For example, cl A is defined in (X, d) in terms of the closed sets, so cl A doesn't change if d is replaced with a different but equivalent metric d' – one that generates the same open sets. Changing d to an equivalent metric d' also doesn't affect interiors, continuity, or convergent sequences. In summary: for many purposes d is logically unnecessary (although d might be a handy tool) after d has done its job in creating the topology $\mathcal{T}_d$.

This suggests a way to generalize our work. For a particular set X we can simply add a topology – that is, a collection of "open" sets given without any mention of a pseudometric that might have generated them. Of course when we do this, we want these "open sets" to "behave the way open sets should behave." This leads us to the definition of a topological space.

2. Topological Spaces

Definition 2.1 A <u>topology</u> $\mathcal{T}$ on a set X is a collection of subsets of X such that

i) $\emptyset, X \in \mathcal{T}$
ii) if $O_\alpha \in \mathcal{T}$ for each $\alpha \in A$, then $\bigcup_{a \in A} O_a \in \mathcal{T}$
iii) if $O_1, ..., O_n \in \mathcal{T}$, then $O_1 \cap ... \cap O_n \in \mathcal{T}$.

A set $O \subseteq X$ is called <u>open</u> if $O \in \mathcal{T}$. The pair $(X, \mathcal{T})$ is called a <u>topological space</u>.

We emphasize that in a topological space there is no distance function d: therefore concepts like "distance between two points" and "ϵ-ball" make no sense in $(X, \mathcal{T})$. There is no preconceived idea about what "open" means; to say "O is open" means nothing more or less than "$O \in \mathcal{T}$."

In a topological space $(X, \mathcal{T})$, we can go on to define closed sets and isolated points just as we did in pseudometric spaces.

Definition 2.2 A set $F \subseteq (X, \mathcal{T})$ is called closed if $X - F$ is open, that is, if $X - F \in \mathcal{T}$.

Definition 2.3 A point $a \in (X, \mathcal{T})$ is called isolated if $\{a\}$ is open, that is, if $\{a\} \in \mathcal{T}$.

The proof of the following theorem is the same as it was for pseudometric spaces; we just take complements and apply properties of open sets.

Theorem 2.4 In any topological space $(X, \mathcal{T})$

i) $\emptyset$ and X are closed
ii) if F_α is closed for each $a \in A$, then $\bigcap_{\alpha \in A} F_\alpha$ is closed
iii) if $F_1, ..., F_n$ are closed, then $\bigcup_{i=i}^{n} F_i$ is closed.

More informally, ii) and iii) state that intersections and finite unions of closed sets are closed.

Proof Read the proof for Theorem II.4.2. $\bullet$

For a particular topological space $(X, \mathcal{T})$, it is sometimes possible to find a pseudometric d on X for which $\mathcal{T}_d = \mathcal{T}$ – that is, a d which generates exactly the same open sets as those already given in $\mathcal{T}$. But this cannot always be done.

Definition 2.5 A topological space $(X, \mathcal{T})$ is called pseudometrizable if there exists a pseudometric d on X such that $\mathcal{T}_d = \mathcal{T}$. If d is a metric, then $(X, \mathcal{T})$ is called metrizable.

Examples 2.6

1) Suppose X is a set and $\mathcal{T} = \{\emptyset, X\}$. $\mathcal{T}$ is called the trivial topology on X and it is the smallest possible topology on X. $(X, \mathcal{T})$ is called a trivial topological space. The only open (or closed) sets are $\emptyset$ and X.

If we put the trivial pseudometric d on X, then $\mathcal{T}_d = \mathcal{T}$. So a trivial topological space turns out to be pseudometrizable.

At the opposite extreme, suppose $\mathcal{T} = \mathcal{P}(X)$. Then $\mathcal{T}$ is called the discrete topology on X and it is the largest possible topology on X. $(X, \mathcal{T})$ is called a discrete topological space. Every subset is open (and also closed). Every point of X is isolated.

If we put the discrete unit metric d (or any equivalent metric) on X, then $\mathcal{T}_d = \mathcal{T}$. So a discrete topological space is metrizable.

2) Suppose $X = \{0, 1\}$ and let $\mathcal{T} = \{\emptyset, \{1\}, X\}$. $(X, \mathcal{T})$ is a topological space called

Sierpinski space. In this case it is not possible to find a pseudometric d on X for which $\mathcal{T}_d = \mathcal{T}$, so Sierpinski space is not pseudometrizable. To see this, consider any pseudometric d on X.

If $d(0,1) = 0$, then d is the trivial pseudometric on X and $\{\emptyset, X\} = \mathcal{T}_d \neq \mathcal{T}$.

If $d(0,1) = \delta > 0$, then the open ball $B_\delta(0) = \{0\} \in \mathcal{T}_d$, so $\mathcal{T}_d \neq \mathcal{T}$.
(*In this case* $\mathcal{T}_d$ *is actually the discrete topology:* d *is just a rescaling of the discrete unit metric.*)

Another possible topology on $X = \{0,1\}$ is $\mathcal{T}' = \{\emptyset, \{0\}, X\}$, although $(X, \mathcal{T})$ and $(X, \mathcal{T}')$ seem very much alike: both are two-point spaces, each with containing exactly one isolated point. One space can be obtained from the other simply renaming "0" and "1" as "1" and "0" respectively. Such "topologically identical" spaces are called "homeomorphic." We will give a precise definition of what this means later in this chapter.

3) For a set X, let $\mathcal{T} = \{O \subseteq X : O = \emptyset$ or $X - O$ is finite$\}$. $\mathcal{T}$ is a topology on X :

i) Clearly, $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$.

ii) Suppose $O_\alpha \in \mathcal{T}$ for each $\alpha \in A$. If $\bigcup_{\alpha \in A} O_\alpha = \emptyset$, then $\bigcup_{\alpha \in A} O_\alpha \in \mathcal{T}$. Otherwise there is at least one $O_{\alpha_0} \neq \emptyset$. Then $X - O_{\alpha_0}$ is finite, so $X - \bigcup_{\alpha \in A} O_\alpha$ $= \bigcap_{\alpha \in A}(X - O_\alpha) \subseteq X - O_{\alpha_0}$. Therefore $X - \bigcup_{\alpha \in A} O_\alpha$ is finite, so $\bigcup_{\alpha \in A} O_\alpha \in \mathcal{T}$.

iii) If $O_1, ..., O_n \in \mathcal{T}$ and some $O_j = \emptyset$, then $\bigcap_{i=1}^n O_i = \emptyset$ so $\bigcap_{i=1}^n O_i \in \mathcal{T}$. Otherwise each O_i is nonempty, so each $X - O_i$ is finite. Then $(X - O_1) \cup ... \cup (X - O_n)$ $= X - \bigcap_{i=1}^n O_i$ is finite, so $\bigcap_{i=1}^n O_i \in \mathcal{T}$.

In $(X, \mathcal{T})$, a set F is closed iff $F = \emptyset$ or F is finite. Because the open sets are $\emptyset$ and the complements of finite sets, $\mathcal{T}$ is called the cofinite topology on X.

If X is a finite set, then the cofinite topology is the same as the discrete topology on X. (*Why?*)
In X is infinite, then no point in $(X, \mathcal{T})$ is isolated.

Suppose X is an infinite set with the cofinite topology $\mathcal{T}$. If U and V are nonempty open sets, then $X - U$ and $X - V$ must be finite so $(X - U) \cup (X - V) = X - (U \cap V)$ is finite. Since X is infinite, this means that $U \cap V \neq \emptyset$ (*in fact,* $U \cap V$ *must be infinite*). Therefore every pair of nonempty open sets in $(X, \mathcal{T})$ has nonempty intersection!

The preceding paragraph lets us see that an infinite cofinite space $(X, \mathcal{T})$ is not pseudometrizable:

i) if d is the trivial pseudometric on X, then certainly $\mathcal{T}_d \neq \mathcal{T}$, and

ii) if d is not the trivial pseudometric on X, then there exist points $a, b \in X$ for which $d(a,b) = \delta > 0$. In that case, $B_{\frac{\delta}{2}}(a)$ and $B_{\frac{\delta}{2}}(b)$ would be disjoint nonempty open sets in $\mathcal{T}_d$, so $\mathcal{T}_d \neq \mathcal{T}$.

4) On $\mathbb{R}$, let $\mathcal{T} = \{(a, \infty): a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$. It is easy to verify that $\mathcal{T}$ is a topology on $\mathbb{R}$, called the right-ray topology. Is $(\mathbb{R}, \mathcal{T})$ metrizable or pseudometrizable?

If $X = \emptyset$, then $\mathcal{T} = \{\emptyset\}$ is the only possible topology on X, and $\mathcal{T} = \{\emptyset, \{a\}\}$ is the only possible topology on a singleton set $X = \{a\}$. But for $|X| > 1$, there are many possible topologies on X. For example, there are four possible topologies on the set $X = \{a, b\}$. These are the trivial topology, the discrete topology, $\{\emptyset, \{a\}, X\}$, and $\{\emptyset, \{b\}, X\}$ although the last two, as we mentioned earlier, can be considered as "topologically identical."

If $\mathcal{T}$ is a topology on X, then $\mathcal{T}$ is a collection of subsets of X, so $\mathcal{T} \subseteq \mathcal{P}(X)$. This means that $\mathcal{T} \in \mathcal{P}(\mathcal{P}(X))$, so $|\mathcal{P}(\mathcal{P}(X))| = 2^{(2^{|X|})}$ is an upper bound for the number of possible topologies on X. For example, there are no more than $2^{2^7} \approx 3.4 \times 10^{38}$ topologies on a set X with 7 elements. But this upper bound is actually very crude, as the following table (given without proof) indicates:

$n = \|X\|$	Actual number of topologies on X
0	1
1	1
2	4
3	29
4	355
5	6942
6	209527
7	9535241 (*many less than* 10^{38})

Counting topologies on finite sets is really a question about combinatorics and we will not pursue that topic.

Each concept we defined for pseudometric spaces can be carried over directly to topological spaces if the concept was defined in topological terms $-$ that is, in terms of open (or closed) sets. This applies, for example, to the definitions of interior, closure, and frontier in pseudometric spaces, so these definitions can also be carried over verbatim to a topological space $(X, \mathcal{T})$.

Definition 2.7 Suppose $A \subseteq (X, \mathcal{T})$. We define

int$_X A$ = the interior of A in X $= \bigcup\{O : O$ is open and $O \subseteq A\}$
cl$_X A$ = the closure of A in X $= \bigcap\{F : F$ is closed and $F \supseteq A\}$
Fr$_X A$ = the frontier (or boundary) of A in X $=$ cl$_X A \cap$ cl$_X (X - A)$

As before, we will drop the subscript "X" when the context makes it clear.

The properties for the operators cl, int, and Fr (except those that mention a pseudometric d or an ϵ-ball) remain true. The proofs in the preceding chapter were deliberately phrased in topological terms so they would carry over to the more general setting of topological spaces.

Theorem 2.8 Suppose $A \subseteq (X, d)$. Then

1) a) int A is the largest open subset of A (that is, if O is open and $O \subseteq A$, then $O \subseteq \text{int } A$).
b) A is open iff $A = \text{ int } A$ (since int $A \subseteq A$, the equality is equivalent to $A \subseteq \text{int } A$).
c) $x \in \text{int } A$ iff there is an open set O such that $x \in O \subseteq A$

2) a) cl A is the smallest closed set containing A (that is, if F is closed and $F \supseteq A$, then $F \supseteq \text{cl } A$).
b) A is closed iff $A = \text{cl } A$ (since $A \subseteq \text{cl } A$, the equality is equivalent to cl $A \subseteq A$).
c) $x \in \text{cl } A$ iff for every open set O containing x, $O \cap A \neq \emptyset$

3) a) Fr A is closed and Fr $A = \text{Fr}\,(X - A)$.
b) $x \in \text{Fr } A$ iff for every open set O containing x, $O \cap A \neq \emptyset$ and $O \cap (X - A) \neq \emptyset$
c) A is clopen iff Fr $A = \emptyset$.

See the proof of Theorem II.4.5

At this point, we add a few additional facts about these operators. Some of the proofs are left as exercises.

Theorem 2.9 Suppose A, B are subsets of a topological space $(X, \mathcal{T})$. Then

1) $\text{cl}\,(A \cup B) = \text{cl}\,(A) \cup \text{cl}\,(B)$
2) $\text{cl}\,A = A \cup \text{Fr}\,A$
3) $\text{int } A = A - \text{Fr}\,A = X - \text{cl}(X - A)$
4) $\text{Fr}\,A = \text{cl}\,A - \text{int}\,A$
5) $X = \text{int}\,A \cup \text{Fr}\,A \cup \text{int}\,(X - A)$, and these 3 sets are disjoint.

Proof 1) $A \subseteq A \cup B$ so, from the definition of closure, we have cl $A \subseteq \text{cl}\,(A \cup B)$. Similarly, cl $B \subseteq \text{cl}\,(A \cup B)$ Therefore cl $A \cup \text{cl}\,B \subseteq \text{cl}\,(A \cup B)$.

On the other hand, cl $A \cup \text{cl}\,B$ is the union of two closed sets, so cl $A \cup$ cl B is closed and cl $A \cup$ cl $B \supseteq A \cup B$, so cl $A \cup$ cl $B \supseteq \text{cl}(A \cup B)$. (*As an exercise, try proving 1) instead using the characterization of closures given above in Theorem 2.8.2c.*)

Is 1) true if " $\cup$ " is replaced by " $\cap$ " ?

2) Suppose $x \in \text{cl}\,A$ but $x \notin A$. If O is any open set containing x, then $O \cap A \neq \emptyset$ (because $x \in \text{cl}\,A$) and $O \cap (X - A) \neq \emptyset$ (because the intersection contains x). Therefore $x \in \text{Fr}\,A$, so $x \in A \cup \text{Fr}\,A$.

Conversely, suppose $x \in A \cup \text{Fr}\,A$. If $x \in A$, then $x \in \text{cl}\,A$. And if $x \notin A$, then $x \in \text{Fr}\,A = \text{cl}\,A \cap \text{cl}\,(X - A)$, so $x \in \text{cl}\,A$. Therefore $\text{cl}\,A = A \cup \text{Fr}\,A$.

The proofs of 3) $-$ 5) are left as exercises. •

Theorem 2.9 shows us that complements, closures, interiors and frontiers are interrelated and therefore some of these operators are redundant. That is, if we wanted to very "economical," we

could discard some of them. For example, we could avoid using "Fr" and "int" and just use "cl" and complement because $\text{Fr}\, A = \text{cl}\, A \cap \text{cl}\,(X - A)$ and $\text{int}\, A = A - \text{Fr}\, A$
$= A - (\text{cl}\, A \cap \text{cl}\,(X - A))$. Of course, the most economical way of doing things is not necessarily the most convenient. (*Could we get by only using complements and "Fr" – that is, can we define "int" and ""cl" in terms of "Fr" and complements? Or could we use just "int" and complements?*)

Here is a famous related problem from the early days of topology: for $A \subseteq (X, \mathcal{T})$, is there an upper bound for the number of different subsets of X which might created from A using only complements and closures, repeated in any order? (*As we just observed, using the interior and frontier operators would not help to create any additional sets.*) For example, one might start with A and then consider such sets as $\text{cl}\, A$, $X - \text{cl}A$, $\text{cl}(X - \text{cl}(X - A))$, and so on. An old theorem of Kuratowski (1922) says that for any set A in any space $(X, \mathcal{T})$, the upper bound is 14. Moreover, this upper bound is "sharp" – there is a set $A \subseteq \mathbb{R}$ from which 14 sets can actually be obtained! Can you find such a set?

Definition 2.10 Suppose $D \subseteq (X, \mathcal{T})$. D is called dense in X if $\text{cl}\, D = X$. The space $(X, \mathcal{T})$ is called separable if there exists a countable dense set D in X.

Example 2.11 Let $\mathcal{T}$ be the cofinite topology on $\mathbb{R}$. Let $\mathbb{E} = \{2, 4, 6, ...\}$.

$\text{int}\, \mathbb{E} = \emptyset$, because any nonempty open set O has finite complement and therefore is not a subset of $\mathbb{E}$. In fact, $\text{int}\, B = \emptyset$ whenever $\mathbb{R} - B$ is infinite.

$\text{cl}\, \mathbb{E} = \mathbb{R}$ because the only closed set containing $\mathbb{E}$ is $\mathbb{R}$. Therefore $(\mathbb{R}, \mathcal{T})$ is separable. In fact, any infinite set B is dense.

$\text{Fr}\, \mathbb{E} = \text{cl}\,(\mathbb{E}) \cap \text{cl}\,(\mathbb{R} - \mathbb{E}) = \mathbb{R} \cap \mathbb{R} = \mathbb{R}$

$(\mathbb{R}, \mathcal{T})$ is not pseudometrizable (*why?*)

Example 2.12 Let $\mathcal{T}$ be the right-ray topology on $\mathbb{R}$. In $(\mathbb{R}, \mathcal{T})$,

$\text{int}\, \mathbb{Z} = \emptyset$
$\text{cl}\, \mathbb{Z} = \mathbb{R}$, so $(\mathbb{R}, \mathcal{T})$ is separable
$\text{Fr}\, \mathbb{Z} = \mathbb{R}$
Any two nonempty open sets intersect, so $(\mathbb{R}, \mathcal{T})$ is not metrizable. Is it pseudometrizable?

3. Subspaces

If A is a subset of a space $(X, \mathcal{T})$, then there is a natural way to make A into a topological space. We use the open sets in X to define open sets in A.

Definition 3.1 Suppose $A \subseteq X$, where $(X, \mathcal{T})$ is a topological space. The subspace topology on A is defined as $\mathcal{T}_A = \{A \cap O : O \in \mathcal{T}\}$. $(A, \mathcal{T}_A)$ is called a subspace of $(X, \mathcal{T})$.
(*Check that $\mathcal{T}_A$ is actually a topology on A. To say that A is a subspace of $(X, \mathcal{T})$ implies that A is given the topology $\mathcal{T}_A$. To emphasize that A is a subspace, we sometimes write $A \subseteq (X, \mathcal{T})$ rather than just $A \subseteq X$.*)

If $A \subseteq X$, we sometimes refer to $O \cap A$ as the "restriction of O to A" or the "trace of O on A"

Example 3.2 Consider $\mathbb{N} \subseteq \mathbb{R}$, where $\mathbb{R}$ has its usual topology. For each $n \in \mathbb{N}$, the interval $(n-1, n+1)$ is open in $\mathbb{R}$. Therefore $(n-1, n+1) \cap \mathbb{N} = \{n\}$ is open in the subspace $\mathbb{N}$, so every point n is isolated in the subspace. The subspace topology is the discrete topology. Notice: the subspace topology on $\mathbb{N}$ is just what we would get if we used the usual metric on $\mathbb{N}$ to generate open sets in $\mathbb{N}$. Similarly, it is easy to check that in $\mathbb{R}^2$ the subspace topology on the x-axis is the same as the usual metric topology on $\mathbb{R}$.

Suppose $A \subseteq (X, d)$. Then we can think of two ways to make A into a topological space.

i) d gives a topology $\mathcal{T}_d$ on X. Take the open sets in $\mathcal{T}_d$ and intersect them with A. This gives us the subspace topology on A, which we call $(\mathcal{T}_d)_A$ and $(A, (\mathcal{T}_d)_A)$ is a subspace of $(X, \mathcal{T})$.

