Chapter IV
Completeness and Compactness

1. Introduction

In Chapter III we introduced topological spaces – a generalization of pseudometric spaces. This allowed us to see how certain ideas – continuity, for example – can be extended into a situation where there is no "distance" between points. Continuity does not really depend on the pseudometric d but only on the topology.

Looking at topological spaces also highlighted the particular properties of pseudometric spaces that were really important for certain purposes. For example, it turned out that first countability is the crucial ingredient for proving that sequences are sufficient to describe a topology, and that Hausdorff property, not the metric d, is what matters to prove that limits of sequences are unique.

We also looked at some properties that are equivalent in pseudometric spaces – for example, second countability and separability – but are not equivalent in topological spaces.

Most of the earlier definitions and theorems are just basic "tools" for our work. Now we look at some deeper properties of pseudometric spaces and some significant theorems related to them.

2. Complete Pseudometric Spaces

Definition 2.1 A sequence (x_n) in a pseudometric space (X, d) is called a Cauchy sequence if $\forall \epsilon > 0$ $\exists M, N \in \mathbb{N}$ such that if $m, n > N$, then $d(x_m, x_n) < \epsilon$.

Informally, a sequence (x_n) is Cauchy if its terms "get closer and closer to each other." It should be intuitively clear that this happens if the sequence converges, and the next theorem confirms this.

Theorem 2.2 If $(x_n) \to x$ in (X, d), then (x_n) is Cauchy.

Proof Let $\epsilon > 0$. Because $(x_n) \to x$, there is an $N \in \mathbb{N}$ such that $d(x_n, x) < \frac{\epsilon}{2}$ for $n > N$. So if $m, n > N$, we get $d(x_m, x_n) \leq d(x_m, x) + d(x, x_n) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. Therefore (x_n) is Cauchy. •

A Cauchy sequence does not always converge. For example, look at the space $(\mathbb{Q}, d)$ where d is the usual metric. Consider a sequence in $\mathbb{Q}$ that converges to $\sqrt{2}$ in $\mathbb{R}$; for example, we could used $(q_n) = (1, 1.4, 1.41, 1.414, \ldots)$. Since (q_n) is convergent in $\mathbb{R}$, (q_n) is a Cauchy sequence in $\mathbb{Q}$ (*"Cauchy" depends only on d and the numbers q_n, not whether we are thinking of these q_n's as elements of $\mathbb{Q}$ or $\mathbb{R}$.*.)

But (q_n) has no limit in $\mathbb{Q}$. (*Why? To say that "$\sqrt{2} \notin \mathbb{Q}$" is part of the answer.*)

Definition 2.3 A pseudometric space (X, d) is called complete if every Cauchy sequence in (X, d) has a limit in X.

Example 2.4 Throughout this example, d denotes the usual metric on subsets of $\mathbb{R}$.

1) $(\mathbb{R}, d)$ is complete. (*For the moment, we take this as a simple fact from analysis; however we will prove it soon.*) But $(\mathbb{Q}, d)$ and $(\mathbb{P}, d)$ are not complete : completeness is not hereditary!

2) Let (x_n) be a Cauchy sequence in $(\mathbb{N}, d)$. Then there is some N such that $|x_m - x_n| < 1$ for $m, n > N$. But the x_n's are integers, so this means that $x_m = x_n$ for $m, n > N$. So every Cauchy sequence is eventually constant and therefore converges. $(\mathbb{N}, d)$ is complete.

3) Consider $\mathbb{N}$ with the metric $d_1(m, n) = |\frac{1}{m} - \frac{1}{n}|$. Then the sequence $(x_n) = (n)$ is Cauchy. If $k \in \mathbb{N}$, then $d_1(x_n, k) = |\frac{1}{n} - \frac{1}{k}| \to \frac{1}{k} \neq 0$ as $n \to \infty$, so (x_n) does not converge to k. The space $(\mathbb{N}, d_1)$ is not complete.

However $(\mathbb{N}, d_1)$ has the discrete topology, so the identity function $f(n) = n$ is a homeomorphism between $(\mathbb{N}, d_1)$ and the complete space $(\mathbb{N}, d)$. So completeness is not a topological property.

A homeomorphism takes convergent sequences to convergent sequences and non convergent sequences to nonconvergent sequences. For example, (n) does not converge in $(\mathbb{N}, d_1)$ and $(f(n))$ does not converge in $(\mathbb{N}, d)$. These two homeomorphic metric spaces have exactly the same convergent sequences but they do not have the same Cauchy sequences. Changing a metric d to a topologically equivalent metric d' does not change the open sets and does not change which sequences converge. But the change may create or destroy Cauchy sequences since this property depends on distance measurements.

Another similar example: we know that any open interval (a, b) is homeomorphic to $\mathbb{R}$. But (a, b) is not complete and $\mathbb{R}$ is complete (where both space have the metric d).

4) In light of the comments in 3) we can ask: if (X, d) is not complete, might there be a different metric $d' \sim d$ for which (X, d') is complete? To be specific: could we find a metric d' on $\mathbb{Q}$ so $d \sim d'$ but where $(\mathbb{Q}, d')$ is complete? We could also ask this question about $(\mathbb{P}, d)$. Later in this chapter we will see that the answer is "no" in one case but "yes" in the other.

5) $(\mathbb{N}, d_1)$ and $(\{\frac{1}{n} : n \in \mathbb{N}\}, d)$ are both countable discrete spaces, so they are homeomorphic. In fact, the function $f(n) = \frac{1}{n}$ is an isometry between these spaces: $d_1(m, n) = |\frac{1}{m} - \frac{1}{n}| = d(f(m), f(n))$. An isometry is a homeomorphism, but since it preserves distances, an isometry also takes Cauchy sequences to Cauchy sequences and non-Cauchy sequences to non-Cauchy sequences. Therefore if there is an isometry between two pseudometric spaces, then one space is complete iff the other space is complete.

For example, the sequence $(x_n) = n$ is Cauchy is $(\mathbb{N}, d_1)$ and the isometry f carries (x_n) to the Cauchy sequence $(f(x_n)) = (\frac{1}{n})$ in $(\{\frac{1}{n} : n \in \mathbb{N}\}, d)$. It is clear that (n) has no limit in the domain and $(\frac{1}{n})$ has no limit in the range.

$(\mathbb{N}, d_1)$ and $(\{\frac{1}{n} : n \in \mathbb{N}\}, d)$ "look exactly alike" $-$ not just topologically but as metric spaces. We can think of f as simply renaming the points in a distance-preserving way.

Theorem 2.5 If x is a cluster point of a Cauchy sequence (x_n) in (X, d), then $(x_n) \to x$.

Proof Let $\epsilon > 0$. Pick N so that $d(x_m, x_n) < \frac{\epsilon}{2}$ when $m, n > N$. Since (x_n) clusters at x, we can pick a $K > N$ so that $x_K \in B_{\frac{\epsilon}{2}}(x)$. Then for this K and for $n > N$,

$$d(x_n, x) \leq d(x_n, x_K) + d(x_K, x) < \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon$$

so $(x_n) \to x$. •

Corollary 2.6 A Cauchy sequence (x_n) in (X, d) converges iff it has a cluster point.

Corollary 2.7 In a metric space (X, d), a Cauchy sequence can have at most one cluster point.

Theorem 2.8 A Cauchy sequence (x_n) in (X, d) is bounded $-$ that is, the set $\{x_1, x_2, ..., x_n, ...\}$ has finite diameter.