2) Or, we could treat A as a pseudometric space: just use d to measure distances in A. To be very precise, $d : X \times X \to \mathbb{R}$, so we make a "new" pseudometric d' defined by $d' = d|A \times A$. Then (A, d') is a pseudometric space and we can use d' to generate open sets in A: the topology $\mathcal{T}_{d'}$. (*Usually, we would be less compulsive about the notation and just use d also to refer to the pseudometric n A. But for just a moment it will help to distinguish d and $d' = d|A \times A$ with different names.*)

It turns out (fortunately) that 1) and 2) produce the same open sets in A: the open sets in (A, d') are just the open sets from (X, d) restricted to A. That's just what Theorem 3.3 says (in "fancier" notation).

Theorem 3.3 Suppose $A \subseteq (X, d)$, where d is a pseudometric: then $(\mathcal{T}_d)_A = \mathcal{T}_{d'}$.

Proof Suppose $U \in (\mathcal{T}_d)_A$, so $U = O \cap A$ where $O \in \mathcal{T}_d$. Let $A \in U$. There is an $\epsilon > 0$ such that $B^d_\epsilon(a) \subseteq O$. Since $d' = d|A \times A$, we get that $B^{d'}_\epsilon(a) = B^d_\epsilon(a) \cap A \subseteq O \cap A = U$, so $U \in \mathcal{T}_{d'}$.

Conversely, suppose $U \in \mathcal{T}_{d'}$. For each $a \in U$ there is an $\epsilon_a > 0$ such that $B^{d'}_{\epsilon_a}(a) \subseteq U$, and $U = \bigcup_{a \in U} B^{d'}_{\epsilon_a}(a)$. Let $O = \bigcup_{a \in U} B^d_{\epsilon_a}(a)$, an open set in $\mathcal{T}_d$. Since $B^{d'}_{\epsilon_a}(a) = B^d_{\epsilon_a}(a) \cap A$, we get $O \cap A = (\bigcup_{a \in O} B^d_{\epsilon_a}(a)) \cap A = (\bigcup_{a \in O} B^d_{\epsilon_a}(a) \cap A)$
$= \bigcup_{a \in U} B^{d'}_{\epsilon_a}(a) = U$. Therefore $U \in (\mathcal{T}_d)_A$. •

Exercise Verify that in any space $(X, \mathcal{T})$

i) If U is open in $(X, \mathcal{T})$ and A is an open set in the subspace U, then A is open in X. (*"An open subset of an open set is open."*)

ii) If F is closed in $(X, \mathcal{T})$ and A is a closed set in the subspace F, then F is closed in X. (*"A closed subset of a closed set is closed."*)

4. Neighborhoods

Definition 4.1 If $N \subseteq (X, \mathcal{T})$ and $x \in \text{int}\, N$, then we say that N is a neighborhood of x. The collection $\mathcal{N}_x = \{N \subseteq X : N \text{ is a neighborhood of } x\}$ is called the neighborhood system at x.

Note that

1) $\mathcal{N}_x \neq \emptyset$, because every point x has at least one neighborhood $-$ for example, $N = X$.
2) If N_1 and $N_2 \in \mathcal{N}_x$, then $x \in \text{int}\, N_1 \cap \text{int}\, N_2 =$ (*why?*) $\text{int}\,(N_1 \cap N_2)$. Therefore $N_1 \cap N_2 \in \mathcal{N}_x$.
3) If $N \in \mathcal{N}_x$ and $N \subseteq N'$, then $x \in \text{int}\, N \subseteq \text{int}\, N'$, so $N' \in \mathcal{N}_x$ (that is, if N' contains a neighborhood of x, then N' is also a neighborhood of x.)

Just as in pseudometric spaces, it is clear that a set O in $(X, \mathcal{T})$ is open iff O is a neighborhood of each of its points.

In a pseudometric space, we use the ϵ-balls centered at x to measure "nearness" to x. For example, saying that "every ϵ-ball in $\mathbb{R}$ centered at x contains an irrational number" tells us that "there are irrational numbers arbitrarily near to x." Of course, we could convey the same information in terms of neighborhoods by saying "every neighborhood of x in $\mathbb{R}$ contains an irrational number." Or instead we could say it in terms of "open sets": "every open set in $\mathbb{R}$ containing x contains an irrational number." These are all equivalent ways to say "there are irrational numbers arbitrarily near to x." *That we can say it all these different ways isn't surprising since, in (X, d), open sets and neighborhoods were defined in terms of ϵ-balls.*

In a topological space $(X, \mathcal{T})$ we don't have ϵ-balls, but we still have open sets and neighborhoods. We now think of the neighborhoods in $\mathcal{N}_x$ (or, if we prefer, the collection of open sets containing x) as the tool we use to talk about "nearness to x."

For example, suppose X has the trivial topology $\mathcal{T}$. For any $x \in X$, the only neighborhood of x is X: therefore every y in X is in every neighborhood of x : the neighborhoods of x are unable to "separate" x and y and that's analogous to having $d(x, y) = 0$ (if we had a pseudometric). In that sense, all points in $(X, \mathcal{T})$ are "very close together": so close together, in fact, that they are "indistinguishable." The neighborhoods of x tell us this.

At the opposite extreme, suppose X has the discrete topology $\mathcal{T}$ and that $x \in X$. If $x \in W$ then (*since W is open*), W is a neighborhood of x. $N = \{x\}$ is the smallest neighborhood of x. So every point x has a neighborhood N that excludes all other points y: for every $y \neq x$, we could say "y is not within the neighborhood N of x." This is analogous to saying "y is not within ϵ of x" (if we had a pseudometric). Because no point y is "within N of x," we call x isolated. The neighborhoods of x tell us this.

(*Of course, if we prefer, we could use "U is an open set containing x" instead of "N is a neighborhood of x" to talk about nearness to x.*)

The complete neighborhood system $\mathcal{N}_x$ of a point x often contains more neighborhoods than we actually need to talk about nearness to x. For example, using just the (open) neighborhoods $B_{\frac{1}{n}}(x)$ in a pseudometric space (X,d) is enough to let us talk about continuous functions at x.

We use the idea of a neighborhood base at x to choose a smaller collection of neighborhoods of x that is i) good enough for all our purposes, and ii) from which all the other neighborhoods of x can be obtained if we want them.

Definition 4.2 A collection $\mathcal{B}_x \subseteq \mathcal{N}_x$ is called a neighborhood base at x if for every neighborhood N of x, there is a neighborhood $B \in \mathcal{B}_x$ such that $x \in B \subseteq N$. We refer to the sets in $\mathcal{B}_x$ as basic neighborhoods of x.

According to the definition, each set in $\mathcal{B}_x$ must be a neighborhood of x, but the collection $\mathcal{B}_x$ may be much simpler than the whole neighborhood system N_x. The crucial thing is that every neighborhood N of x must contain "basic neighborhood" B of x.

Example 4.3 At a point $x \in (X,d)$, let

i) $\mathcal{B}_x$ = the collection of all balls $B_\epsilon(x)$

ii) $\mathcal{B}_x$ = the collection of all balls $B_\epsilon(x)$, where ϵ is a positive rational

iii) $\mathcal{B}_x$ = the collection of all balls $B_{\frac{1}{n}}(x)$ for $n \in \mathbb{N}$

iv) $\mathcal{B}_x = \mathcal{N}_x$ (*The neighborhood system is always a base for itself, but it is not an "efficient" choice; the goal is to get a base $\mathcal{B}_x$ that's much simpler than N_x.*)

Which $\mathcal{B}_x$ to use is our choice: each of i)-iv) gives a neighborhood base at x. But ii) and iii) might be more convenient: each of those gives a countable family $\mathcal{B}_x$ as a neighborhood base. (*If $X = \mathbb{R}$, for example, with the usual metric d, then the collections $\mathcal{B}_x$ in i) and iv) are uncountable collections.*) Of the four, iii) is probably the simplest choice for $\mathcal{B}_x$.

Suppose we want to check whether some property that involving neighborhoods of x is true. Often all we need to do is to check whether the property holds for the simpler neighborhoods in $\mathcal{B}_x$. For example, in (X,d) : O is open iff O contains a neighborhood N of each $x \in O$. But that is true iff O contains a set $B_{\frac{1}{n}}(x)$ from $\mathcal{B}_x$.

Similarly, $x \in \text{cl}\,A$ iff $N \cap A \neq \emptyset$ for every $N \in \mathcal{N}_x$ iff $B \cap A \neq \emptyset$ for every $B \in \mathcal{B}_x$. For example, suppose we want to check, in $\mathbb{R}$, that $1 \in \text{cl}\,\mathbb{P}$. It is sufficient just to check that $B_{\frac{1}{n}}(1) \cap \mathbb{P} \neq \emptyset$ for each $n \in \mathbb{N}$, because this implies that $N \cap \mathbb{P} \neq \emptyset$ for every $N \in \mathcal{N}_1$.

Therefore an "efficient" choice of the "simplest possible" neighborhood base $\mathcal{B}_x$ at each point x is desirable.

Definition 4.4 We say that a space $(X, \mathcal{T})$ satisfies the first axiom of countability (or, more simply, that $(X, \mathcal{T})$ is first countable) if at each point $x \in X$, it is possible to choose a countable neighborhood base $\mathcal{B}_x$.

Example 4.5

1) The preceding Example 4.3 shows that every pseudometric space is first countable.

2) If $\mathcal{T}$ is the discrete topology on X, then $(X, \mathcal{T})$ is first countable. In fact, at each point x, we can choose a neighborhood base consisting of a single set: $\mathcal{B}_x = \{\,\{x\}\,\}$.

3) Let $\mathcal{T}$ be the cofinite topology on an uncountable set X. For any $x \in X$, there cannot be a countable neighborhood base $\mathcal{B}_x$ at x.

To see this, suppose that we had a countable neighborhood base at x : $\mathcal{B}_x = \{B_1, ..., B_n, ...\}$. For any $y \neq x$, $\{y\}$ is closed, so $X - \{y\}$ is a neighborhood of x. Therefore for some k, $x \in \text{int}\, B_k \subseteq B_k \subseteq X - \{y\}$, so $y \notin \bigcap_{n=1}^{\infty} \text{int}\, B_n$. But $x \in \bigcap_{n=1}^{\infty} \text{int}\, B_n$ so $\bigcap_{n=1}^{\infty} \text{int}\, B_n = \{x\}$.

Taking complements, we get $X - \{x\} = X - \bigcap_{n=1}^{\infty} \text{int}\, B_n$ $= \bigcup_{n=1}^{\infty} (X - \text{int}\, B_n)$. Since $X - \text{int}\, B_n$ is finite (*why*?), this would mean that $X - \{x\}$ is countable $-$ which is impossible.

Since any pseudometric space is first countable, the example gives us another way to see that this space $(X, \mathcal{T})$ is not pseudometrizable.

For a given topology $\mathcal{T}$ on X, all the neighborhood systems $\mathcal{N}_x$ are completely determined by the topology $\mathcal{T}$, but $\mathcal{B}_x$ is not. As the preceding examples illustrate, there are usually many possible choices for $\mathcal{B}_x$. (*Can you describe all the spaces* $(X, \mathcal{T})$ *for which* $\mathcal{B}_x$ *is uniquely determined at each point* x?)

On the other hand, if we were given $\mathcal{B}_x$ at each point $x \in X$ we could

1) "reconstruct" the whole neighborhood system $\mathcal{N}_x$:
$\mathcal{N}_x = \{N \subseteq X : \exists B \in \mathcal{B}_x \text{ such that } x \in B \subseteq N\}$, and then we could

2) "reconstruct" the whole topology $\mathcal{T}$:
$\mathcal{T} = \{O : O \text{ is a neighborhood of each of its points}\}$
$= \{O : \forall x \in O\ \exists B \in \mathcal{B}_x\ \ x \in B \subseteq O\}$, that is,
O is open iff O contains a basic neighborhood of each of its points.

5. Describing Topologies

How can a topological space be described? If $X = \{0, 1\}$, it is simple to give a topology by just writing $\mathcal{T} = \{\emptyset, \{0\}, X\}$. However, describing all the sets in $\mathcal{T}$ explicitly is often not the easiest way to go.

In this section we look at three important, alternate ways to define a topology on a set. All of these will be used throughout the remainder of the course. A fourth method – by using a "closure operator" – is not used much nowadays. It is included just as an historical curiosity.

A. Basic Neighborhoods

Suppose that at each point $x \in (X, \mathcal{T})$ we have picked a neighborhood base $\mathcal{B}_x$. As mentioned above, the collections $\mathcal{B}_x$ contain implicitly all the information about the topology: a set O is in $\mathcal{T}$ iff O contains a basic neighborhood of each of its points. This suggests that if we start with a set X, then we could define a topology on X if we begin by saying what the $\mathcal{B}_x$'s should be. Of course, we can't just put "random" sets in $\mathcal{B}_x$: the sets in each $\mathcal{B}_x$ must "act the way basic neighborhoods are supposed to act." And how is that? The next theorem describes the crucial behavior of a collection of basic neighborhoods at x.

Theorem 5.1 Suppose $(X, \mathcal{T})$ is a topological space and that $\mathcal{B}_x$ is a neighborhood base at x for each $x \in X$. Then

1) $\mathcal{B}_x \neq \emptyset$ and $B \in \mathcal{B}_x \Rightarrow x \in B \subseteq X$

2) if B_1 and $B_2 \in \mathcal{B}_x$, then $\exists\ B_3 \in \mathcal{B}_x$ such that $x \in B_3 \subseteq B_1 \cap B_2$

3) if $B \in \mathcal{B}_x$, then $\exists\ I$ such that $x \in I \subseteq B$ and,
$\forall\ y \in I$, $\exists\ B_y \in \mathcal{B}_y$ such that $y \in B_y \subseteq I$

4) $O \in \mathcal{T} \Leftrightarrow\ \forall x \in O\ \ \exists\ B \in \mathcal{B}_x$ such that $x \in B \subseteq O$.

Proof 1) Since X is a neighborhood of x, there is a $B \in \mathcal{B}_x$ such that $x \in B \subseteq X$. Therefore $\mathcal{B}_x \neq \emptyset$. If $B \in \mathcal{B}_x \subseteq \mathcal{N}_x$, then B is a neighborhood of x, so $x \in \operatorname{int} B \subseteq B$.

2) The intersection of the two neighborhoods B_1 and B_2 of x is again a neighborhood of x. Therefore, by the definition of a neighborhood base, there is a set $B_3 \in \mathcal{B}_x$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

3) Let $I = \operatorname{int} B$. Then $x \in I \subseteq B$ and because I is open, I is a neighborhood of each its points y. Since $\mathcal{B}_y$ is a neighborhood base at y, there is a set $B_y \in \mathcal{B}_y$ such that $y \in B_y \subseteq I$.

4) $\Rightarrow$: If O is open, then O is a neighborhood of each of its points x. Therefore for each $x \in O$ there must be a set $B \in \mathcal{B}_x$ such that $x \in B \subseteq O$.

$\Leftarrow$: The condition implies that O contains a neighborhood of each of its points. Therefore O is a neighborhood of each of its points, so O is open. •

The features of a neighborhood base in Theorem 5.1 are the crucial ones about the behavior of a neighborhood base at x. The next theorem tells us that, with this information, we can put a topology on a set by giving a collection of sets to become the basic neighborhoods at each point x.

Theorem 5.2 (The Neighborhood Base Theorem) Let X be a set. Suppose that for each $x \in X$ we assign a collection $\mathcal{B}_x$ of subsets of X in such a way that conditions 1) - 3) of Theorem 5.1 are true. Define $\mathcal{T} = \{O \subseteq X : \forall x \in O\ \exists B \in \mathcal{B}_x$ such that $x \in B \subseteq O\}$.

Then $\mathcal{T}$ is a topology on X and $\mathcal{B}_x$ is now a neighborhood base at x in $(X, \mathcal{T})$.

Note: In Theorem 5.2, we do not ask that the $\mathcal{B}_x$'s satisfy condition 4) of Theorem 5.1 – because the set X has no topology and condition 4) would be meaningless. Here, Condition 4) is our motivation for how to define $\mathcal{T}$ using the B_x's.

Proof We need to prove three things: a) that $\mathcal{T}$ is a topology, that b) in $(X, \mathcal{T})$, each $B \in \mathcal{B}_x$ is now a neighborhood of x, and that c) the collection $\mathcal{B}_x$ is now a neighborhood base at x.

a) Clearly, $\emptyset \in \mathcal{T}$. If $x \in X$ then, by condition 1), we can choose a $B \in \mathcal{B}_x$ and $x \in B \subseteq X$. Therefore $X \in \mathcal{T}$.

Suppose $O_\alpha \in \mathcal{T}$ for all $\alpha \in A$. If $x \in \bigcup\{O_\alpha : \alpha \in A\}$, then $x \in O_{\alpha_0}$ for some $\alpha_0 \in A$. By definition of $\mathcal{T}$, there is a set $B \in \mathcal{B}_x$ such that $x \in B \subseteq O_{\alpha_0} \subseteq \bigcup\{O_\alpha : \alpha \in A\}$, so $\bigcup\{O_\alpha : \alpha \in A\} \in \mathcal{T}$.

To finish a), it is sufficient to show that if O_1 and $O_2 \in \mathcal{T}$, then $O_1 \cap O_2 \in \mathcal{T}$. Suppose $x \in O_1 \cap O_2$. By the definition of $\mathcal{T}$, there are sets B_1 and $B_2 \in \mathcal{B}_x$ such that $x \in B_1 \subseteq O_1$ and $x \in B_2 \subseteq O_2$, so $x \in B_1 \cap B_2 \subseteq O_1 \cap O_2$. By condition 2), there is a set $B_3 \in \mathcal{B}_x$ such that $x \in B_3 \subseteq B_1 \cap B_2 \subseteq O_1 \cap O_2$. Therefore $O_1 \cap O_2 \in \mathcal{T}$.

Therefore $\mathcal{T}$ is a topology on X – so now we have a topological space $(X, \mathcal{T})$ – and we must show that in this space $\mathcal{B}_x$ is now a neighborhood base at x. Doing that involves the awkward-looking condition 3) – which we have not yet used.

b) If $B \in \mathcal{B}_x$, then $x \in B$ (by condition 1) and (by condition 3), there is a set $I \subseteq X$ such that $x \in I \subseteq B$ and $\forall\ y \in I,\ \exists\ B_y \in \mathcal{B}_y$ such that $y \in B_y \subseteq I$. The underlined phrase states that I satisfies the condition for $I \in \mathcal{T}$, so I is open. Since I is open and $x \in I \subseteq B$, B is a neighborhood of x, that is, $\mathcal{B}_x \subseteq \mathcal{N}_x$.

c) To complete the proof, we have to check that $\mathcal{B}_x$ is a neighborhood base at x. If N is a neighborhood of x, then $x \in \operatorname{int} N$. Since $\operatorname{int} N$ is open, $\operatorname{int} N$ must satisfy the criterion for membership in $\mathcal{T}$, so there is a set $B \in \mathcal{B}_x$ such that $x \in B \subseteq \operatorname{int} N \subseteq N$. Therefore $\mathcal{B}_x$ forms a neighborhood base at x. ∙

Example 5.3 For each $x \in \mathbb{R}$, let $\mathcal{B}_x = \{[x, b) : b > x\}$. We can easily check the conditions 1) - 3) from Theorem 5.2:

1) For each $x \in \mathbb{R}$, certainly $\mathcal{B}_x \neq \emptyset$ and $x \in [x,b)$ for each set $[x,b) \in \mathcal{B}_x$
2) If $B_1 = [x,b_1)$ and $B_2 = [x,b_2)$ are in $\mathcal{B}_x$, then (in this example) we can choose $B_3 = B_1 \cap B_2 = [x,b_3) \in \mathcal{B}_x$, where $b_3 = \min\{b_1,b_2\}$.
3) If $B = [x,b) \in \mathcal{B}_x$, then (in this example) we can let $I = B$.
If $y \in I = [x,b)$, pick c so $y < c < b$. Then $B_y = [y,c) \in \mathcal{B}_y$ and $y \in [y,c) \subseteq I$.

According to Theorem 5.2, $\mathcal{T} = \{O \subseteq \mathbb{R} : \forall x \in O\ \exists\, [x,b) \in \mathcal{B}_x$ such that $x \in [x,b) \subseteq O\}$ is a topology on $\mathbb{R}$ and $\mathcal{B}_x$ is a neighborhood base at x in $(\mathbb{R},\mathcal{T})$. The space $(\mathbb{R},\mathcal{T})$ is called the <u>Sorgenfrey line</u>.

Notice that <u>in this example</u> each set $[x,b) \in \mathcal{T}$: that is, the sets in $\mathcal{B}_x$ turn out to be open (not merely neighborhoods of x. (*This does not always happen.*)

It is easy to check that sets of form $(-\infty,\ x)$ and $[b,\infty)$ are open, so $(-\infty,x) \cup [b,\infty)$ $= \mathbb{R} - [x,b)$ is open. Therefore $[x,b)$ is also closed. So, in the Sorgenfrey line, there is a neighborhood base $\mathcal{B}_x$ consisting of clopen sets at each point x.

We can write $(a,c) = \bigcup_{n=1}^{\infty}[a+\frac{1}{n},c)$, so (a,c) is open in the Sorgenfrey line. Because every <u>usual</u> open set in $\mathbb{R}$ is a union of sets of the form (a,c), we conclude that <u>every usual open set in $\mathbb{R}$ is also open in the Sorgenfrey line</u>. The usual topology on $\mathbb{R}$ is strictly smaller that the Sorgenfrey topology: $\mathcal{T}_d \underset{\neq}{\subseteq} \mathcal{T}$.

$\mathbb{Q}$ is dense in the Sorgenfrey line: if $x \in \mathbb{R}$, then every basic neighborhood $[x,b)$ of x intersects $\mathbb{Q}$, so $x \in \operatorname{cl}\mathbb{Q}$. Therefore the Sorgenfrey line is separable. It is also clear that the Sorgenfrey line is first countable: at each point x the collection $\{[x,x+\frac{1}{n}) : n \in \mathbb{N}\}$ is a countable neighborhood base.

Example 5.4 Similarly, we can define the <u>Sorgenfrey plane</u> by putting a new topology on $\mathbb{R}^2$. At each point $(x,y) \in \mathbb{R}^2$, let $\mathcal{B}_{(x,y)} = \{[x,b) \times [y,c) : b > x,\ c > y\}$. The families $\mathcal{B}_{(x,y)}$ satisfy the conditions in the Neighborhood Base Theorem, so they give a topology $\mathcal{T}$ for which $\mathcal{B}_{(x,y)}$ is a neighborhood base at (x,y). A set $O \subseteq \mathbb{R}^2$ is open iff: for each $(x,y) \in O$, there are $b > x$ and $c > y$ such that $(x,y) \in [x,b) \times [y,c) \subseteq O$. (*Make a sketch!*) You should check that the sets $[x,b) \times [y,c) \in \mathcal{B}_{(x,y)}$ are actually clopen in the Sorgenfrey plane. It is also easy to check the usual topology $\mathcal{T}_d$ on the plane is strictly smaller that the Sorgenfrey topology. It is clear that $\mathbb{Q}^2$ is dense, so $(\mathbb{R}^2,\mathcal{T})$ is separable. Is the Sorgenfrey plane first countable?

Example 5.5 At each point $p \in \mathbb{R}^2$, let $C_\epsilon(p) = \{x \in \mathbb{R}^2 : d(x,p) \le \epsilon\}$ and define $\mathcal{B}_p = \{C_\epsilon(p) :\ p \in \mathbb{R}^2\}$. It is easy to check that the conditions 1) - 3) of Theorem 5.2 are satisfied. (*For $C_\epsilon(p)$ in condition 3), let $I = B_\epsilon(p)$.*) The topology generated by the $\mathcal{B}_p$'s is just the usual topology; the sets in $\mathcal{B}_p$ are basic <u>neighborhoods</u> of p in the usual topology – as they should be – but the sets in $\mathcal{B}_p$ did <u>not</u> turn out to be open sets.

Example 5.6 Let $\Gamma = \{(x,y) \in \mathbb{R}^2 : y \geq 0\}$ = the "closed upper half-plane."

For a point $p = (x,y) \in \Gamma$ with $y > 0$, let $\mathcal{B}_p = \{B_\epsilon(p) : \epsilon < |\, y\,|\}$
For a point $p = (x,0) \in \Gamma$, let

$\mathcal{B}_p = \{\{p\} \cup A : A$ is a usual open disc in the upper half-plane, tangent to the x-axis at $p\}$.

It is easy to check that the collections $\mathcal{B}_p$ satisfy the conditions 1) - 3) of Theorem 5.2 and therefore give a topology on Γ. In this topology, the sets in $\mathcal{B}_p$ turn out to be open neighborhoods of p.

The space Γ, with this topology, is called the "Moore plane." Notice that Γ is separable and 1st countable. The subspace topology on the x-axis is the discrete topology. (*Verify these statements!*)

B. Base for the topology

Definition 5.7 A collection of open sets in $(X, \mathcal{T})$ is called a base for the topology $\mathcal{T}$ if each $O \in \mathcal{T}$ is a union of sets from $\mathcal{B}$. More precisely, $\mathcal{B}$ is a base if $\mathcal{B} \subseteq \mathcal{T}$ and for each $O \in \mathcal{T}$, there exists a subfamily $\mathcal{A} \subseteq \mathcal{B}$ such that $O = \bigcup \mathcal{A}$. We also call $\mathcal{B}$ a base for the open sets and we refer to the open sets in $\mathcal{B}$ as basic open sets.

If $\mathcal{B}$ is a base, then it is easy to see that: $O \in \mathcal{T}$ iff $\forall x \in O\ \exists B \in \mathcal{B}$ such that $x \in B \subseteq O$. This means that if we were given $\mathcal{B}$, we could use it to decide which sets are open and thus "reconstruct" $\mathcal{T}$.

Of course, we could choose $\mathcal{B} = \mathcal{T}$: every topology $\mathcal{T}$ is a base for itself. But there usually are many ways to choose a base, and the idea is that a simpler base $\mathcal{B}$ would be easier to work with. For example, the set of all balls is a base for the topology $\mathcal{T}_d$ in any pseudometric space (X,d); a different base would be the set containing only the balls with positive rational radii.
(*Can you describe those topological spaces for which $\mathcal{T}$ is the only base for $\mathcal{T}$* ?)

The following theorem tells us the crucial properties of a base $\mathcal{B}$ in $(X, \mathcal{T})$.

Theorem 5.8 If $(X, \mathcal{T})$ is a topological space with a base $\mathcal{B}$ for $\mathcal{T}$, then

1) $X = \bigcup\{B : B \in \mathcal{B}\}$

2) if B_1 and $B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, then there is a set $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

Proof 1) Certainly $\bigcup \mathcal{B} \subseteq X$. But by definition of base, the open set X is a union of a subfamily of $\mathcal{B}$. Therefore $\bigcup \mathcal{B} = X$.

2) If B_1 and $B_2 \in \mathcal{B}$, then $B_1 \cap B_2$ is open. If $x \in B_1 \cap B_2$, the definition of base implies that there must be a set $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$. ∙

The next theorem tells us that if we are given a collection $\mathcal{B}$ of subsets of a set X with properties 1) and 2), we can use it to define a topology.

Theorem 5.9 (The Base Theorem) Suppose X is a set and that $\mathcal{B}$ is a collection of subsets of X that satisfies conditions 1) and 2) in Theorem 5.8. Define $\mathcal{T} = \{O \subseteq X : O$ is a union of sets from $\mathcal{B}\} = \{O \subseteq X : \forall x \in O\ \exists B \in \mathcal{B}$ such that $x \in B \subseteq O\}$.
Then $\mathcal{T}$ is a topology on X and $\mathcal{B}$ is a base for $\mathcal{T}$.

Proof First we show that $\mathcal{T}$ is a topology on X. Since $\emptyset$ is the union of the empty subfamily of $\mathcal{B}$, we get that $\emptyset \in \mathcal{T}$, and condition 1) simply states that $X \in \mathcal{T}$.

If $O_\alpha \in \mathcal{T}$ $(\alpha \in A)$, then $\bigcup\{O_\alpha : \alpha \in A\}$ is a union of sets O_α each of which is a union of sets from $\mathcal{B}$. So clearly $\bigcup\{O_\alpha : \alpha \in A\}$ is a union of sets from $\mathcal{B}$. So $\bigcup\{O_\alpha : \alpha \in A\} \in \mathcal{T}$.

Suppose O_1 and $O_2 \in \mathcal{T}$ and that $x \in O_1 \cap O_2$. For each such x we can use 2) to pick a set $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq O_1 \cap O_2$. Then $O_1 \cap O_2$ is the union of all the B_x's chosen in this way, so $O_1 \cap O_2 \in \mathcal{T}$.

Now we know that we have a topology, $\mathcal{T}$, on X. By definition of $\mathcal{T}$ it is clear that $\mathcal{B} \subseteq \mathcal{T}$ and that each set in $\mathcal{T}$ is a union of sets from $\mathcal{B}$. Therefore $\mathcal{B}$ is a base for $\mathcal{T}$. •

Example 5.10 The collections

$$\mathcal{B} = \{B_p(\epsilon) : \ p \in \mathbb{R}^2,\ \epsilon > 0\}$$
$$\mathcal{B}' = \{B_p(\tfrac{1}{k}) : \ p \in \mathbb{Q}^2,\ k \in \mathbb{N}\} \text{ and}$$
$$\mathcal{B}'' = \{(a,b) \times (c,d) : a,b,c,d \in \mathbb{R},\ a < b,\ c < d\}$$

each satisfy the conditions 1) - 2) in Theorem 5.9, so each collection is the base for a topology on $\mathbb{R}^2$. In fact, all three are bases for the same topology on $\mathbb{R}^2$ $-$ that is, the usual topology (*check this!*) Of the three, $\mathcal{B}'$ is the simplest choice $-$ it is a countable base for the usual topology.

Example 5.11 Suppose $(X, \mathcal{T}')$ and $(Y, \mathcal{T}'')$ are topological spaces. Let $\mathcal{B} =$ the set of "open boxes" in $X \times Y = \{U \times V : U \in \mathcal{T}'$ and $V \in \mathcal{T}''\}$. (*Verify that $\mathcal{B}$ satisfies conditions 1) and 2) of The Base Theorem.*) The product topology on the set $X \times Y$ is the topology for which $\mathcal{B}$ is a base. We always assume that $X \times Y$ has the product topology unless something else is stated.

Therefore a set $A \subseteq X \times Y$ is open (in the product topology) iff : for all $(x, y) \in A$, there are open sets $U \subseteq X$ and $V \subseteq Y$ such that $(x, y) \in U \times V \subseteq A$. (*Note that A itself might not be a "box."*)

Let $\pi_x : X \times Y \to X$ be the "projection" defined by $\pi_x(x, y) = x$. If U is any open set in X, then $\pi_x^{-1}[U] = U \times Y \in \mathcal{B}$. Therefore $\pi_x^{-1}[U]$ is open. Similarly, for the other projection map $\pi_y : X \times Y \to Y$ defined by $\pi_y(x, y) = y$: if V is open in Y, then $\pi_y^{-1}[V] = X \times V$ is open in $X \times Y$. (*As we see in Section 8, this means that the projection maps are continuous. It is not hard to show that the projection maps π_x and π_y are open maps.: that is, the images aof open sets in the product are open.*)

If D_1 is dense in X and D_2 is dense in Y, we claim that $D_1 \times D$ is dense in $X \times Y$. If $(x, y) \in A$, where A is open, then there are nonempty open sets $U \subseteq X$ and $V \subseteq Y$ for which $(x, y) \in U \times V \subseteq A$. Since $x \in cl$ D_1, $U \cap D_1 \neq \emptyset$; and similarly $V \cap D_2 \neq \emptyset$. Therefore $(U \times V) \cap (D_1 \times D_2) = (U \cap D_1) \times (V \cap D_2) \neq \emptyset$, so $A \cap (D_1 \times D_2) \neq \emptyset$. Therefore $(x, y) \in \text{cl}\,(D_1 \times D_2)$. So $D_1 \times D_2$ is dense in $X \times Y$. In particular, this shows that the product of two separable spaces is separable.

Example 5.12 The open intervals (a, b) form a base for the usual topology in $\mathbb{R}$, so each set $(a, b) \times (c, d)$ is in the base $\mathcal{B}$ for the product topology on $\mathbb{R} \times \mathbb{R}$. It is easy to see that every "open box" $U \times V$ in $\mathcal{B}$ can be written as a union of simple "open boxes" like $(a, b) \times (c, d)$. Therefore $\mathcal{B}' = \{(a, b) \times (c, d) : a < b, c < d\}$ also is a (simpler) base for the product topology on $\mathbb{R} \times \mathbb{R}$. From this, it is clear that the product topology on $\mathbb{R} \times \mathbb{R}$ is the usual topology on the plane $\mathbb{R}^2$ (*see Example 5.10*).

In general, the open sets U and V in the base for the product topology on $X \times Y$ can be replaced by sets "U chosen from a base for X" and "V chosen from a base for Y," as in this example. So in the definition of the product topology, it is sufficient to say that basic open sets are of the form $U \times V$, where U and V are basic open sets from X and from Y.

Definition 5.13 A space $(X, \mathcal{T})$ is said to satisfy the second axiom of countability (or, more simply, to be a second countable space) if it is possible to find a countable base $\mathcal{B}$ for the topology $\mathcal{T}$.

For example, $\mathbb{R}$ is second countable because, for example, $\mathcal{B} = \{(a, b) : a, b \in \mathbb{Q}\}$ is a countable base. Is $\mathbb{R}^2$ is second countable (*why or why not*) ?

Example 5.14 The collection $\mathcal{B} = \{[x, x + \frac{1}{n}) : x \in \mathbb{R}, n \in \mathbb{N}\}$ is a base for the Sorgenfrey topology on $\mathbb{R}$. But the collection $\{[x, x + \frac{1}{n}) : x \in \mathbb{Q}, n \in \mathbb{N}\}$ is not a countable base for the Sorgenfrey topology. Why not?

Since the sets in a base may be simpler than arbitrary open sets, they are often more convenient to work with, and working with the basic open sets is often all that is necessary – not a surprise since all the information about the open sets in contained in the base $\mathcal{B}$. For example, you should check that

1) If $\mathcal{B}$ is a base for $\mathcal{T}$, then $x \in \text{cl}\, A$ iff each basic open set B containing x satisfies $B \cap A \neq \emptyset$.

2) If $f : (X, d) \to (Y, d')$ and $\mathcal{B}$ is a base for the topology $\mathcal{T}_{d'}$ on Y, then f is continuous iff $f^{-1}[B]$ is open for each $B \in \mathcal{B}$. This means that we needn't check the inverse images of all open sets to verify that f is continuous.

C. Subbase for the topology

Definition 5.15 Suppose $(X, \mathcal{T})$ is a topological space. A family $\mathfrak{S}$ of open sets is called a subbase for the topology $\mathcal{T}$ if the collection $\mathcal{B}$ containing all finite intersections of sets from $\mathfrak{S}$ is a base for $\mathcal{T}$. (*It is clear that if $\mathcal{B}$ is a base for $\mathcal{T}$, then $\mathcal{B}$ is also a subbase for $\mathcal{T}$.*)

Examples i) The collection $\mathfrak{S} = \{I : I = (-\infty, b)$ or $I = (a, \infty),\ a, b \in \mathbb{R}\}$ is a subbase for a topology on $\mathbb{R}$. All intervals of the form $(a, b) = (-\infty, b) \cap (a, \infty)$ are in $\mathcal{B}$, and $\mathcal{B}$ is a base for the usual topology on $\mathbb{R}$.

ii) The collection $\mathfrak{S}$ of all sets $U \times Y$ and $V \times X$ (for U open in X and V open in Y) is a subbase for the product topology on $X \times Y$: the collection $\mathcal{B}$ of finite intersections from $\mathfrak{S}$ includes all open boxes $U \times V = (U \times Y) \cap (X \times V)$.

We can define a topology on a set X by giving a collection $\mathfrak{S}$ of subsets as the subbase for a topology. Surprisingly, any collection $\mathfrak{S}$ can be used: no special conditions on $\mathfrak{S}$ are required.

Theorem 5.16 (The Subbase Theorem) Suppose X is a set and $\mathfrak{S}$ is any collection of subsets of X. Let $\mathcal{B}$ be the collection of all finite intersections of sets from $\mathfrak{S}$. Then $\mathcal{B}$ is a base for a topology $\mathcal{T}$, and $\mathfrak{S}$ is a subbase for $\mathcal{T}$.

Proof First we show that $\mathcal{B}$ satisfies conditions 1) and 2) of The Base Theorem.

1) $X \in \mathcal{B}$ since X is the intersection of the empty subcollection of $\mathfrak{S}$ (*this follows the convention that the intersection of an empty family of subsets of X is X itself. See Example I.4.5.5*). Since $X \in \mathcal{B}$, certainly $X = \bigcup\{B\colon B \in \mathcal{B}\}$.

2) Suppose B_1 and $B_2 \in \mathcal{B}$, and $x \in B_1 \cap B_2$. We know that $B_1 = S_1 \cap ... \cap S_m$ and $B_2 = S_{m+1} \cap ... \cap S_{m+k}$ for some $S_1, ..., S_m, ..., S_{m+k} \in \mathfrak{S}$, so

$$x \in B_3 = B_1 \cap B_2 = S_1 \cap ... \cap S_m \cap S_{m+1} \cap ... \cap S_{m+k} \in \mathcal{B}$$

Therefore $\mathcal{T} = \{O : O$ is a union of sets from $\mathcal{B}\}$ is a topology, and $\mathcal{B}$ is a base for $\mathcal{T}$. By definition of $\mathcal{B}$ and $\mathcal{T}$, we have $\mathfrak{S} \subseteq \mathcal{B} \subseteq \mathcal{T}$ and each set in $\mathcal{T}$ is a union of finite intersections of sets from $\mathfrak{S}$. Therefore $\mathfrak{S}$ is a subbase for $\mathcal{T}$. •

Example 5.17

1) Let $\mathbb{E} = \{2, 4, 6, ...\} \subseteq \mathbb{N}$ and $\mathfrak{S} = \{\{1\}, \{2\}, \mathbb{E}\}$. $\mathfrak{S}$ is a subbase for a topology on $\mathbb{N}$. A base for this topology is the collection $\mathcal{B}$ of all finite intersections of sets in $\mathfrak{S}$:

$$\mathcal{B} = \{\emptyset, \mathbb{N}, \{2\}, \{1\}, \mathbb{E}\}.$$

and the (not very interesting) topology $\mathcal{T}$ generated the set of all possible unions of sets from $\mathcal{B}$:

$$\mathcal{T} = \{\emptyset, \mathbb{N}, \{1\}, \{2\}, \{1,2\}, \mathbb{E}, \mathbb{E} \cup \{1\}\}$$

2) For each $n \in \mathbb{N}$, let $S_n = \{n, n+1, n+2, ...\}$. The collection $\mathfrak{S} = \{S_n : n \in \mathbb{N}\}$ is a subbase for a topology on $\mathbb{N}$. Here, $S_n \cap S_m = S_k$ where $k = \max\{m, n\}$, so the collection of all finite intersections from $\mathfrak{S}$ is just $\mathfrak{S}$ itself. So $\mathcal{B} = \mathfrak{S}$ is actually a base for a topology. The topology is $\mathcal{T} = \mathfrak{S} \cup \{\emptyset\}$.