Proof Pick N so that $d(x_m, x_n) < 1$ when $m, n \geq N$. Then $x_n \in B_1(x_N)$ for all $n \geq N$. Let $r = \max\{1, d(x_1, x_N), d(x_2, x_N), ..., d(x_{N-1}, x_N)\}$. Therefore, for all m, n we have $d(x_m, x_n) \leq d(x_n, x_N) + d(x_N, x_m) \leq 2r$, so diam $\{x_1, x_2, ... , x_n, ...\} \leq 2r$. •

Now we will prove that $\mathbb{R}$ (with its usual metric d) is complete. The proof depends on the completeness property (also known as the "least upper bound property") of $\mathbb{R}$. We start with two lemmas which might be familiar from analysis. A sequence (x_n) is called weakly increasing if $x_n \leq x_{n+1}$ for all n, and weakly decreasing if $x_n \geq x_{n+1}$ for all n. A monotone sequence is one that is either weakly increasing or weakly decreasing.

Lemma 2.9 If (x_n) is a bounded monotone sequence in $\mathbb{R}$, then (x_n) converges.

Proof Suppose (x_n) is weakly increasing. Since (x_n) is bounded, $\{x_n : n \in \mathbb{N}\}$ has a least upper bound in $\mathbb{R}$. Let $x = \sup\{x_n : n \in \mathbb{N}\}$. We claim $(x_n) \to x$.

Let $\epsilon > 0$. Since $x - \epsilon < x$, we know that $x - \epsilon$ is not an upper bound for $\{x_n : n \in \mathbb{N}\}$. Therefore $x - \epsilon < x_N \leq x$ for some N. Since (x_n) is weakly increasing and x is an upper bound for $\{x_n : n \in \mathbb{N}\}$, it follows that $x - \epsilon < x_N \leq x_n \leq x$ for all $n \geq N$. Therefore $|x_n - x| \leq \epsilon$ for all $n \geq N$, so $(x_n) \to x$.

If (x_n) is weakly decreasing, then $(-x_n)$ is a weakly increasing sequence so $(-x_n) \to a$ for some a; then $(x_n) \to -a$. •

Lemma 2.10 Every sequence (x_n) in $\mathbb{R}$ has a monotone subsequence.

Proof Call x_k a peak point of (x_n) if $x_k \geq x_n$ for all $n \geq k$. We consider two cases.

i) If (x_n) has only finitely many peak points, then there is a last peak point x_{n_0} in (x_n). Then x_n is not a peak point for $n > n_0$. Pick an $n_1 > n_0$. Since x_{n_1} is not a peak point, there is an $n_2 > n_1$ with $x_{n_1} < x_{n_2}$. Since x_{n_2} is not a peak point, there is an $n_3 > n_2$ with $x_{n_2} < x_{n_3}$. We continue in this way to pick an increasing subsequence (x_{n_k}) of (x_n).

ii) If (x_n) has infinitely many peak points, then list the peak points as a subsequence $x_{n_1}, x_{n_2}, ...$ (where $n_1 < n_2 < ...$). Since each of these points is a peak point, we have $x_{n_k} \geq x_{n_{k+1}}$ for each k, so (x_{n_k}) is a weakly decreasing subsequence of (x_n). •

Theorem 2.11 $(\mathbb{R}, d)$ is complete.

Proof Let (x_n) be a Cauchy sequence in $(\mathbb{R}, d)$. By Theorem 2.8, (x_n) is bounded and by the Lemma 2.10, (x_n) has a monotone subsequence (x_{n_k}). Of course (x_{n_k}) is also bounded. By Lemma 2.9, (x_{n_k}) converges to some point $x \in \mathbb{R}$. Therefore x is a cluster point of the Cauchy sequence (x_n), so $(x_n) \to x$. •

Corollary 2.12 $(\mathbb{R}^n, d)$ is complete.

Proof Exercise (*Hint: For $n = 2$, the sequence $((x_n, y_n))$ in $\mathbb{R}^2$ is Cauchy iff the sequences (x_n) and (y_n) both are Cauchy sequences in $\mathbb{R}$. If $(x_n) \to x$ and $(y_n) \to y$, then $((x_n, y_n) \to (x, y)$ in $\mathbb{R}^2$.*)

3. Subspaces of Complete Spaces

Theorem 3.1 Let d be a metric and suppose $A \subseteq (X, d)$.