3) Let $\mathfrak{S} = \{\ell : \ell$ is a straight line in $\mathbb{R}^2\}$ in $\mathbb{R}^2$. If $p \in \mathbb{R}^2$, then $\{p\}$ is the intersection of two sets from $\mathfrak{S}$, so $\{p\} \in \mathcal{B}$. $\mathfrak{S}$ is a subbase for the discrete topology on $\mathbb{R}^2$.

4) Let $\mathfrak{S} = \{\ell : \ell$ is a vertical line in $\mathbb{R}^2\}$. $\mathfrak{S}$ generates a topology on $\mathbb{R}^2$ for which $\mathcal{B} = \{\mathbb{R}^2, \emptyset\} \cup \mathfrak{S}$ is a base and $\mathcal{T} = \{O : O$ is a union of vertical lines$\}$.

5) Suppose $\mathcal{T}'$ is <u>any</u> topology on X for which $\mathfrak{S} \subseteq \mathcal{T}'$. $\mathcal{T}'$ must contain the finite intersections of sets in $\mathfrak{S}$, and therefore must contain all possible unions of those intersections. Therefore $\mathcal{T}'$ contains the topology $\mathcal{T}$ for which $\mathfrak{S}$ is a subbase. To put it another way, <u>the topology for which $\mathfrak{S}$ is a subbase is in the smallest topology on X containing the collection $\mathfrak{S}$</u>. In fact, $\mathcal{T} = \bigcap\{\mathcal{T}' : \mathcal{T}' \supseteq \mathfrak{S}$ and $\mathcal{T}'$ is a topology on $X\}$.

Caution We said earlier that for some purposes it is sufficient to work with <u>basic</u> open sets, rather than <u>arbitrary</u> open sets – for example to check whether $x \in \text{cl}\, A$, or to check whether a function f between metric spaces is continuous. However, it is <u>not always</u> sufficient to work with <u>subbasic</u> open sets. Some caution is necessary.

For example, $\mathfrak{S} = \{I : I = (-\infty, b)$ or $I = (a, \infty),\ a, b \in \mathbb{R}\}$ is a subbase for the usual topology on $\mathbb{R}$. We have $I \cap \mathbb{Z} \neq \emptyset$ for every <u>subbasic</u> open set I containing $\frac{1}{2}$, but $\frac{1}{2} \notin \text{cl}\, \mathbb{Z}$.

<u>D. The closure operator</u>

Usually we describe a topology $\mathcal{T}$ by giving a subbase $\mathfrak{S}$, a base $\mathcal{B}$, or by giving collections $\mathcal{B}_x$ to be the basic neighborhoods at each point x. In the early history of general topology, one other method was sometimes used. We will never use it, but we include it here as a curiosity.

Let $\text{cl}_{\mathcal{T}}$ be the closure operator in $(X, \mathcal{T})$ (*normally, we would just write "cl" for the closure operator; here we write "$\text{cl}_{\mathcal{T}}$" to emphasize that this closure operator comes from the topology $\mathcal{T}$ on X*). It gives us all the information about $\mathcal{T}$. That is, using $\text{cl}_{\mathcal{T}}$, we can decide whether any set A is closed (by asking "is $\text{cl}_{\mathcal{T}} A = A$?") and therefore can decide whether any set B is open (by asking "is $X - B$ closed ?"). It should be not be a surprise, then, that we can define a topology on a <u>set</u> X if we are start with an "operator" which "behaves like a closure operator." How is that? Our first theorem tells us the crucial properties of a closure operator.

Theorem 5.18 Suppose $(X, \mathcal{T})$ is a topological space and A, B are subsets of X. Then

1) $\text{cl}_{\mathcal{T}}\emptyset = \emptyset$
2) $A \subseteq \text{cl}_{\mathcal{T}} A$
3) A is closed iff $A = \text{cl}_{\mathcal{T}} A$
4) $\text{cl}_{\mathcal{T}} A = \text{cl}_{\mathcal{T}}(\text{cl}_{\mathcal{T}} A)$
5) $\text{cl}_{\mathcal{T}}(A \cup B) = \text{cl}_{\mathcal{T}} A \cup \text{cl}_{\mathcal{T}} B$

Proof 1) Since $\emptyset$ is closed, $\text{cl}_{\mathcal{T}}\emptyset = \emptyset$

2) $A \subseteq \bigcap\{F : F \text{ is closed and } A \subseteq F\} = \text{cl}_{\mathcal{T}} A$

3) $\Rightarrow$: If A is closed, then A is one of the closed sets F used in the definition $\text{cl}_{\mathcal{T}} A = \cap\{F : F \text{ is closed and } F \supseteq A\}$, so $A = \text{cl}_{\mathcal{T}} A$.
$\Leftarrow$: If $A = \text{cl}_{\mathcal{T}} A$, then A is closed because $\text{cl}_{\mathcal{T}} A$ is an intersection of closed sets.

4) $\text{cl}_{\mathcal{T}} A$ is closed, so by 3), $\text{cl}_{\mathcal{T}} A = \text{cl}_{\mathcal{T}}(\text{cl}_{\mathcal{T}} A)$

5) $A \cup B \supseteq A$, so $\text{cl}_{\mathcal{T}}(A \cup B) \supseteq \text{cl}_{\mathcal{T}} A$. Likewise, $\text{cl}_{\mathcal{T}}(A \cup B) \supseteq \text{cl}_{\mathcal{T}} B$. Therefore $\text{cl}_{\mathcal{T}}(A \cup B) \supseteq \text{cl}_{\mathcal{T}}(A) \cup \text{cl}_{\mathcal{T}}(B)$.
On the other hand, $\text{cl}_{\mathcal{T}} A \cup \text{cl}_{\mathcal{T}} B$ is a closed set that contains $A \cup B$, and therefore $\text{cl}_{\mathcal{T}} A \cup \text{cl}_{\mathcal{T}} B \supseteq \text{cl}_{\mathcal{T}}(A \cup B)$. •

The next theorem tells us that we can use an operator "cl" to create a topology on a set.

Theorem 5.19 (The Closure Operator Theorem) Suppose X is a set and that for each $A \subseteq X$, a subset $\text{cl}\, A$ is defined (*that is, we have a function cl* : $\mathcal{P}(X) \to \mathcal{P}(X)$) in such a way that conditions 1), 2), 4) and 5) of Theorem 5.18 are satisfied. Define $\mathcal{T} = \{O : \text{cl}(X - O) = X - O\}$. Then $\mathcal{T}$ is a topology on X, and cl is now the closure operator for this topology (*that is, cl* $=$ *cl*$_{\mathcal{T}}$).

Note: i) Such a function cl is called a "Kuratowski closure operator," or just "closure operator" for short.
ii) The Closure Operator Theorem does not ask that cl satisfy condition 3): since there is no topology on the given set X, 3) would be meaningless. But 3) motivates the definition of $\mathcal{T}$ as the collection of sets whose complements are unchanged when "cl" is applied..

Proof (*Numbers in parentheses refer to properties of "cl"*)

First note that:

$$(*) \qquad \text{if } A \subseteq B, \text{ then } \text{cl}\, A \subseteq \text{cl}\, A \cup \text{cl}\, B \overset{(5)}{=} \text{cl}(A \cup B) = \text{cl}\, B$$

$\mathcal{T}$ is a topology on the set X :

i) $X \overset{(2)}{\subseteq} \text{cl}\, X$, so $X = \text{cl}\, X$. Therefore $\text{cl}\,(X - \emptyset) = \text{cl}\, X = X = X - \emptyset$, so $\emptyset \in \mathcal{T}$.
Since $\text{cl}(X - X) = \text{cl}\,\emptyset \overset{(1)}{=} \emptyset = X - X$, we have that $X \in \mathcal{T}$.

ii) Suppose $O_\alpha \in \mathcal{T}$ for all $\alpha \in A$. For each $\alpha_0 \in A$, we have

$\text{cl}(X - \bigcup O_\alpha) \overset{(*)}{\subseteq} \text{cl}(X - O_{\alpha_0}) = X - O_{\alpha_0}$ because $O_{\alpha_0} \in \mathcal{T}$. This is true for <u>every</u> $\alpha_0 \in \Lambda$, so $\text{cl}(X - \bigcup O_\alpha) \subseteq \bigcap (X - O_\alpha) = X - \bigcup O_\alpha$.

But by 2), we know that $X - \bigcup O_\alpha \subseteq \text{cl}\,(X - \bigcup O_\alpha)$.

Therefore $\text{cl}(X - \bigcup O_\alpha) = X - \bigcup O_\alpha$, so $\bigcup O_\alpha \in \mathcal{T}$.

iii) If O_1 and O_2 are in $\mathcal{T}$, then $\text{cl}(X - (O_1 \cap O_2)) = \text{cl}((X - O_1) \cup (X - O_2))$
$\overset{(5)}{=} \text{cl}(X - O_1) \cup \text{cl}(X - O_2) = (X - O_1) \cup (X - O_2)$ (since O_1 and $O_2 \in \mathcal{T}$)
$= X - (O_1 \cap O_2)$. Therefore $O_1 \cap O_2 \in \mathcal{T}$. Therefore $\mathcal{T}$ is a topology on X.

With the topology $\mathcal{T}$ we now have a closure operator $\text{cl}_\mathcal{T}$. We want to show that $\text{cl}_\mathcal{T} = \text{cl}$. First, observe that:

$$\begin{aligned} (**)\ \text{cl}_\mathcal{T} B = B \ &\text{iff}\ B \text{ is closed in } (X, \mathcal{T}) \ \text{iff}\ X - B \in \mathcal{T} \\ &\text{iff}\ \text{cl}\,(X - (X - B)) = X - (X - B) \\ &\text{iff}\ \text{cl}\, B = B. \end{aligned}$$

To finish, we must show that $\text{cl}\, A = \text{cl}_\mathcal{T} A$ for every $A \subseteq X$.

$A \subseteq \text{cl}_\mathcal{T} A$, so (*) gives that $\text{cl}\, A \subseteq \text{cl}(\text{cl}_\mathcal{T} A)$. But $\text{cl}_\mathcal{T} A$ is closed in $(X, \mathcal{T})$, so using $B = \text{cl}_\mathcal{T} A$ in (**) gives $\text{cl}\,\text{cl}_\mathcal{T} A = \text{cl}_\mathcal{T} A$. Therefore $\text{cl}\, A \subseteq \text{cl}_\mathcal{T} A$.

On the other hand, $\text{cl}\,\text{cl}\, A \overset{(4)}{=} \text{cl}\, A$, so using $B = \text{cl}\, A$ in (**) gives that $\text{cl}\, A$ is closed in $(X, \mathcal{T})$.

But $\text{cl}\, A \overset{(2)}{\supseteq} A$, so $\text{cl}\, A$ is one of the closed sets in the intersection that defines $\text{cl}_\mathcal{T} A$. Therefore $\text{cl}\, A \supseteq \text{cl}_\mathcal{T} A$. Therefore $\text{cl}\, A = \text{cl}_\mathcal{T} A$. •

Example 5.20

1) Let X be a set. For $A \subseteq X$, define $\text{cl}\, A = \begin{cases} A & \text{if } A \text{ is finite} \\ X & \text{if } A \text{ is infinite} \end{cases}$.

Then cl satisfies the conditions in the Closure Operator Theorem. Since $\text{cl}\, A = A$ iff A is finite or $A = X$, the closed sets in the topology generated by cl are precisely X and the finite sets – that is, cl generates the cofinite topology on X.

2) For each subset A of $\mathbb{R}$, define

$$\text{cl}\, A = \{x \in \mathbb{R} : \text{ there is a sequence } (a_n) \text{ in } A \text{ with each } a_n \geq x \text{ and } |a_n - x| \to 0\}.$$

It is easy to check that cl satisfies the hypotheses of the Closure Operator Theorem. Moreover, a set A is open in the corresponding topology iff $\forall x \in A \ \exists b > x$ such that $[x, b) \subseteq A$. Therefore the topology generated by cl is the Sorgenfrey topology on $\mathbb{R}$. What happens in this example if " $\geq$ " is replaced by " $>$ " in the definition of cl ?

Since closures, interiors and Frontiers are all related, it shouldn't be surprising that we can also describe a topology by defining an appropriate "int" operator or "Fr" operator on a set X.

Exercises

E1. Let $X = \{0, 1, 2, ., ., .\} = \{0\} \cup \mathbb{N}$. For $O \subseteq X$, let $\phi_O(n) =$ the number of elements in $O \cap [1, n] \,| = |\, O \cap [1, n]\,|$. Define

$$\mathcal{T} = \{\ O :\ 0 \notin O \ \text{ or } \ (0 \in O \ \text{ and } \lim_{n \to \infty} \tfrac{\phi_O(n)}{n} = 1)\ \}.$$

a) Prove that $\mathcal{T}$ is a topology on X.

b) *In any space* $(Y, \mathcal{T})$: *a point* x *is called a* *limit point of the set A if* $N \cap (A - \{x\}) \neq \emptyset$ *for every neighborhood* N *of* x. *Informally, this means that* x *is a limit point of* A *if there are points of* A *other than* x *itself arbitrarily close to* x.)

Prove that in any $(Y, \mathcal{T})$, a subset B is closed iff B contains all of its limit points.

c) For $(X, \mathcal{T})$ as defined above, prove that x is a limit point of X if and only if $x = 0$.

E2. Suppose that $A_\alpha \subseteq (X, \mathcal{T})$ for each $\alpha \in A$.

a) Suppose that $\bigcup \{\text{cl}\, A_\alpha : \alpha \in A\}$ is closed. Prove that

$$\bigcup \{\text{cl}\, A_\alpha : \alpha \in A\} \ = \ \text{cl}\,(\bigcup \{A_\alpha : \alpha \in A\})$$

(*Note that "* $\subseteq$ *" is true for any collection of sets* A_α.)

b) A family $\{B_\alpha : \alpha \in A\}$ of subsets of $(X, \mathcal{T})$ is called locally finite if each point $x \in X$ has a neighborhood N such that $N \cap B_\alpha \neq \emptyset$ for only finitely many α's. Prove that if $\{B_\alpha : \alpha \in A\}$ is locally finite, then

$$\bigcup \{\text{cl}\, B_\alpha : \alpha \in A\} \ = \ \text{cl}\,(\bigcup \{B_\alpha : \alpha \in A\})$$

c) Prove that in $(X, \mathcal{T})$, the union of a locally finite family of closed sets is closed.

E3. Suppose $f : (X, d) \to (Y, s)$. Let $\mathcal{B}$ be a base for the topology $\mathcal{T}_s$ and let $\mathfrak{S}$ be a subbase for $\mathcal{T}_s$. Prove or disprove: f is continuous iff $f^{-1}[O]$ is open for all $O \in \mathcal{B}$ iff $f^{-1}[O]$ is open for all $O \in \mathfrak{S}$.

E4. A space $(X, \mathcal{T})$ is called a T_1-*space* if $\{x\}$ is closed for every $x \in X$.

a) Give an example of a space $(X, \mathcal{T})$ with a nontrivial topology and which is not a T_1-space.
b) Prove that X is a T_1-space if and only if, given any two distinct points $x, y \in X$, each point is contained in an open set not containing the other point.
c) Prove that in a T_1-space, each set $\{x\}$ can be written as an intersection of open sets.
d) Prove that a subspace of a T_1-space is a T_1-space.
e) Prove that if a pseudometric space (X, d) is a T_1-space, then d must in fact be a metric.
f) Prove that if X and Y are T_1-spaces, so is $X \times Y$.

E5. Prove that every infinite T_2-space contains an infinite discrete subspace (*that is, a subset which is discrete in the subspace topology*).

E6. Suppose that $(X, \mathcal{T})$ and $(Y, \mathcal{T}\,')$ are topological spaces. Recall that the product topology on $X \times Y$ is the topology for which the collection of "open boxes"

$$\mathcal{B} = \{U \times V : U \in \mathcal{T},\ V \in \mathcal{T}\,'\} \text{ is a base.}$$

Therefore a set $O \subseteq X \times Y$ is open in the product topology iff for all $(x, y) \in O$, there exist open sets U in X and V in Y such that $(x, y) \in U \times V \subseteq O$. (*Note that the product topology on* $\mathbb{R} \times \mathbb{R}$ *is the usual topology on* $\mathbb{R}^2$. *We always assume that the "product topology" is the topology on* $X \times Y$ *unless something different is explicitly stated.*)

a) Verify that $\mathcal{B}$ is, in fact, a base for a topology on $X \times Y$.

b) Prove that the projection map $\pi_x : X \times Y \to X$ is an open map.

c) Prove that if $A \subseteq X$ and $B \subseteq Y$, then $\text{cl}_{X \times Y}(A \times B) = \text{cl}_X A \times \text{cl}_Y B$. Use this to explain why "the product of two closed sets is closed in $X \times Y$."

d) Show that X and Y each has a countable base iff $X \times Y$ has a countable base. Show that there is a countable neighborhood base at $(x, y) \in X \times Y$ iff there is a countable neighborhood base $\mathcal{B}_x$ at $x \in X$ and a countable neighborhood base $\mathcal{B}_y$ at $y \in Y$.

e) Suppose (X, d_1) and (Y, d_2) are pseudometric spaces. Define a pseudometric d on the set $X \times Y$ by

$$d((x_1, y_1), (x_2, y_2)) = d_1(x_1, x_2) + d_2(y_1, y_2).$$

Prove that the product topology on $X \times Y$ is the same as the topology $\mathcal{T}_d$.

Note: d *is the analogue of the taxicab metric in* $\mathbb{R}^2$. *Of course, there are other equivalent pseudometrics producing the product topology on* $X \times Y$, *e.g.,*

$$d\,'((x_1, y_1), (x_2, y_2)) = (d_1(x_1, x_2)^2 + d_2(y_1, y_2)^2)^{1/2}, \text{ or}$$
$$d\,''((x_1, y_1), (x_2, y_2)) = \max\{d_1(x_1, x_2), d_2(y_1, y_2)\}.$$

E7. $(X, \mathcal{T})$ is called a T_2-*space* (or Hausdorff space) if whenever $x, y \in X$ and $x \neq y$, then there exist disjoint open sets U and V with $x \in U$ and $y \in V$.

a) Give an example of a space $(X, \mathcal{T})$ which is a T_1-space but not a T_2-space.

b) Prove that a subspace of a Hausdorff space is Hausdorff.

c) Prove that if X and Y are Hausdorff, then so is $X \times Y$.

E8. Suppose $A \subseteq (X, \mathcal{T})$. The set A is called *regular open* if $A = \text{int}\,(\text{cl}\,A)$ and A is *regular closed* if $A = \text{cl}\,(\text{int}\,A)$.

a) Show that for any subset B

i) $X - \text{cl}\,B = \text{int}\,(X - B)$
ii) $X - \text{int}\,B = \text{cl}\,(X - B)$

b) Give an example of a closed subset of $\mathbb{R}$ which is not regular closed.
c) Show that the complement of a regular open set in $(X, \mathcal{T})$ is regular closed and vice-versa.
d) Show that the interior of any closed set in $(X, \mathcal{T})$ is regular open.
e) Show that the intersection of two regular open sets in $(X, \mathcal{T})$ is regular open.
f) Give an example of the union of two regular open sets that is not regular open.

E9. Prove or give a counterexample:

a) For any x in $(X, \mathcal{T})$, $\{x\}$ is equal to the intersection of all open sets containing x.

b) In $(X, \mathcal{T})$, a finite set must be closed.

c) If for each $\alpha \in A$, each $\mathcal{T}_\alpha$ is a topology on X, then $\bigcap\{\mathcal{T}_\alpha \colon \alpha \in A\}$ is a topology on X.

d) If $\mathcal{T}_1$ and $\mathcal{T}_2$ are topologies on X, then there is a unique smallest topology $\mathcal{T}_3$ on X such that $\mathcal{T}_3 \supseteq \mathcal{T}_1 \cup \mathcal{T}_2$.

e) Suppose, for each $\alpha \in A$, that $\mathcal{T}_\alpha$ is a topology on X. Then there is a unique smallest topology $\mathcal{T}$ on X such that for each α, $\mathcal{T} \supseteq \mathcal{T}_\alpha$.

E10. Assume that each integer (except is divisible by a prime but nothing else about prime numbers. (This assumption is equivalent to assuming that each natural number bigger than can be factored into primes). For $a \in \mathbb{Z}$ and $d \in \mathbb{N}$, let

$$B_{a,d} = \{..., a-2d, a-d, a, a+d, a+2d, ...\} = \{a + kd : k \in \mathbb{Z}\}$$

and let $\mathcal{B} = \{B_{a,d} : a \in \mathbb{Z}, d \in \mathbb{N}\}$ (*so $\mathcal{B}$ is the set of all arithmetic progressions in $\mathbb{Z}$*)

a) Prove that $\mathcal{B}$ is a base for a topology $\mathcal{T}$ on $\mathbb{Z}$.

b) Show that each set is closed in $(\mathbb{Z}, \mathcal{T})$.

c) What is the set $\bigcup\{B_{0,p} : p \text{ a prime number}\}$? Explain why this set is not closed in $(\mathbb{Z}, \mathcal{T})$.

d) What does part c) tell you about the set of prime numbers?

6. Countability Properties of Spaces

Countable sets are generally easier to work with than uncountable sets, so it is not surprising that spaces with certain "countability properties" are viewed as desirable. Most of these properties have already been defined, but the definitions are collected together here for convenience.

Definition 6.1 $(X, \mathcal{T})$ is called

first countable (or, is said to satisfy the *first axiom of countability*) if we can choose a countable neighborhood base $\mathcal{B}_x$ at every point $x \in X$

second countable (or, is said to satisfy the *second axiom of countability*) if there is a countable base $\mathcal{B}$ for the topology $\mathcal{T}$

separable if there is a countable dense set D in X

Lindelöf if whenever $\mathcal{U}$ is a collection of open sets for which $\bigcup\mathcal{U} = X$, then there is a countable $\mathcal{U}' = \{U_1, U_2, ..., U_n, ...\} \subseteq \mathcal{U}$ for which $\bigcup\mathcal{U}' = X$.

$\mathcal{U}$ is called an open cover of X, and $\mathcal{U}'$ is called a subcover from $\mathcal{U}$. Thus, X is Lindelöf if "*every open cover has a countable subcover*.")

Example 6.2

1) A countable discrete space $(X, \mathcal{T})$ is second countable because $\mathcal{B} = \{\{x\} : x \in X\}$ is a countable base.

2) $\mathbb{R}$ is second countable because $\mathcal{B} = \{(a, b) : a, b \in \mathbb{Q}\}$ is a countable base. Similarly, $\mathbb{R}^n$ is second countable since the collection $\mathcal{B}$ of boxes $(x_1, y_1) \times (x_2, y_2) \times ... \times (x_n, y_n)$ with rational endpoints is a countable base (*check!*)

3) Let X be a countable set with the cofinite topology $\mathcal{T}$. X has only countably many finite subsets (*see Theorem I.11.1*) so there are only countably many sets in $\mathcal{T}$. $(X, \mathcal{T})$ is second countable because we could choose $\mathcal{B} = \mathcal{T}$ as a countable base.

The following theorem implies that each of the spaces in the preceding example is also first countable, separable, and Lindelöf. (*However, it is really worthwhile to try to verify each of those assertions directly from the definitions.*)

Theorem 6.3 A second countable topological space $(X, \mathcal{T})$ is also first countable, separable, and Lindelöf.

Proof Let $\mathcal{B} = \{O_1, O_2, ...O_n, ...\}$ be a countable base for $\mathcal{T}$.

i) For each n, pick a point $x_n \in O_n$ and let $D = \{x_n : n \in \mathbb{N}\}$. The countable set D is dense. To see this, notice that if U is any nonempty open set in X, then for some n, $x_n \in O_n \subseteq U$ so $U \cap D \neq \emptyset$. So X is separable.

ii) For each $x \in X$, let $\mathcal{B}_x = \{O \in \mathcal{B} : x \in O\}$. Clearly $\mathcal{B}_x$ is a neighborhood base at x, so X is first countable.

iii) Let $\mathcal{U}$ be any open cover of X. If $x \in X$, then $x \in$ some set $U_x \in \mathcal{U}$. For each x, we can then pick a basic open set $O_x \in \mathcal{B}$ such that $x \in O_x \subseteq U_x$. Let $\mathcal{V} = \{O_x : x \in X\}$. Since each $O_x \in \mathcal{B}$, there can be only countably many different sets O_x: that is, $\mathcal{V}$ may contain "repeats." Eliminate repeats and list only the different sets in $\mathcal{V}$, so $\mathcal{V} = \{O_{x_1}, O_{x_2}, ...O_{x_n}, ...\}$ where $O_{x_n} \subseteq U_{x_n} \in \mathcal{U}$. Every x is in one of the sets O_{x_n}, so $\mathcal{U}' = \{U_{x_1}, U_{x_2}, ...U_{x_n}, ...\}$ is a countable subcover from $\mathcal{U}$. Therefore X is Lindelöf. $\bullet$

The following examples show that no other implications exist among the countability properties mentioned in Theorem 6.3.

Example 6.4

1) Suppose X is uncountable and let $\mathcal{T}$ be the cofinite topology on X.

$(X, \mathcal{T})$ is separable since any infinite set is dense.

$(X, \mathcal{T})$ is Lindelöf. To see this, let $\mathcal{U}$ be an open cover of X. Pick any one nonempty set $U \in \mathcal{U}$. Then $X - U$ is finite, say $X - U = \{x_1, ..., x_n\}$. For each x_i, pick a set $U_i \in \mathcal{U}$ with $x_i \in U_i$. Then $\mathcal{U}' = \{U\} \cup \{U_i : i = 1, ..., n\}$ is a countable (actually, finite) subcover chosen from $\mathcal{U}$.

However X is not first countable (Example 4.5.3), so Theorem 6.3, X is also not second countable.

2) Suppose X is uncountable. Define $\mathcal{T} = \{O \subseteq X : O = \emptyset \text{ or } X - O \text{ is countable}\}$.

$\mathcal{T}$ is a topology on X (*check!*) called the cocountable topology. A set $C \subseteq X$ is closed iff $C = X$ or C is countable. (*This is an "upscale" analogue of the cofinite topology*.)

An argument very similar to the one in the preceding example shows that $(X, \mathcal{T})$ is Lindelöf. But $(X, \mathcal{T})$ is not separable $-$ every countable subset is closed and therefore not dense. By Theorem 6.3, $(X, \mathcal{T})$ also cannot be second countable.

3) Suppose X is uncountable set and choose a particular point $p \in X$.
Define $\mathcal{T} = \{O \subseteq X : O = \emptyset \text{ or } p \in O\}$. (*Check that $\mathcal{T}$ is a topology*.)

$(X, \mathcal{T})$ is separable because $\{p\}$ is dense.

$(X, \mathcal{T})$ is not Lindelöf $-$ because the cover $\mathcal{U} = \{\{x, p\} : x \in X\}$ has no countable subcover.

Is $(X, \mathcal{T})$ first countable?

4) Suppose X is uncountable and let $\mathcal{T}$ be the discrete topology on X.

$(X, \mathcal{T})$ is first countable because $\mathcal{B}_x = \{\{x\}\}$ is a neighborhood base at x

$(X, \mathcal{T})$ is not second countable because each open set $\{x\}$ would have to be in any base $\mathcal{B}$.

$(X, \mathcal{T})$ is not separable: for every countable set D, cl $D = D \neq X$.

$(X, \mathcal{T})$ is not Lindelöf because the cover $\mathcal{U} = \{\{x\} : x \in X\}$ has no countable subcover. (*In fact, not a single set can be omitted from* $\mathcal{U}$*:* $\mathcal{U}$ *has no proper subcovers at all.*)

For "special" topological spaces – pseudometrizable ones, for example – it turns out that things are better behaved. For example, we noted earlier that <u>every</u> pseudometric space (X, d) is <u>first countable</u>. The following theorem shows that in (X, d) the other three countability properties are equivalent to each other: that is, either all of them are true in (X, d) or none are true.

Theorem 6.5 Any pseudometric space (X, d) is first countable. (X, d) is second countable iff (X, d) is separable iff (X, d) is Lindelöf.

Proof i) Second countable $\Rightarrow$ Lindelöf: by Theorem 6,3, this implication is true in any topological space.

ii) Lindelöf $\Rightarrow$ separable: suppose (X, d) is Lindelöf. For each $n \in \mathbb{N}$, let $\mathcal{U}_n = \{B_{\frac{1}{n}}(x) : n \in \mathbb{N},\, x \in X\}$. For each n, $\mathcal{U}_n$ is an open cover so $\mathcal{U}_n$ has a countable subcover – that is, for each n we can find countably many $\frac{1}{n}$-balls that cover X: say $X = \bigcup_{k=1}^{\infty} B_{\frac{1}{n}}(x_{n,k})$. Let D be the set of centers of all these balls: $D = \{x_{n,k} : n, k \in \mathbb{N}\}$. For any $x \in X$ and every n, we have $x \in B_{\frac{1}{n}}(x_{n,k})$ for some k, so $d(x, x_{n,k}) < \frac{1}{n}$ – so x can be approximated arbitrarily closely by points from D. Therefore D is dense, so (X, d) is separable.

iii) Separable $\Rightarrow$ second countable: suppose (X, d) is separable and that $D = \{x_1, x_2, ..., x_k, ...\}$ is a countable dense set. Let $\mathcal{B} = \{B_{\frac{1}{n}}(x_k) : n, k \in \mathbb{N}\}$. $\mathcal{B}$ is a countable collection of open balls and we claim $\mathcal{B}$ is a base for the topology $\mathcal{T}_d$.

Suppose $y \in O \in T_d$. By the definition of "open," we have $B_\epsilon(y) \subseteq O$ for some $\epsilon > 0$. Pick n so that $\frac{1}{n} < \frac{\epsilon}{2}$, and pick $x_k \in D$ so that $d(x_k, y) < \frac{1}{n}$. Then $y \in B_{\frac{1}{n}}(x_k) \subseteq B_\epsilon(y) \subseteq O$ (because $z \in B_{\frac{1}{n}}(x_k) \Rightarrow d(z, y) \leq d(z, x_k) + d(x_k, y)$ $< \frac{1}{n} + \frac{1}{n} < 2(\frac{\epsilon}{2}) = \epsilon$). ∙

It's <u>customary</u> to call a metric space that has these three equivalent properties a "separable metric space" rather than a "second countable metric space" or "Lindelöf metric space."

Theorem 6.5 <u>implies</u> that the spaces in parts 1), 2), 3) of Example 6.4 are not pseudometrizable. In general, to show a space $(X, \mathcal{T})$ is <u>not</u> pseudometrizable we can i) show that it fails to have some property common to all pseudometric spaces (for example, first countability), or ii) show that it has <u>one but not all</u> of the properties "second countable," "Lindelöf," or "separable."

Exercises

E11. Define $\mathcal{T} = \{U \cup V : U$ is open in the usual topology on $\mathbb{R}$ and $V \subseteq \mathbb{P}\}$.

a) Show that $\mathcal{T}$ is a topology on $\mathbb{R}$. If x is irrational, describe an "efficient" neighborhood base at x. Do the same if x is rational.

b) Is $(\mathbb{R}, \mathcal{T})$ first countable? second countable? Lindelöf ? separable?

The space $(\mathbb{R}, \mathcal{T})$ is called the "scattered line." We could change the definition of $\mathcal{T}$ by replacing $\mathbb{P}$ with some other set $A \subseteq \mathbb{R}$, creating a topology in which the set A is "scattered."
Hint: See Example I.7.9.6. It is possible to find open intervals I_n such that $\bigcup_{n=1}^{\infty} I_n \supseteq \mathbb{Q}$ and for which $\sum_{n=1}^{\infty} \text{length}(I_n) < 1$.

E12. A point $x \in (X, \mathcal{T})$ is called a <u>condensation point</u> if every neighborhood of x is uncountable.

a) Let C be the set of all condensation points in X. Prove that C is closed.
b) Prove that if X is second countable, then $X - C$ is countable.

E13. Suppose $(X, \mathcal{T})$ is a second countable space and let $\mathcal{B}$ be a countable base for the topology. Suppose $\mathcal{B}'$ is another base (<u>not</u> <u>necessarily countable</u>) for $\mathcal{T}$ containing open sets all of which have some property P. (For example, "P" could be "clopen" or "separable.") Show that there is a <u>countable</u> base $\mathcal{B}''$ consisting of open sets with property P.
Hint: think about the Lindelöf property.)

E14. A space $(X, \mathcal{T})$ is called *hereditarily Lindelöf* if every subspace of X is Lindelöf.

a) Prove that a second countable space is hereditarily Lindelöf.

In any space, a point x is called a <u>limit point of the set A</u> if $N \cap (A - \{x\}) \neq \emptyset$ for every neighborhood N of x. Informally, x is a limit point of A if there are points in A different from x and arbitrarily close to x.)

b) Suppose X is hereditarily Lindelöf.
Prove that $A = \{x \in X : x$ is not a limit point of $X\}$ is countable.

E15. A space $(X, \mathcal{T})$ is said to satisfy the *countable chain condition* ($=$ *CCC*) if every family of disjoint open sets must be countable.

a) Prove that a separable space $(X, \mathcal{T})$ satisfies the CCC.
b) Give an example of a space that satisfies the CCC but that is not separable. (*It is not necessary to do so, but can you find an example which is a metric space?)*

E16. Suppose $\mathcal{P}$ and $\mathcal{B}$ are two bases for the topology in $(X, \mathcal{T})$, and that $\mathcal{P}$ and $\mathcal{B}$

are infinite.

a) Prove that there is a subfamily $\mathcal{B}' \subseteq \mathcal{B}$ such that $\mathcal{B}'$ is also a base and $|\mathcal{B}'| \leq |\mathcal{P}|$.
(*Hint: For each pair* $t = (U, V) \in \mathcal{P} \times \mathcal{P}$, *pick, if possible, a set* $W_t \in \mathcal{B}$ *such that* $U \subseteq W \subseteq V$; *otherwise set* $W_t = \emptyset$.)

b) Use part a) to prove that the Sorgenfrey line is not second countable.
(*Hint: Show that otherwise there would be a countable base of sets of the form* $[a, b)$, *but that this is impossible.*)

7. More About Subspaces

Suppose $A \subseteq X$, where $(X, \mathcal{T})$ is a topological space. We defined the subspace topology $\mathcal{T}_A$ on A in Definition 3.1: $\mathcal{T}_A = \{A \cap O : O \in \mathcal{T}\}$.

In this section we explore some simple but important properties of subspaces.

If $A \subseteq B \subseteq X$, there are two ways to put a topology on A :

1) we can give A the subspace topology $\mathcal{T}_A$ from X, or

2) we can give B the subspace topology $\mathcal{T}_B$, and then give A the subspace topology from the space $(B, \mathcal{T}_B)$ – that is, we can give A the topology $(\mathcal{T}_B)_A$.

In other words, we can think of A as a subspace of X or as a subspace of the subspace B. Fortunately, the next theorem says that these two topologies are the same. More informally, Theorem 7.1 says that "a subspace of a subspace is a subspace."

Theorem 7.1 If $A \subseteq B \subseteq X$, and $\mathcal{T}$ is a topology on X, then $\mathcal{T}_A = (\mathcal{T}_B)_A$.

Proof $U \in \mathcal{T}_A$ iff $U = O \cap A$ for some $O \in \mathcal{T}$ iff $U = O \cap (B \cap A)$ iff $U = (O \cap B) \cap A$. But $O \cap B \in \mathcal{T}_B$, so the last equation holds iff $O \in (\mathcal{T}_B)_A$. •

We always assume a subset A has the subspace topology (unless something else is explicitly stated). The notation $A \subseteq (X, \mathcal{T})$ emphasizes that A is considered a sub<u>space</u>, not merely a sub<u>set</u> $A \subseteq X$.

By definition, a set is open in the subspace topology on A iff it is the intersection with A of an open set in X. We can prove the same is also true for closed sets.

Theorem 7.2 Suppose $A \subseteq (X, \mathcal{T})$. F is closed in A iff $F = A \cap C$ where C is closed in X.

Proof F is closed in A iff $A - F$ is open in A iff $A - F = O \cap A$ (for some open set O in X) iff $F = A - (O \cap A) = (X - O) \cap A = C \cap A$ (where $C = X - O$ is a closed set in X). •

Theorem 7.3 Suppose $A \subseteq (X, \mathcal{T})$.

1) Let $a \in A$. If $\mathcal{B}_a$ is a neighborhood base at a in X, then $\{B \cap A : B \in \mathcal{B}_a\}$ is a neighborhood base at a in A.

2) If $\mathcal{B}$ is a base for $\mathcal{T}$, then $\{B \cap A : B \in \mathcal{B}\}$ is a base for $\mathcal{T}_A$.

Slightly abusing notation, we can informally write these collections as $\mathcal{B}_a \cap A$ and.$\mathcal{B} \cap A$. Why is this an "abuse?" What do $\mathcal{B}_a \cap A$ and *$\mathcal{B} \cap A$ mean if taken literally*?

Proof 1) Suppose $a \in A$ and that N is a neighborhood of a in A. Then $a \in \text{int}_A N \subseteq N$, so there is an open set O in X such that $a \in \text{int}_A N = O \cap A \subseteq O$.

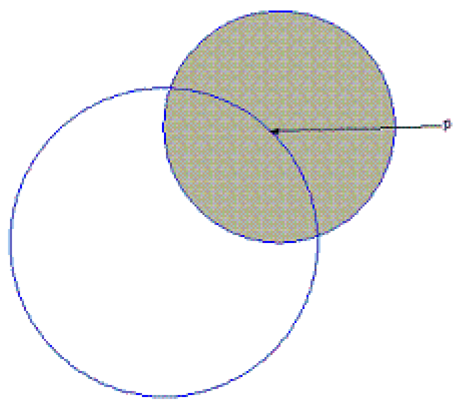
Since $\mathcal{B}_a$ is a neighborhood base at a in X, there is a neighborhood $B \in \mathcal{B}_a$ such that $a \in B \subseteq O$. Then $a \in (\text{int}_X B) \cap A \subseteq B \cap A \in \mathcal{B}_a \cap A$. Since $(\text{int}_X B) \cap A$ is open in A, we see that $B \cap A$ is a neighborhood of a in A. And since $a \in B \cap A \subseteq O \cap A = \text{int}_A N \subseteq N$, we see that $\mathcal{B}_a \cap A$ is a neighborhood base at a in A. $\bullet$

2) Exercise

Theorem 7.3 tells us that we can get a neighborhood base at a point $a \in A$ by choosing a neighborhood base at a in X are then restricting all its sets to A; and that the same applies to a base for the subspace topology.

Corollary 7.4 Every subspace of a first countable (or second countable) space $(X, \mathcal{T})$ is first countable (or second countable).