1) If (X, d) is complete and A is closed, then (A, d) is complete.
2) If (A, d) is complete, then A is closed in (X, d).

Proof 1) Let (a_n) be a Cauchy sequence in A. Then (a_n) is also Cauchy in the complete space (X, d), so $(a_n) \to x$ for some $x \in X$. But A is closed, so this limit x must be in A, that is, $(a_n) \to x \in A$. Therefore (A, d) is complete.

2) If $x \in \text{cl}\, A$, we can pick a sequence in A such that $(a_n) \to x$. Since (a_n) converges, it is a Cauchy sequence in A. But (A, d) is complete, so we know $(a_n) \to a$ for some $a \in A \subseteq X$. Since limits of sequences are unique in metric spaces, we conclude that $x = a$. Therefore if $x \in \text{cl}\, A$, then $x \in A$. So A is closed. •

Part 1) of the proof is valid when d is a pseudometric, but the proof of part 2) requires that d be a metric. Can you provide an example of a pseudometric space (X, d) for which the part 2) of the theorem is false?

The definition of completeness is stated in terms of the existence of limits for Cauchy sequences. The following theorem characterizes complete spaces differently: in terms of the existence of points in certain intersections. This illustrates an important idea: completeness for (X, d) can be expressed in different ways, but all equivalent characterizations of completeness are statements that guarantee the existence of certain points.

Theorem 3.2 (The Cantor Intersection Theorem) The following are equivalent for a metric space (X, d)

1) (X, d) is complete
2) Whenever $F_1 \supseteq F_2 \supseteq ... \supseteq F_n \supseteq ...$ is a decreasing sequence of nonempty closed sets with diam$(F_n) \to 0$, then $\bigcap_{n=1}^{\infty} F_n = \{x\}$ for some $x \in X$.

Proof $1) \Rightarrow 2)$ For each k, pick a point $x_k \in F_k$. Since diam$(F_k) \to 0$, the sequence (x_k) is Cauchy so $(x_k) \to x$ for some $x \in X$.

For any n, the subsequence $(x_{n+k}) = (x_{n+1}, x_{n+2}, ...) \to x$. Since the F_n's are decreasing, this subsequence is inside the closed set F_n, so $x \in F_n$. Therefore $x \in \bigcap_{n=1}^{\infty} F_n$.

If $x, y \in \bigcap_{n=1}^{\infty} F_n$, then $d(x, y) \leq$ diam(F_n) for every n. Since diam$(F_n) \to 0$, this means that $d(x, y) = 0$. Therefore $x = y$, and so $\bigcap_{n=1}^{\infty} F_n = \{x\}$.

$2) \Rightarrow 1)$ Suppose 1) is false and let (x_n) be a nonconvergent Cauchy sequence (x_n). Without loss of generality, we may assume that all the x_n's are distinct. (*If any value $x_k = y$ were repeated infinitely often, then y would be a cluster point of (x_n) and we would have $(x_n) \to y$. So, since each term occurs only finitely often, it is possible to pick a subsequence from (x_n) whose terms are all distinct, and this subsequence is also a nonconvergent Cauchy sequence.*) We will construct sets F_n which violate condition 2).

Let $F_n = \{x_n, x_{n+1}, ..., x_{n+k}, ...\}$ = the "n^{th} tail" of the sequence (x_n). For each n, $F_n \neq \emptyset$, $F_n \supseteq F_{n+1}$ and $\bigcap_{n=1}^{\infty} F_n = \emptyset$. Also, for $\epsilon > 0$, we can pick N so that $d(x_m, x_n) < \frac{\epsilon}{2}$ if $m, n \geq N$. This means that for $n > N$, diam$(F_n) \leq$ diam$(F_N) \leq \frac{\epsilon}{2} < \epsilon$ and therefore diam$(F_n) \to 0$.