Example 7.5 Suppose S^1 is a circle in $\mathbb{R}^2$ and that $p \in S^1 \subseteq \mathbb{R}^2$. $\mathcal{B}_p = \{B_\epsilon(p) : \epsilon > 0\}$ is a neighborhood base at p in $\mathbb{R}^2$, and therefore $\mathcal{B}_p \cap S^1$ is a neighborhood base at p in the subspace S^1. The sets in $\mathcal{B}_p \cap S^1$ are "open arcs on S^1 containing p." (See the figure,)



The following theorem relates closures in subspaces to closures in the larger space. It is a useful technical tool.

Theorem 7.6 Suppose $A \subseteq B \subseteq (X, \mathcal{T})$, then $\text{cl}_B A = B \cap \text{cl}_X A$.

Proof $B \cap \text{cl}_X A$ is a closed set in B that contains A, so $B \cap \text{cl}_X A \supseteq \text{cl}_B A$.

On the other hand, suppose $b \in B \cap \text{cl}_X A$. To show that $b \in \text{cl}_B A$, pick an open set U in B that contains b. We need to show $U \cap A \neq \emptyset$. There is an open set O in X such that $O \cap B = U$. Since $b \in \text{cl}_X A$, we have that $\emptyset \neq O \cap A = O \cap (B \cap A) = (O \cap B) \cap A$
$= U \cap A$. $\bullet$

Example 7.7 1) $\mathbb{Q} = \text{cl}_{\mathbb{Q}}\mathbb{Q} = \text{cl}_{\mathbb{R}}\mathbb{Q} \cap \mathbb{Q}$

2) $\text{cl}_{(0,2]}\,(0,1) = \text{cl}_{\mathbb{R}}(0,1) \cap (0,2] = [0,1] \cap (0,2] = (0,1]$

3) The analogous results are not true for interiors and boundaries. For example:

$\mathbb{Q} = \text{int}_{\mathbb{Q}}\mathbb{Q} \neq \text{int}_{\mathbb{R}}\mathbb{Q} \cap \mathbb{Q} = \emptyset$, and

$\emptyset = \text{Fr}_{\mathbb{Q}}\,\mathbb{Q} \neq \text{Fr}_{\mathbb{R}}\mathbb{Q} \cap \mathbb{Q} = \mathbb{Q}$.

Why does "cl" have a privileged role here? Is there a "reason" why you would expect a better connection between closures in A and closures in X than you would expect between interiors in A and interiors in X?

Definition 7.8 A property P of topological spaces is called hereditary if whenever a space X has property P, then every subspace A also has property P.

For example, Corollary 7.4 tells us that first and second countability are hereditary properties. Other hereditary properties include "finite cardinality" and "pseudometrizability." On the other hand, "infinite cardinality" is not a hereditary property.

Example 7.9

1) Separability is not a hereditary property. For example, consider the Sorgenfrey plane X (*see Example 5.4*). X is separable because $\mathbb{Q}^2$ is dense.

Consider the subspace $D = \{(x,y) : x + y = 1\}$. The set $U = [a, a+1) \times [b, b+1)$ is open in X so if $(a,b) \in D$, then $U \cap D = \{(a,b)\}$ is open in the subspace D. Therefore D is a discrete subspace of X, and an uncountable discrete space is not separable.

Similarly, the Moore place Γ is separable (*see Example 5.6*); the x-axis in Γ is an uncountable discrete subspace which is not separable.

2) The Lindelöf property is not heredity. Let A be an uncountable set and let p be an additional point not in A. Define $X = A \cup \{p\}$. Put a topology on X by describing a neighborhood base at each point.

$$\begin{cases} \mathcal{B}_a = \{\{a\}\} \text{ for } a \in A \\ \mathcal{B}_p = \{B : p \in B \text{ and } X - B \text{ is finite}\} \end{cases}$$

(*Check that the collections $\mathcal{B}_x$ satisfy the hypotheses of the Neighborhood Base Theorem 5.2.*)

If $\mathcal{V}$ is an open cover of X, then $p \in V$ for some $V \in \mathcal{V}$. By ii), every neighborhood of p in X has a finite complement, so $X - V$ is finite. For each y in the finite set $X - V$, we can choose a set $V_y \in \mathcal{V}$ with $y \in V_y$. Then $\mathcal{V}' = \{V\} \cup \{V_y : y \in X - V\}$ is a countable (in fact, finite) subcover from $\mathcal{V}$, so X is Lindelöf.

The definition of $\mathcal{B}_a$ implies that each point of A is isolated in A; that is, A is an uncountable discrete subspace and $\mathcal{U} = \{\{a\} : a \in A\}$ is an open cover of A that has no countable subcover. Therefore the subspace A is not Lindelöf.

Even if a property is not hereditary, it is sometimes "inherited" by certain subspaces as the next theorem illustrates.

Theorem 7.10 A closed subspace of a Lindelöf space is Lindelöf (W*e say that the Lindelöf property is "closed hereditary."*)

Proof Suppose K is a closed subspace of the Lindelöf space X. Let $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ be a cover of K by sets U_α that are open in K. For each α, there is an open set V_α in X such that $V_\alpha \cap K = U_\alpha$.

Since K is closed, $X - K$ is open and $\mathcal{V} = \{\{X - K\}\} \cup \{V_\alpha : \alpha \in A\}$ is an open cover of X. But X is Lindelöf, so $\mathcal{V}$ has a countable subcover from $\mathcal{V}$, say $\mathcal{V}' = \{X - K, V_1, ..., V_n, ...\}$. (*The set $X - K$ may not be needed in $\mathcal{V}'$, but it can't hurt to include it.*) Clearly then, the collection $\{U_1, ..., U_n, ...\}$ is a countable subcover of K from $\mathcal{U}$. •

(*A little reflection on the proof shows that to prove K is Lindelöf, it would be equivalent to show that every cover of K by sets open in X has a countable subcover*.)

8. Continuity

To define continuous functions between pseudometric spaces, we began by using the distance function d. But then we proved that our definition is equivalent to other formulations stated in terms of open sets, closed sets, or neighborhoods. In the context of topological spaces we do not have distance functions to describe "nearness." But we can still use neighborhoods of a to take about "nearness" to a.

Definition 8.1 A function $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ is <u>continuous</u> <u>at</u> <u>a</u> $\in X$ if whenever N is a neighborhood of $f(a)$, then $f^{-1}[N]$ is a neighborhood of a. We say f is <u>continuous</u> if f is continuous at each point of X.

The statement that f is continuous at a is clearly equivalent to each of the following statements: :

i) for each neighborhood N of $f(a)$ there is a neighborhood W of a such that $f[W] \subseteq N$

ii) for each open set V containing $f(a)$ there is an open set U containing a such that $f[U] \subseteq V$

iii) for each basic open set V containing $f(a)$ there is a basic open set U containing a such that $f[U] \subseteq V$.

The conditions i) - iii) for continuity in the following theorem are the same as those in Theorem II.5.6 for pseudometric spaces. Condition iv) was not mentioned in Chapter II, but it is sometimes handy.

Theorem 8.2 Suppose $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$. The following are equivalent.

i) f is continuous
ii) if $O \in \mathcal{T}'$, then $f^{-1}[O] \in \mathcal{T}$ (*the inverse image of an open set is open*)
iii) if F is closed in Y, then $f^{-1}[F]$ is closed in X (*the inverse image of a closed set is closed*)
iv) for every $A \subseteq X$: $f[\mathrm{cl}_X(A)] \subseteq \mathrm{cl}_Y(f[A])$.

Proof The proof that i) - iii) are equivalent is identical to the proof for Theorem II.5.6 for pseudometric spaces. That proof was deliberately worded in terms of open sets, closed sets, and neighborhoods so that it would carry over to this new situation.

iii) $\Rightarrow$ iv) $f^{-1}[\mathrm{cl}_Y(f[A])] \supseteq f^{-1}[f[A]] \supseteq A$. Since $\mathrm{cl}_Y(f[A])$ is closed in Y, iii) gives that $f^{-1}[\mathrm{cl}_Y(f[A])]$ is a closed set in X that contains A. Therefore $f^{-1}[\mathrm{cl}_Y(f[A])] \supseteq \mathrm{cl}_X(A)$, so $[\mathrm{cl}_Y(f[A])]] \supseteq f[\mathrm{cl}_X(A)]$.

iv) $\Rightarrow$ i) Suppose $a \in X$ and that N is a neighborhood of $f(a)$. Let $K = X - f^{-1}[N]$ and $U = X - \mathrm{cl}_X K \subseteq f^{-1}[N]$. U is open, and we claim that $a \in U$ – which will show that $f^{-1}[N]$ is a neighborhood of a, completing the proof. So we need to show that $a \notin \mathrm{cl}_X K$. But this is clear: if $a \in \mathrm{cl}_X K$, we would have $f(a) \in f[\mathrm{cl}_X K] \subseteq \mathrm{cl}_Y f[K]$, which is impossible since $N \cap f[K] = \emptyset$. ∙

Example 8.3 Sometimes we want to know whether a certain property is "preserved by continuous functions" – that is, if X has property P and $f : X \to Y$ is continuous and onto, must Y also have the property P?

Condition iv) in Theorem 8.2 implies that continuous maps preserve separability. Suppose D is a countable dense set in X. Then $f[D]$ is countable and $f[D]$ is dense in Y because $Y = f[X] = f[\text{cl}\, D] \subseteq \text{cl}\, f[D]$.

By contrast, continuous maps do not preserve first countability: for example, let $(Y, \mathcal{T})$ be any topological space. Let $\mathcal{T}'$ be the discrete topology on Y. $(Y, \mathcal{T}')$ is first countable and the identity map $i : (Y, \mathcal{T}') \to (Y, \mathcal{T})$ is continuous and onto. Thus, every space $(Y, \mathcal{T})$ is the continuous image of a first countable space.

Do continuous maps preserve other properties – such as Lindelöf, second countable, or metrizable?

The following theorem makes a few simple and useful observations about continuity.

Theorem 8.4 Suppose $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$.

1) Let $B = \text{ran}(f) \subseteq Y$. Then f is continuous iff $f : (X, \mathcal{T}) \to (B, \mathcal{T}'_B)$ is continuous.
In other words, $B =$ the range of f (a subspace of the codomain Y) is what matters for the continuity of f. Points (if any) in $Y - B$ are irrelevant. For example, the function $\sin : \mathbb{R} \to \mathbb{R}$ *is continuous iff the function* $\sin : \mathbb{R} \to [-1, 1]$ *is continuous.*

2) Let $A \subseteq X$. If f is continuous, then $f|A = g : A \to Y$ is continuous.
That is, the restriction of a continuous function to a subspace is continuous.
For example, sin $: \mathbb{Q} \to [-1, 1]$ *is continuous.)*

3) If $\mathcal{S}$ is a subbase for $\mathcal{T}'$ (in particular, if $\mathcal{S}$ is a base), then f is continuous iff $f^{-1}[U]$ is open whenever $U \in \mathcal{S}$.
To check continuity, it is sufficient to show that the inverse image of every subbasic open set is open.

Proof 1) Exercise: the crucial observation is that if $O \subseteq Y$, then $f^{-1}[O] = f^{-1}[B \cap O]$.
2) If O is open in Y, then $g^{-1}[O] = f^{-1}[O] \cap A$ – which is an open set in A.
3) Exercise: it depends only on the definition of a subbase and set theory:

$$f^{-1}[\bigcup U_\alpha : \alpha \in A] = \bigcup\{f^{-1}[U_\alpha] : \alpha \in A\} \text{ and}$$
$$f^{-1}[\bigcap U_\alpha : \alpha \in A] = \bigcap\{f^{-1}[U_\alpha] : \alpha \in A\} \quad \bullet$$

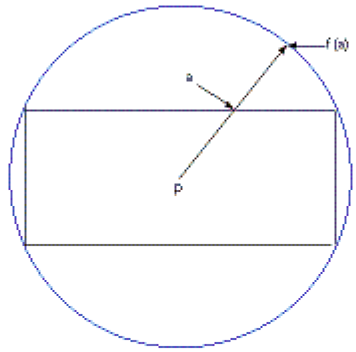
Example 8.5

1) If X has the discrete topology and Y is any topological space, then every function $f : X \to Y$ is continuous.

2) Suppose X has the trivial topology and that $f : X \to \mathbb{R}$. If f is constant, then f is continuous. If f is not constant, then there are points $a, b \in X$ for which $f(a) \neq f(b)$. Let I be an open set in $\mathbb{R}$ containing $f(a)$ but not $f(b)$. Then $f^{-1}[I]$ is not open in X so f is not continuous. We conclude that f is continuous iff f is constant.
(*In this example, we could replace* $\mathbb{R}$ *by any metric space* (Y, d), *or by any topological space* $(Y, \mathcal{T})$ *that has what property?*

3) Let X be a rectangle inscribed inside a circle Y centered at P. For $a \in X$, let

$f(a)$ be the point where the ray from P through a intersects Y. (*The function f is called a "central projection."*). Then both $f : X \to Y$ and $f^{-1} : Y \to X$ are continuous bijections.



Example 8.6 (Weak topologies) Suppose X is a set. Let $\mathcal{F} = \{f_\alpha : \alpha \in A\}$ be a collection of functions where each $f_\alpha : X \to \mathbb{R}$. If we put the discrete topology on X, then all of the function f_α will be continuous. But a topology on X smaller than the discrete topology might also make all the f_α's continuous. The smallest topology on X that makes all the f_α's continiuous is called the weak topology $\mathcal{T}$ on X generated by the collection $\mathcal{F}$.

How can we describe that topology more directly? $\mathcal{T}$ makes all the f_α's continuous iff for each open $O \subseteq \mathbb{R}$ and each $\alpha \in A$, the set $f_\alpha^{-1}[O]$ is in $\mathcal{T}$. Therefore the weak topology generated by $\mathcal{F}$ is the smallest topology that contains all these sets. According to Example 5.17.5, this means that weak topology $\mathcal{T}$ is the one for which the collection $\mathfrak{S} = \{f_\alpha^{-1}[O] : O \text{ open in } \mathbb{R}, \alpha \in A\}$ is a subbase. (*It is clearly sufficient here to use only basic open sets O from $\mathbb{R}$ – that is, open intervals (a, b) : why ? Would using all open sets O put any additional sets into $\mathcal{T}$?*)

For example, suppose $X = \mathbb{R}^2$ and that $\mathcal{F} = \{\pi_x, \pi_y\}$ contains the two projection maps $\pi_x(x, y) = x$ and $\pi_y(x, y) = y$. For an open interval $U = (a, b) \subseteq \mathbb{R}$, $\pi_x^{-1}[U]$ is the "open vertical strip" $U \times \mathbb{R}$; and $\pi_y^{-1}[V]$ is the "open horizontal strip" $\mathbb{R} \times V$. Therefore a subbase for the weak topology on $\mathbb{R}^2$ generated by $\mathcal{F}$ consists of all such open horizontal or vertical strips. Two such strips intersect in an "open box" $(a, b) \times (c, d)$ in $\mathbb{R}^2$, so it is easy to see that the weak topology is the product topology on $\mathbb{R} \times \mathbb{R}$, that is, the usuall topology of $\mathbb{R}^2$.

Suppose $A \subseteq \mathbb{R}$ and that $i : A \to \mathbb{R}$ is the identity function $i(x) = x$. What is the weak topology on the domain A generated by the collection $\mathcal{F} = \{i\}$?

Definition 8.7 $f:(X,\mathcal{T}) \to (Y,\mathcal{T}')$ is called open if whenever O is open in X, then $f[O]$ is open in Y, and f is called closed if $f[F]$ is closed in Y whenever F is closed in X.

Suppose $|X| > 1$. Let $\mathcal{T}$ be the discrete topology on X and let $\mathcal{T}'$ be the trivial topology on X. The identity map $i:(X,\mathcal{T}) \to (X,\mathcal{T}')$ is continuous but neither open nor closed, and $i:(X,\mathcal{T}') \to (X,\mathcal{T})$ is both open and closed but not continuous. Open (closed) maps are quite different from continuous maps – even if f is a bijection! Here are some examples that are more interesting.

Example 8.8

1) $f:[0,2\pi) \to S^1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ given by $f(\theta) = (\cos\theta, \sin\theta)$. It is easy to check that f is continuous, one-to-one, and onto. $F = [\pi, 2\pi)$ is closed in $[0,2\pi)$ but $f[\,[\pi,\ 2\pi)]$ is not closed in S^1. Also, $[0,\pi)$ is open in $[0,2\pi)$ but $f[\,[0,\pi)\,]$ is not open in S^1. A continuous, one-to-one, onto mapping does not need to be open or closed!

2) Suppose X and Y are topological spaces and that $f: X \to Y$ is a bijection so there is an inverse function $f^{-1}: Y \to X$. Then f^{-1} is continuous iff f is open. To check this, let $g = f^{-1}$. For an open set $O \subseteq X$, we have $y \in g^{-1}[O]$ iff $g(y) \in O$ iff $y = f(g(y)) \in f[O]$, so $f[O] = g^{-1}[O]$. So $f[O]$ is open iff $g^{-1}[O]$ is open. For a bijection f, "f is open" is equivalent to "f^{-1} is continuous."

If O is replaced by a closed set $F \subseteq X$, then a similar argument also shows that a bijection f is closed iff f^{-1} is continuous.

Definition 8.9 A mapping $f:(X,\mathcal{T}) \to (Y,\mathcal{T}')$ is called a homeomorphism if f is a bijection and f and f^{-1} are both continuous. If a homeomorphism f exists, we say that X and Y are homeomorphic and write $X \simeq Y$.

> *Note: The term is "homeomorphism," not "homomorphism" (a term from algebra). The etymologies are closely related: "-morphism" comes from the Greek word $(\mu o\rho\phi\eta)$ for "shape" or "form." The prefixes "homo" and "homeo" come from Greek words meaning "same" and "similar" respectively. There was a major dispute in western religious history, mostly during the 4th century AD, that hinged on the distinction between "homeo" and "homo."*

As noted in the preceding example, we could also describe a homeomorphism as a continuous open bijection or a continuous closed bijection.

It is obvious that in a collection of topological spaces, the homeomorphism relation " $\simeq$ " is an equivalence relation: if X, Y, and Z are topological spaces, then

i) $X \simeq X$
ii) if $X \simeq Y$, then $Y \simeq X$
iii) if $X \simeq Y$ and $Y \simeq Z$, then $X \simeq Z$.

Example 8.10

1) The function $f : [0, 2\pi) \to S^1$ given by $f(\theta) = (\cos\theta, \sin\theta)$ is not a homeomorphism even though f is continuous, $1-1$, and onto.

2) The "central projection" from the rectangle to the circle (*Example 8.5.3*) is a homeomorphism.

3) Using a linear map, it is easy to see that any two open intervals (a, b) in $\mathbb{R}$ are homeomorphic. The mapping $\tan : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ is a homeomorphism, so that each nonempty open interval in $\mathbb{R}$ is actually homeomorphic to $\mathbb{R}$ itself.

3) If $f : (X, d) \to (Y, s)$ is an isometry (onto) between metric spaces, then both f and f^{-1} are continuous, so f is a homeomorphism.

4) If d and d' are equivalent metrics (so $\mathcal{T}_d = \mathcal{T}_{d'}$), then the identity map $i : (X, d) \to (X, d')$ is a homeomorphism. A homeomorphism f between metric spaces need not preserve distances, that is, f need not be an isometry.

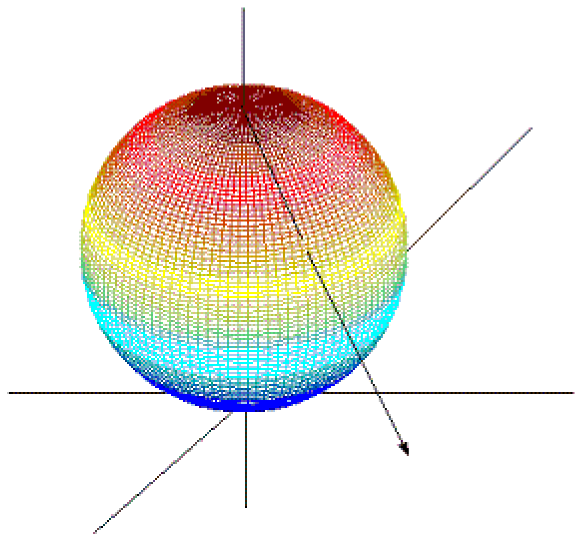
5) The function $f : \{\frac{1}{n} : n \in \mathbb{N}\} \to \mathbb{N}$ given by $f(\frac{1}{n}) = n$ is a homeomorphism (*both spaces have the discrete topology!*) Topologically, these spaces are the identical: both are just countable infinite sets with the discrete topology. f is not an isometry

Any two discrete spaces X and Y with the same cardinality are homeomorphic: any bijection between them is a homeomorphism.

6) Let P denote the "north pole" of the sphere

$$S^2 = \{(x, y, z) : x^2 + y^2 + z^2 = 1\} \subseteq \mathbb{R}^3.$$

The function f illustrated below is a "stereographic projection". The arrow starts at P, runs inside the sphere and exits through the surface of the sphere at a point (x, y, z). $f(x, y, z)$ is the point where the tip of the arrow hits the xy-plane $\mathbb{R}^2$. Thus f maps each point in $S^2 - \{P\}$ to a point in $\mathbb{R}^2$. The function f is a homeomorphism. (*See the figure below.*)



What is the significance of a homeomorphism $f : X \to Y$?

i) f is a bijection so it sets up a perfect one-to-one correspondence between the points in X and Y: $x \leftrightarrow y = f(x)$. We can imagine that f just "renames" the points in X. There is also a perfect one-to-one correspondence between the subsets X_α and Y_α of X and Y: $X_\alpha \leftrightarrow Y_\alpha = f[X_\alpha]$. Because f is a bijection, each subset $Y_\alpha \subseteq Y$ corresponds in this way to one and only one subset $X_\alpha \subseteq X$.

ii) f is a bijection, so f treats unions, intersections and complements "nicely":

a) $f^{-1}[\bigcup Y_\alpha : \alpha \in A] = \bigcup\{f^{-1}[Y_\alpha] : \alpha \in A\}$
b) $f^{-1}[\bigcap Y_\alpha : \alpha \in A] = \bigcap\{f^{-1}[Y_\alpha] : \alpha \in A\}$, and
c) $f[\bigcup X_\alpha : \alpha \in A] = \bigcup\{f[X_\alpha] : \alpha \in A\}$
d) $f[\bigcap X_\alpha : \alpha \in A] = \bigcap\{f[X_\alpha] : \alpha \in A\}$, and
e) $f[X - C] = f[X] - f[C] = Y - f[C]$

(*Actually a),b), c) are true for* <u>*any*</u> *function; but d) and e) depend on f being a bijection.*)

These properties say that this correspondence between subsets preserves unions: if each $X_\alpha \leftrightarrow Y_\alpha$, then $\bigcup X_\alpha \leftrightarrow \bigcup Y_\alpha = \bigcup f[X_\alpha] = f[\bigcup X_\alpha]$. Similarly, f preserves intersection and complements.

iii) Finally if f and f^{-1} are continuous, then open (closed) sets in X correspond to open(closed) sets in Y and vice-versa.

The total effect is all the "topological structure" in X is exactly "duplicated" in Y and vice versa: we can think of points, subsets, open sets and closed sets in Y are just "renamed copies" of their counterparts in X. Moreover f preserves unions, intersections and complements, so f also preserves all properties of X that can defined be using unions, intersections and complements of open sets. For example, we can check that if f is a homeomorphism and $A \subseteq X$, then $f[\text{int}_X A] = \text{int}_Y f[A]$, that $f[\text{cl}_X A] = \text{cl}_Y f[A]$, and that $f[\,\text{Fr}_X A] = \text{Fr}_Y f[A]$. That is, f takes interiors to interiors, closures to closures, and boundaries to boundaries.

Definition 8.11 A property P of topological spaces is called a topological property if, whenever a space X has property P and $Y \simeq X$, then the space Y also has property P.

By definition: if X and Y are homeomorphic, then X and Y have the same topological properties. Conversely, if two topological spaces X and Y have the same topological properties, then X and Y must be homeomorphic. (*Why? Let $P =$ "is homeomorphic to X." P is a topological property because if Y has P (that is, $Y \simeq X$) and $Z \simeq Y$, then Z also has P. And X has this property P, because $X \simeq X$. So if we are assuming that X and Y have the same topological properties, then Y has the property P, that is, Y is homeomorphic to X.*)

So we think of two homeomorphic spaces as "topologically identical" $-$ they are homeomorphic iff they have exactly the same topological properties. We can show that two spaces are not homeomorphic by showing a topological property of one space that the other space doesn't possess.

Example 8.12 Let P be the property that "every continuous real-valued function achieves a maximum value." Suppose a space X property P and that $h : X \to Y$ is a homeomorphism. We claim that Y also has property P.

Let f be any continuous real-valued function defined on Y.
Then $f \circ h : X \to \mathbb{R}$ is continuous :

$$\begin{array}{ccc} & & \mathbb{R} \\ & {}^{g}\nearrow & \uparrow f \\ X & \xrightarrow[h]{} & Y \end{array}$$

By assumption, $g = f \circ h$ achieves a maximum value at some point $a \in X$, and we claim that f must achieve a maximum value at the point $b = h(a) \in Y$. If not, then there is a point $y \in Y$ where $f(y) > f(b)$. Let $x = h^{-1}(y)$. Then

$$g(x) = f(h(x))) = f(h(h^{-1}(y)))) = f(y) > f(b) = f(h(a)) = g(a),$$

which contradicts the fact that g achieves a maximum value at a.

Therefore P is a topological property.

For example, the closed interval $[0,1]$ has property P discussed in Example 8.12 (*this is a well-known fact from elementary analysis – which we will prove later*). But $(0,1)$ and $[0,1)$ do not have this property P (*why?*). So we can conclude that $[0,1]$ is not homeomorphic to either $(0,1)$ or $[0,1)$.

Some simple examples of topological properties are: cardinality, first and second countability, Lindelöf, separability, and (pseudo)metrizability are topological properties. In the case of metrizability, for example:

> If (X,d) is a metric space and $f:(X,d)\to(Y,\mathcal{T})$ is a homeomorphism, then we can define a metric d' on Y as $d'(a,b)=d(f^{-1}(a),f^{-1}(b))$ for $a,b\in Y$. You then need to check that $\mathcal{T}_{d'}=\mathcal{T}$ (using the properties of a homeomorphism and the definition of d'). This shows that $(Y,\mathcal{T})$ is metrizable. Be sure you can do this!

9. Sequences

In Chapter II we saw that sequences are a useful tool for working with pseudometric spaces. In fact, sequences are sufficient to describe the topology in a pseudometric space (X,d) – because convergent sequences determine the closure of a set.

We can easily define convergent sequences in any topological space $(X,\mathcal{T})$. But as we shall see, sequences need to be used with more care in spaces that are not pseudometrizable. Whether or not a sequence converges to a certain point x is a "local" question – it depends on "getting near x" and we "measure nearness" to x by using the neighborhood system $\mathcal{N}_x$. If $\mathcal{N}_x$ is "too large" or "too complicated," then it may be impossible for a sequence to "get arbitrarily close" to x. We will see a specific example soon where such a problem occurs. But first, we look at some of the things that do work out just as nicely for topological spaces as they do in pseudometric spaces.

Definition 9.1 Suppose (x_n) is a sequence in $(X,\mathcal{T})$. We say that (x_n) converges to x if, for every neighborhood W of x, $\exists k\in\mathbb{N}$ such that $x_n\in W$ when $n\geq k$. In this case we write $(x_n)\to x$. More informally, we can say that $(x_n)\to x$ if (x_n) is eventually in every neighborhood W of x.

Clearly, we can replace "every neighborhood W of x" in the definition with "every basic neighborhood B of x" or "every open set O containing x." Be sure you are convinced of this.

In a pseudometric space a sequence can converge to more than one point, but we proved that in a metric space limits of convergent sequences must be unique. A similar situation holds in any space: the important issue is whether we can "separate points by open sets."

Definition 9.2 A space $(X, \mathcal{T})$ is a T_1-space if for every pair of points $x \neq y \in X$ there exist open sets U and V such that $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$ (*that is, each of the points is an open set that does not contain the other point*).

$(X, \mathcal{T})$ is called a T_2-space (or Hausdorff space) if whenever $x \neq y \in X$ there exist disjoint open sets U and V such that $x \in U$ and $y \in V$.

It is easy to check that X is a T_1-space iff for every $x \in X$, $\{x\}$ is closed.

> *There are several "separation axioms" called $T_0, T_1, T_2, T_3,$ and T_4 that a topological space might satisfy. We will look at all of them eventually. Each condition is stronger than the earlier ones in the list $($for example, $T_2 \Rightarrow T_1)$. The letter "T" is used here because in the early (German) literature, the word for "separation axioms" was "Trennungsaxiome."*

Theorem 9.3 In a Hausdorff space $(X, \mathcal{T})$, a sequence can converge to at most one point.

Proof Suppose $x \neq y \in X$. Choose disjoint open sets U and V with $x \in U$ and $y \in V$. If $(x_n) \to x$, then (x_n) is eventually in U, so (x_n) is not eventually in V. Therefore (x_n) does not also converge to y. •

When we try to generalize results from pseudometric spaces to topological spaces, we often get a better insight about where the heart of a proof lies. For example, to prove that limits of sequences are unique it is the Hausdorff property that is important, not the presence of a metric. For a pseudometric space (X, d) we proved that $x \in \text{cl}\, A$ iff x is a limit of a sequence (a_n) in A. That proof (*see Theorem II.5.18*) used the fact that there was a countable neighborhood base $\{B_{\frac{1}{n}}(x) : n \in \mathbb{N}\}$ at each point x. We can see now that the countable neighborhood base was the crucial fact: we can prove the same result in any first countable topological space $(X, \mathcal{T})$.

First, two technical lemmas are helpful.

Lemma 9.4 Suppose $\{V_1, V_2, ..., V_k, ...\}$ is a countable neighborhood base at $a \in X$ and define $U_k = \text{int}\,(V_1 \cap ... \cap V_k)$. Then $\{U_1, U_2, ..., U_k, \,...\}$ is also a neighborhood base at a.

Proof $V_1 \cap ... \cap V_k$ is a neighborhood of a, so $a \in \text{int}\,(V_1 \cap ... \cap V_k) = U_k$. Therefore U_k is a open neighborhood of a. If N is any neighborhood of a, then $a \in V_k \subseteq N$ for some k, so then $a \in U_k \subseteq V_k \subseteq N$. Therefore the U_k's are a neighborhood base at a. •

The important thing is that we can get an "improved" neighborhood base $\{U_1, U_2, ..., U_k, \,...\}$ in which the U_k's are open and $U_1 \supseteq U_2 \supseteq ... \supseteq U_k \supseteq ...$; the exact formula for the U_k's in Lemma 9.4 doesn't matter. This new neighborhood base at a plays a role like the neighborhood base $B_1(a) \supseteq B_{\frac{1}{2}}(a) \supseteq ... \supseteq B_{\frac{1}{k}}(a) \supseteq ...$ in a pseudometric space. We call $\{U_1, U_2, ..., U_k, ...\}$ an open, shrinking neighborhood base at a.

Lemma 9.5 Suppose $\{U_1, U_2, ..., U_k, ...\}$ is a shrinking neighborhood base at x and that $a_n \in U_n$ for each n. Then $(a_n) \to x$.

Proof If W is any neighborhood of x, then there is a k such that $x \in U_k \subseteq W$. Since the U_k's are a shrinking neighborhood base, we have that for any $n \geq k$, $a_n \in U_n \subseteq U_k \subseteq W$. So $(a_n) \to x$. ∙

Theorem 9.6 Suppose $(X, \mathcal{T})$ is first countable and $A \subseteq X$. Then $x \in \text{cl}\, A$ iff there is a sequence (a_n) in A such that $(a_n) \to x$. (*More informally, "sequences are sufficient" to describe the topology in a first countable topological space.*)

Proof ($\Leftarrow$) Suppose (a_n) is a sequence in A and that $(a_n) \to x$. For each neighborhood W of x, (a_n) is eventually in W. Therefore so $W \cap A \neq \emptyset$, so $x \in \text{cl}\, A$. (*This half of the proof works in any topological space: it does not depend on first countability*.)

($\Rightarrow$) Suppose $x \in \text{cl}\, A$. Using Lemma 9.4, choose a countable shrinking neighborhood base $\{U_1, ..., U_n, ...\}$ at x. Since $x \in \text{cl}\, A$, we can choose a point $a_n \in U_n \cap A$ for each n. By Lemma 9.5, $(a_n) \to x$. ∙

We can use Theorem 9.6 to get an upper bound on the size of certain topological spaces, analogous to what we did for pseudometric spaces. This result is not very important, but it illustrates that in Theorem II.5.21 the properties that are really important are "first countability" and "Hausdorff," not the actual presence of a metric d.

Corollary 9.7 If D is a dense subset in a first countable Hausdorff space $(X, \mathcal{T})$, then $|X| \leq |D|^{\aleph_0}$. In particular, If X is a separable, first countable Hausdorff space, then $|X| \leq \aleph_0^{\aleph_0} = c$.

Proof X is first countable, so for each $x \in X$ we can pick a sequence (d_n) in D such that $(d_n) \to x$; formally, this sequence is a function $f_x : \mathbb{N} \to D$, so $f_x \in D^{\mathbb{N}}$. Since X is Hausdorff, a sequence cannot converge to two different points: so if $x \neq y \in X$, then $f_x \neq f_y$. Therefore the function $\Phi : X \to D^{\mathbb{N}}$ given by $\Phi(x) = f_x$ is one-to-one, so $|X| \leq |D^{\mathbb{N}}| = |D|^{\aleph_0}$. ∙

The conclusion in Theorem 9.6 may not be true if X is not first countable: sequences are not always "sufficient to determine the topology" of X – that is, convergent sequences cannot always describe the closure of a set.

Example 9.8 (the space *L*)

Let $L = \{(m, n) : m, n \in \mathbb{Z},\, m, n \geq 0\}$, and let C_j be "the j^{th} column of L," that is $C_j = \{(j, n) \in L : n = 0, 1, ...\}$. We put a topology on L by giving a neighborhood base at each point p :

$$\mathcal{B}_p = \begin{cases} \{(m,n)\} & \text{if } p = (m,n) \neq (0,0) \\ \{B : (0,0) \in B \text{ and } C_j - B \text{ is finite for all but finitely many } j\} & \text{if } p = (0,0) \end{cases}$$

(*Check that this definition satisfies the conditions in the Neighborhood Base Theorem 5.2 and* *therefore does describe a topology for* L.) If $p \neq (0,0)$ then p is isolated in L. A basic neighborhood of $(0,0)$ is a set which contains $(0,0)$ and which, we could say, contains "most of the points from most of the columns." With this topology, L is a Hausdorff space.

Certainly $(0,0) \in \text{cl}\,(L - \{(0,0)\})$, but no sequence from $L - \{(0,0)\}$ converges to $(0,0)$. To see this, consider any sequence (a_n) in $L - \{(0,0)\}$:

i) if there is a column C_{j_0} that contains infinitely many of the terms a_n, then $N = (L - C_{j_0}) \cup \{(0,0)\}$ is a neighborhood of $(0,0)$ and (a_n) is not eventually in N.

ii) if every column C_j contains only finitely many a_n's, then $N = L - \{a_n : n \in \mathbb{N}\}$ is a neighborhood of $(0,0)$ and (a_n) is not eventually in N (in fact, the sequence is never in N).

In L, sequences are not sufficient to describe the topology: convergent sequences can't show us that $(0,0) \in \text{cl}\,(L - \{(0,0)\})$. According to Theorem 9.6, this means that L cannot be first countable $-$ there is a countable neighborhood base at each point $p \neq (0,0)$ but not at $(0,0)$.

The neighborhood system at $(0,0)$ "measures nearness to" $(0,0)$ but the ordering relationship ($\supseteq$) among the basic neighborhoods at $(0,0)$ is very complicated $-$ much more complicated than the neat, simple nested chain of neighborhoods $U_1 \supseteq U_2 \supseteq \ldots \supseteq U_n \supseteq \ldots$ that would form a base at x in a first countable space. Roughly, the complexity of the neighborhood system is the reason why the terms of a sequence can't get "arbitrarily close" to $(0,0)$.

Sequences do suffice to describe the topology in a first countable space, so it is not surprising that we can use sequences to decide whether a function defined on a first countable space X is continuous.

Theorem 9.9 Suppose $(X, \mathcal{T})$ is first countable and $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$. Then f is continuous at $a \in X$ iff whenever $(x_n) \to a$, then $(f(x_n)) \to f(a)$.

Proof ($\Rightarrow$) If f is continuous at a and W is a neighborhood of $f(a)$, then $f^{-1}[W]$ is a neighborhood of a. Therefore (x_n) is eventually in $f^{-1}[W]$, so $f((x_n))$ is eventually in W. (*This half of the proof is valid in any topological space* X.)

($\Leftarrow$) Let $\{U_1, \ldots, U_n, \ldots\}$ be a shrinking neighborhood base at a. If f is not continuous at a, then there is a neighborhood W of $f(a)$ such that for every n, $f[U_n] \nsubseteq W$. For each n, choose a point $x_n \in U_n - f^{-1}[W]$. Then (since the U_n's are shrinking) we have $(x_n) \to a$ but $(f(x_n))$ fails to converge to $f(a)$ because $f(x_n)$ is never in W. •

(*Compare this to the proof of Theorem II.5.22.*)

10. Subsequences

Definition 10.1 Suppose $f : \mathbb{N} \to X$ is a sequence in X and that $\phi : \mathbb{N} \to \mathbb{N}$ is strictly increasing. The composition $f \circ \phi : \mathbb{N} \to X$ is called a <u>subsequence</u> of f.

$$\begin{array}{ccc} & f & \\ & \mathbb{N} \longrightarrow X & \\ \phi \uparrow & \nearrow f \circ \phi & \\ & \mathbb{N} & \end{array}$$

If we write $f(n) = x_n$ and $\phi(k) = n_k$, then $(f \circ \phi)(k) = f(n_k) = x_{n_k.}$ We write the sequence f informally as (x_n) and the subsequence $f \circ \phi$ as (x_{n_k}). Since ϕ is increasing, we have that $n_k \to \infty$ as $k \to \infty$.

For example, if $\phi(k) = n_k = 2k$, then $f \circ \phi$ is the subsequence informally written as $(x_{n_k}) = (x_{2k})$, that is, the subsequence $(x_2, x_4, x_6, ..., x_{2k}, ...)$. But if $\phi(n) = 1$ for all n, then $f \circ \phi$ is <u>not</u> a subsequence: informally, $(x_1, x_1, x_1, ..., x_1, ...)$ is not a subsequence of $(x_1, x_2, x_3, ..., x_n, ...)$. A sequence f is a subsequence of itself: just let $\phi(n) = n$.

Theorem 10.2 Suppose $x \in (X, \mathcal{T})$. Then $(x_n) \to x$ iff every subsequence $(x_{n_k}) \to x$.