We claim that the F_n's are closed – which contradicts 2) and completes the proof. Suppose that some F_{n_0} is not closed. Then there is a point $x \in \text{cl}\, F_{n_0} - F_{n_0}$ and we can find a sequence of distinct terms in $\{x_{n_0}, x_{n_0+1}, x_{n_0+2}, ...\} = F_{n_0}$ that converges to x. This sequence is a subsequence $(x_{n_k}) \to x$ of the original sequence (x_n), so x is a cluster point of (x_n). But this is impossible – a nonconvergent Cauchy sequence (x_n) cannot have a cluster point. •

How would Theorem 3.2 be different if d were only a pseudometric? How would the proof change? Can you locate where the fact that d is a metric is used in each half of the proof ?

Example 3.3

1) In the complete space $(\mathbb{R}, d)$ consider:

a) $F_n = [n, \infty)$
b) $F_n = (0, \frac{1}{n})$
c) $F_n = [n,\, n + \frac{1}{n}]$

In each case, $\bigcap_{n=1}^{\infty} F_n = \emptyset$. Why are these examples not inconsistent with the Theorem 3.2?

2) Let $F_n = \{x \in \mathbb{Q} : [\sqrt{2} - \frac{1}{n} \leq x \leq \sqrt{2} + \frac{1}{n}]\}$. The F_n's satisfy all the hypotheses in Theorem 3.2), but $\bigcap_{n=1}^{\infty} F_n = \emptyset$. Why is this not inconsistent with Theorem 3.2?

Example 3.4 Here is a proof, using the Cantor Intersection Theorem, that the closed interval $[0, 1]$ is uncountable.

If $[0,1]$ were countable, we could list its elements as a sequence (x_n). Pick a subinterval $[a_1, b_1]$ of $[0,1]$ with length less than $\frac{1}{2}$ and excluding x_1. Then pick an interval $[a_2, b_2] \subseteq [a_1, b_1]$ of length less than $\frac{1}{4}$ and excluding x_2. Continuing inductively, choose an interval $[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$ of length less than $\frac{1}{2^{n+1}}$ and excluding x_{n+1}. Then $[a_1, b_1] \supseteq [a_2, b_2] \supseteq ... \supseteq [a_n, b_n] \supseteq ...$ and these sets clearly satisfy the conditions in part 2) of the Cantor Intersection Theorem. But $\bigcap_{n=1}^{\infty} [a_n, b_n] = \emptyset$. This is impossible since $[0,1]$ is complete.

The following theorem has some interesting consequences. In addition, the proof is a very nice application of Cantor's Intersection Theorem. The completeness is used to prove the existence of "very many" points in X.

Theorem 3.6 Suppose (X, d) is a nonempty complete metric space with no isolated points. Then $|X| \geq c$.

To prove Theorem 3.5, we will use the following lemma.

Lemma 3.5 Suppose (X, d) is a metric space that $A \subseteq X$. If $a, b \in \text{int}\, A$ and $a \neq b$, then we can find disjoint closed balls F_a and F_b (centered at a and b) with $F_a \subseteq \text{int}\, A$ and $F_b \subseteq \text{int}\, A$.

Proof Let $\epsilon = d(a, b)$. Since $a, b \in \text{int}\, A$, we can choose positive radii ϵ_a and ϵ_b so that $B_{\epsilon_a}(a) \subseteq \text{int}\, A$ and $B_{\epsilon_b}(b) \subseteq \text{int}\, A$. We may choose both ϵ_a and ϵ_b less than $\frac{\epsilon}{2}$. Let $\epsilon\,' = \min\{\epsilon_a, \epsilon_b\}$. Then we can use the closed balls $F_a = \{x : d(x, a) \leq \epsilon\,'\}$ and $F_b = \{x : d(x, b) \leq \epsilon\,'\}$. $\bullet$

Proof of Theorem 3.6 The idea of the proof is to inductively construct c different descending sequences of closed sets, each of which satisfies condition 2) in the Cantor Intersection Theorem, and to do this in a way that the intersection of each sequence gives a <u>different</u> point in X. It then follows that $|X| \geq c$. The idea is simple but the notation gets a bit complicated. First we will give the idea of how the construction is done. The actual details of the induction step are relegated to the end of the proof.

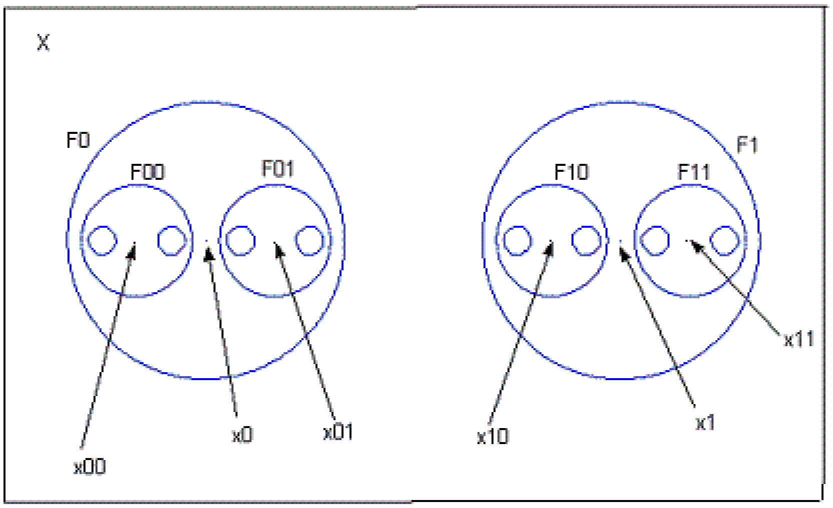
<u>Stage 1</u> X is nonempty and has no isolated points, so there must exist two points $x_0 \neq x_1 \in X$. Pick disjoint closed balls F_0 and F_1, centered at x_0 and x_1, each with diameter $< \frac{1}{2}$.

<u>Stage 2</u> Since $x_0 \in \text{int}\, F_0$ and x_0 is not isolated, we can pick distinct points x_{00} and x_{01} (both $\neq x_0$) in $\text{int}\, F_0$ and use Lemma 3.5 to pick disjoint closed balls F_{00} and F_{01} (centered at x_{00} and x_{01}) and both $\subseteq \text{int}\, F_0$. We can then shrink the balls, if necessary, so that each has diameter $< \frac{1}{2^2}$.

We can repeat the same steps inside F_1 : since $x_1 \in \text{int}\, F_1$ and x_1 is not isolated, we can pick distinct points x_{10} and x_{11} (both $\neq x_1$) in $\text{int}\, F_1$ and use the Lemma 3.5 to pick disjoint closed balls F_{10} and F_{11} (centered at x_{10} and x_{11}) and both $\subseteq \text{int}\, F_1$. We can then shrink the balls, if necessary, so that each has diameter $< \frac{1}{2^2}$. At the end of Stage 2, we have 4 disjoint closed balls: $F_{00}, F_{01}, F_{10}, F_{11}$.

<u>Stage 3</u> We now repeat the same construction inside <u>each</u> of the 4 sets $F_{00}, F_{01}, F_{10}, F_{11}$. For example, we can pick distinct points x_{000} and x_{001} (both $\neq x_{00}$) in $\text{int}\, F_{00}$ and use the Lemma

3.5 pick disjoint closed balls F_{000} and F_{001} (centered at x_{000} and x_{001}) and both $\subseteq \operatorname{int} F_{00}$. Then we can shrink the balls, if necessary, so that each has diameter $< \frac{1}{2^3}$. *See the figure below.*