Proof ($\Leftarrow$) This is clear because (x_n) is a subsequence of itself.
($\Rightarrow$) Suppose $(x_n) \to x$ and that (x_{n_k}) is a subsequence. If W is any neighborhood of x, then $x_n \in W$ whenever $n >$ some n_0. Since the n_k's are strictly increasing, $n_k > n_0$ for all $k >$ some k_0. Therefore (x_{n_k}) is eventually in W, so $(x_{n_k}) \to x$. •

Definition 10.3 Suppose $x \in (X, \mathcal{T})$. We say that x is a <u>cluster point</u> of the sequence (x_n) if for each neighborhood W of x and <u>for each $k \in \mathbb{N}$, there is an $n > k$ for which $x_n \in W$</u>. More informally, we say that x is a cluster point of (x_n) if the sequence is <u>frequently in every neighborhood W of x</u>. (*The underlined phrases mean the same thing.*)

Definition 10.4 Suppose $x \in (X, \mathcal{T})$ and $A \subseteq X$. We say that x is a <u>limit point</u> of A if $N \cap (A - \{x\}) \neq \emptyset$ for every neighborhood N of x – that is, every neighborhood of x contains points of "arbitrarily close" to x but different from x.

Example 10.5.

1) Suppose $X = [0, 1] \cup \{2\}$ and $A = \{2\}$. Then $W \cap A \neq \emptyset$ for every neighborhood of 2, but 2 is not a limit point of A since $W \cap (A - \{2\}) = \emptyset$. Each $x \in [0, 1]$ is a limit point of $[0, 1]$ and also a limit point of X. Since $\{2\}$ is open, 2 is not a limit point of X.

2) In $\mathbb{R}$, every point r is a limit point of $\mathbb{Q}$. If $A \subseteq \mathbb{N}$, then A has no limit points in $\mathbb{N}$ and no limit points in $\mathbb{R}$.

3) If $(x_n) \to x$, then x is a cluster point of (x_n). More generally, if (x_n) has a subsequence $(x_{n_k}) \to x$, then x is a cluster point of (x_n). (*Why?*)

4) A sequence can have many cluster points. For example, if the sequence (q_n) lists all the elements of $\mathbb{Q}$, then every $r \in \mathbb{R}$ is a cluster point of (q_n).

5) The only cluster points in $\mathbb{R}$ for the sequence $(x_n) = ((-1)^n)$ are -1 and 1. But the set $\{x_n : n \in \mathbb{N}\} = \{-1, 1\}$ has no limit points in $\mathbb{R}$. The set of cluster points of a sequence is not always the same as the set of limit points of the set of terms in the sequence! (*Is one of these sets always a subset of the other?*)

Theorem 10.6 Suppose $(X, \mathcal{T})$ is first countable and that a is a cluster point of (x_n). Then there is a subsequence $(x_{n_k}) \to a$.

Proof Let $\{U_1, U_2, ..., U_n, ...\}$ be a countable shrinking neighborhood base at a. Since (x_n) is frequently in U_1, we can pick n_1 so that $x_{n_1} \in U_1$. Since (x_n) is frequently in U_2, we can pick an $n_2 > n_1$ so that $x_{n_2} \in U_2 \subseteq U_1$. Continue inductively: having chosen $n_1 < ... < n_k$ so that $x_{n_k} \in U_k \subseteq ... \subseteq U_1$, we can then choose $n_{k+1} > n_k$ so that $x_{n_{k+1}} \in U_{k+1} \subseteq U_k$. Then (x_{n_k}) is a subsequence of (x_n) and $(x_n) \to a$. ∙

Example 10.7 (the space L, revisited)

Let L be the space in Example 9.8 and let (x_n) be a sequence which lists all the elements of $L - \{(0,0)\}$.

Every basic neighborhood B of $(0,0)$ is infinite, so B must contain terms x_n for arbitrarily large n. This means that (x_n) is frequently in B, so $(0,0)$ is a cluster point of (x_n).

But no subsequence of (x_n) can converge to $(0,0)$ – because we showed earlier that no sequence whatsoever from $L - \{(0,0)\}$ can converge to $(0,0)$. Therefore Theorem 10.6 may not be true if the space X is not first countable.

Consider any sequence $(x_n) \to (0,0)$ in L. If there were infinitely many $x_n \neq (0,0)$, then we could form the subsequence that contains those terms, and that subsequence would be a sequence in $L - \{(0,0)\}$ that converges to $(0,0)$ – which is impossible. Therefore we conclude that eventually $x_n = (0,0)$. Now let $f : L \to \mathbb{N}$ be any bijection. If $(x_n) \to (0,0)$ in L, then $f(x_n) = f((0,0)) = k$ eventually, so $(f(x_n)) \to k = f((0,0))$ in $\mathbb{N}$.

The topology on $\mathbb{N}$ is discrete, so $\{k\}$ is a neighborhood of $f(0,0)$, but $f^{-1}[\{k\}] = \{(0,0)\}$ is not a neighborhood of $(0,0)$. We conclude that f is not continuous at $(0,0)$ even though $(f(x_n)) \to f((0,0))$ for every sequence $(x_n) \to (0,0)$.

Theorem 9.9 does not apply to L: if a space is not first countable, sequences may be inadequate to check whether a function f is continuous at a point.

Exercises

E17. Suppose $f, g : (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2)$ are continuous functions, that D is dense in X, and that $f|D = g|D$. Prove that is Y is Hausdorff, then $f = g$. (*This generalizes the result in Chapter 2, Theorem 5.12.)*

E18. A function $f : (X, \mathcal{T}) \to \mathbb{R}$ is called *lower semicontinuous* if

$$f^{-1}[(b, \infty)] = \{x : f(x) > b\} \text{ is open for every } b \in \mathbb{R},$$

and f is called *upper semicontinuous* if

$$f^{-1}[(-\infty, b)\,] = \{x : f(x) < b\} \text{ is open for each } b \in \mathbb{R}.$$

a) Show that f is continuous iff f is both upper and lower semicontinuous.

b) Give an example of a lower semicontinuous $f : \mathbb{R} \to \mathbb{R}$ which is not continuous. Do the same for upper semicontinuous.

c) Prove that the characteristic function χ_A of a set $A \subseteq (X, \mathcal{T})$ is lower semicontinuous if A is open in X and upper semicontinuous if A is closed in X.

E19. Suppose X is an infinite set with the cofinite topology, and that Y has the property that every singleton $\{y\}$ is a closed set. (*Such a space Y is called a T_1-space.*) Let $f: X \to Y$ be continuous and onto. Prove that either f is constant or X is homeomorphic to Y.

1) Note: the problem does <u>not</u> say that if f is not constant, then f is a homeomorphism.
2) Hint: Prove first that if f is not constant, then $|X| = |Y|$. Then examine the topology of Y.)

E20. Suppose X is a countable set with the cofinite topology. State and prove a theorem that completely answers the question: "what sequences in $(X, \mathcal{T})$ converge to what points?"

E21. Suppose that $(X, \mathcal{T})$ and $(Y, \mathcal{T}\,')$ are topological spaces. Recall that the <u>product topology</u> on $X \times Y$ is the topology for which the collection of "open boxes"

$$\mathcal{B} = \{U \times V : U \in \mathcal{T},\ V \in \mathcal{T}\,'\} \text{ is a base.}$$

a) The "projection maps" $\pi_x : X \times Y \to X$ and $\pi_y : X \times Y \to Y$ are defined by

$$\pi_x(x, y) = x \text{ and } \pi_y(x, y) = y.$$

We showed in Example 5.11 that π_x and π_y are continuous. Prove that π_x and π_y are open maps. Give examples to show that π_x and π_y might not be closed.

b) Suppose that $(Z, \mathcal{T}\,'')$ is a topological space and that $f : Z \to X \times Y$. Prove that

f is continuous iff both compositions $\pi_x \circ f : Z \to$ and $\pi_y \circ f : Z \to Y$ are continuous.

c) Prove that $((x_n, y_n)) \to (x, y) \in X \times Y$ iff $(x_n) \to x$ in X and $(y_n) \to y$ in Y. (*For this reason, the product topology is sometimes called the "topology of coordinatewise convergence."*)

d) Prove that $X \times Y$ is homeomorphic to $Y \times X$. (*Topological products are commutative.*)

e) Prove that $(X \times Y) \times Z$ is homeomorphic to $X \times (Y \times Z)$ (*Topological products are associative.*)

E22. Let $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ be topological spaces. Suppose $f : X \to Y$. Let

$$\Gamma(f) = \{(x, y) \in X \times Y \colon y = f(x)\} = \text{"the graph of } f.\text{"}$$

Prove that the map $h \colon X \to \Gamma(f)$ defined by $h(x) = (x, f(x))$ is a homeomorphism if and only if f is continuous.

Note: if we think of f as a set of ordered pairs, the "graph of f" is f. More informally, $however$, the problem states that the graph of a function is homeomorphic to its domain iff the function is continuous.)

E23. In $(X, \mathcal{T})$, a family of sets $\mathcal{F} = \{B_\alpha : \alpha \in \Lambda\}$ is called locally finite if each point $x \in X$ has a neighborhood N such that $N \cap B_\alpha \neq \emptyset$ for only finitely many α's. (*Part b) was also in Exercise E2.*)

a) Suppose (X, d) is a metric space and that $\mathcal{F}$ is a family of closed sets. Suppose there is an $\epsilon > 0$ such that $d(B_1, B_2) \geq \epsilon$ for all $B_1, B_2 \in \mathcal{F}$. Prove that $\mathcal{F}$ is locally finite.

b) Prove that if $\mathcal{F}$ is a locally finite family of sets in $(X, \mathcal{T})$, then $\text{cl}\,(\bigcup_{\alpha \in A} B_\alpha) = \bigcup_{\alpha \in A} \text{cl}\,(B_\alpha)$. Explain why this implies that if all the B_α's are closed, then $\bigcup_{\alpha \in A} B_\alpha$ is closed. (*This would apply, for example, to the sets in part a*).

c) (*The Pasting Lemmas: compare Exercise II.E24*)

Let $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$ be topological spaces. For each $\alpha \in A$, suppose $B_\alpha \subseteq X$, that $f_\alpha : B_\alpha \to Y$ is a continuous function, and that $f_\alpha | (B_\alpha \cap B_\beta) = f_\beta | (B_\alpha \cap B_\beta)$ for all $\alpha, \beta \in A$ (*that is, f_α and f_β agree wherever their domains overlap*). Then $\bigcup_{\alpha \in A} f_\alpha = f$ is a function and $f : \bigcup B_\alpha \to Y$. (*The function f is built by "pasting together" all the "function pieces" f_α.*)

i) Show that if all the B_α's are open, then f is continuous.

ii) Give an example to show that f might not be continuous when there are infinitely many B_α's all of which are closed.

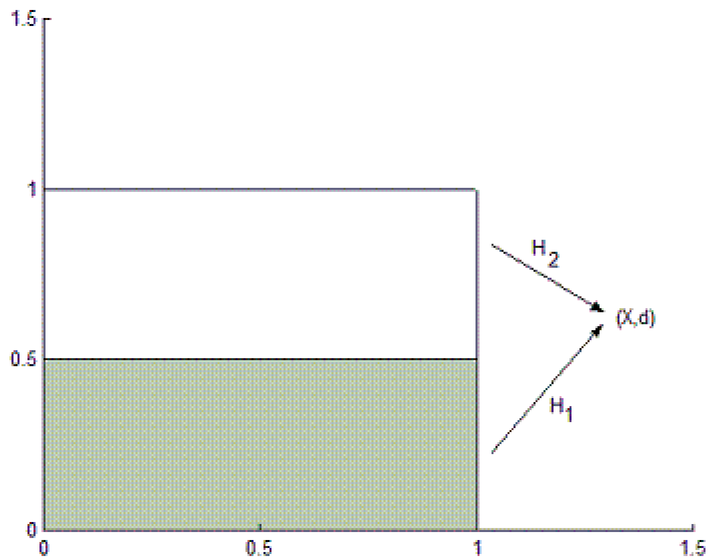
iii) Show that if there are only finitely many B_α's and they are all closed, then f

is continuous. (*Hint: use a characterization of continuity in terms of closed sets.*)

iv) Show that if $\mathcal{F}$ is a locally finite family of closed sets, then f is continuous. (*Of course, iv)* $\Rightarrow$ *iii)*).

Note: the most common use of the Pasting Lemma is when the index set A *is finite. For example, suppose*

$H_1 : [0,1] \times [0, \frac{1}{2}] \to (X,d)$ *is continuous, and*
$H_2 : [0,1] \times [\frac{1}{2}, 1] \to (X,d)$ *is continuous, and*
$H_1(t, \frac{1}{2}) = H_2(t, \frac{1}{2})$ *for all* $t \in [0,1]$



H_1 *is defined on the lower closed half of the box* $[0,1]^2$, H_2 *is defined on the upper closed half, and they agree on the "overlap" – that is, on the horizontal line segment* $[0,1] \times \{\frac{1}{2}\}$. *Part b) (or part c)) says that the two functions can be pieced together into a continuous function* $H : [0,1]^2 \to (X,d)$, *where* $H = H_1 \cup H_2$.

Chapter III Review

Explain why each statement is true, or provide a counterexample. If nothing else is mentioned, X and Y are topological spaces with no other special properties assumed.

1. For every possible topology $\mathcal{T}$, the space $(\{0,1,2\},\mathcal{T})$ is pseudometrizable.

2. A convergent sequence in a first countable topological space has at most one limit.

3. If $x \in (X,\mathcal{T})$, then $\{x\}$ is the intersection of a family of open sets.

4. A one point set $\{x\}$ in a pseudometric space (X,d) is closed.

5. Suppose $\mathcal{T}$ and $\mathcal{T}'$ are topologies on X and that for every subset A of X, $\mathrm{cl}_{\mathcal{T}}(A) = \mathrm{cl}_{\mathcal{T}'}(A)$. Then $\mathcal{T} = \mathcal{T}'$.

6. Suppose $f\colon X \to Y$ and $A \subseteq X$. If $f|A : A \to Y$ is continuous, then f is continuous at each point of A.

7. Suppose $\mathcal{S}$ is a subbase for the topology on X and that $D \subseteq X$. If $S \cap D \neq \emptyset$ for every $S \in \mathcal{S}$, then D is dense in X.

8. If $A \subseteq X$ and D is dense in X, then $A \cap D$ is dense in A.

9. If D is dense in $(X,\mathcal{T})$ and $\mathcal{T}^*$ is another topology on X with $\mathcal{T} \subseteq \mathcal{T}^*$, then D is dense in $(X,\mathcal{T}^*)$.

10. If $f\colon X \to Y$ is both continuous and open, then f is also closed.

11. Every space is a continuous image of a first countable space.

12. Let $X = \{0,1\}$ with the topology $\mathcal{T} = \{\emptyset, X, \{1\}\}$. There are exactly 3 continuous functions $f : (X,\mathcal{T}) \to (X,\mathcal{T})$.

13. If A and B are subspaces of $(X,\mathcal{T})$ and both A and B are discrete in the subspace topology, then $A \cup B$ is discrete in the subspace topology.

14. If $D \subseteq \mathbb{R}$ and D is discrete in the subspace topology, then D is closed in $\mathbb{R}$.

15. If $A \subseteq (X, \mathcal{T})$ and X is separable, then A is separable.

16. Suppose $\mathcal{T}$ is the cofinite topology on X. Then any bijection $f: (X, \mathcal{T}) \to (X, \mathcal{T})$ is a homeomorphism.

17. If *X* has the cofinite topology, then the closure of every open set in *X* is open.

18. If $(X, \mathcal{T})$ is metrizable, and $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ is continuous and onto, then $(Y, \mathcal{T}')$ is also metrizable.

19. If *D* is dense in X, then each point of X is a limit of a sequence of points from *D*.

20. A continuous bijection from $\mathbb{R}$ to $\mathbb{R}$ is a homeomorphism.

21. For every cardinal number m, there is a separable topological space $(X, \mathcal{T})$ with $|X| = m$.

22. If every family of disjoint open sets in $(X, \mathcal{T})$ is countable, then $(X, \mathcal{T})$ is separable.

23. For $a \in \mathbb{R}^2$ and $\epsilon > 0$, let $C_\epsilon(a) = \{x \in \mathbb{R}^2\colon\ d(x, a) = \epsilon\}$.
Let $\mathcal{T}$ be the topology on $\mathbb{R}^2$ for which the collection $\{C_\epsilon(a)\colon\ \epsilon > 0, a \in \mathbb{R}^2\}$ is a subbase.
Let $\mathcal{U}$ be the usual topology on $\mathbb{R}^2$.
Then the function $f: (\mathbb{R}^2, \mathcal{T}) \to (\mathbb{R}^2, \mathcal{U})$ given by $f(x, y) = (\sin x, \sin y)$ is continuous.

24. If $D \subseteq \mathbb{R}$ and each point of D is isolated in D, then cl D must be countable.

25. Consider the property S: "every minimal nonempty closed set is a singleton". Then $T_1 \Rightarrow$ S.

26. A one-to-one, continuous map $f : (X, \mathcal{T}) \to (Y, \mathcal{T})$ is a homeomorphism iff f is onto.

27. An uncountable closed set in $\mathbb{R}$ must contain an interval of positive length.

28. A countable metric space has a base consisting of clopen sets.

29. If *D* is dense in $(X, \mathcal{T})$ and $\mathcal{T}^*$ is topology on X with $\mathcal{T}^* \subseteq \mathcal{T}$, then *D* is dense in $(X, \mathcal{T}^*)$.

30. If $D \subseteq \mathbb{R}$ and D is discrete in the subspace topology, then D is countable.

31. The Sorgenfrey plane has a subspace homeomorphic to $\mathbb{R}$ (with its usual topology).

32. Let $\mathcal{B}_n = \{B \subseteq \mathbb{N} : n \in B$ and $B \subseteq \{1, 2, ..., n\}\}$. The $\mathcal{B}_n$'s satisfy the conditions in the Neighborhood Base Theorem and therefore describe a topology on $\mathbb{N}$.

33. At each point $n \in \mathbb{N}$, let $\mathcal{B}_n = \{B : B \supseteq \{n, n+1, n+2, ...\}\}$. The $\mathcal{B}_n$'s satisfy the conditions in the Neighborhood Base Theorem and therefore describe a topology on $\mathbb{N}$.

34. Suppose $(X, \mathcal{T})$ has a base $\mathcal{B}$ with $|\mathcal{B}| = c$. Then every open cover $\mathcal{U}$ of X has a subcover $\mathcal{U}'$ for which $|\mathcal{U}'| \leq c$.

35. Suppose $(X, \mathcal{T})$ has a base $\mathcal{B}$ with $|\mathcal{B}| = c$. Then X has a dense set D with $|D| \leq c$.

36. Suppose $(X, \mathcal{T})$ has a base $\mathcal{B}$ with $|\mathcal{B}| = c$. Then at each point $x \in X$, there is a neighborhood base with $|\mathcal{B}_x| \leq c$.

37. If $(X, \mathcal{T})$ has a base $\mathcal{B}$ where $|\mathcal{B}| = m$ is finite, then $|\mathcal{T}| \leq 2^m$.

38. Suppose X is an infinite set. Let $\mathcal{T}_1$ be the cofinite topology on X and let $\mathcal{T}_2$ be the discrete topology on X. If $f : \mathbb{R} \to (X, \mathcal{T}_1)$ is continuous, then $f : \mathbb{R} \to (X, \mathcal{T}_2)$ is also continuous.

39. Consider $\mathbb{R}$ with the topology $\mathcal{T} = \{(a, \infty) : a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$. $(\mathbb{R}, \mathcal{T})$ is first countable.