After Stage 3, we have 8 disjoint closed balls: $F_{000}, F_{001}, F_{010}$, ... We have the beginnings of 8 descending sequences of closed balls at this stage. In each sequence, at the n^{th} "stage," the sets have diameter $< \frac{1}{2^n}$.

$$\begin{array}{ccc} & & F_{000} \\ & F_{00} \supseteq & \\ & & F_{001} \\ F_0 \supseteq & & \\ & & F_{010} \\ & F_{01} \supseteq & \\ & & F_{011} \end{array}$$

and

$$\begin{array}{ccc} & & F_{100} \\ & F_{10} \supseteq & \\ & & F_{101} \\ F_1 \supseteq & & \\ & & F_{110} \\ & F_{11} \supseteq & \\ & & F_{111} \end{array}$$

We continue inductively (*see the details below*) in this way – at a given stage, each descending sequence of closed balls splits into two "disjoint branches." The sequence "splits" when we choose two new nonempty closed balls inside the current one, making sure that their diameters keep shrinking toward 0.

In the end, each binary sequence $s = (n_1, n_2, ..., n_k, ...) \in \{0,1\}^{\mathbb{N}}$ will correspond to a descending sequence of nonempty closed sets whose diameters $\to 0$:

$$F_{n_1} \supseteq F_{n_2 n_2} \supseteq F_{n_1 n_2 n_3} \supseteq ... \supseteq F_{n_1 n_2 ... n_k} \supseteq ...$$

For example, the binary sequence $s = (0,1,1,0,0,1...) \in \{0,1\}^{\mathbb{N}}$ corresponds to the descending sequence of closed sets

$$F_0 \supseteq F_{01} \supseteq F_{011} \supseteq F_{0110} \supseteq F_{01100} \supseteq F_{011001} \supseteq ...$$

The Cantor Intersection theorem tells us for each such s, there is an $x_s \in X$ such that

$$F_{n_1} \cap F_{n_2 n_2} \cap F_{n_1 n_2 n_3} \cap ... \cap F_{n_1 n_2 ... n_k} \cap ... = \bigcap_{k=1}^{\infty} F_{n_1 n_2 ... n_k} = \{x_s\}$$

For two different binary sequences, say $t = (m_1, m_2, ..., m_k, ...) \neq (n_1, n_2, ..., n_k, ...) = s$, there is a smallest k for which $m_k \neq n_k$. Then $x_s \in F_{n_1 n_2 ... n_{k-1} n_k}$ and $x_t \in F_{n_1 n_2 ... n_{k-1} m_k}$. Since these sets are disjoint, we have $x_s \neq x_t$. Thus, mapping $s \mapsto x_s$ gives a $1-1$ function from $\{0,1\}^{\mathbb{N}}$ into X. We conclude that $c = 2^{\aleph_0} = |\{0,1\}|^{\mathbb{N}} \leq |X|$.

Here are the details of the formal induction step in the proof..

Induction Hypothesis: Suppose we have completed k stages $-$ that is, for each $i = 1, ..., k$ and for each i-tuple $(n_1, ..., n_i) \in \{0,1\}^i$ we have defined points $x_{n_1 ... n_i}$ and closed balls $F_{n_1 ... n_i}$ centered at $x_{n_1 ... n_i}$, with diam$(F_{n_1 ... n_i}) < \frac{1}{2^i}$ and so that

$$\text{for each } (n_1, ..., n_i), \ F_{n_1} \supseteq F_{n_1 n_2} \supseteq ... \supseteq F_{n_1 n_2 ... n_i}$$

Induction step: We must construct the sets for stage $k+1$. For each $(k+1)$-tuple $(n_1, ..., n_k, n_{k+1}) \in \{0,1\}^{k+1}$, we need to define a point $x_{n_1 ... n_k n_{k+1}}$ and a closed ball $F_{n_1 ... n_k n_{k+1}}$ in such a way that the conditions in the induction hypothesis remain true with $k+1$ replacing k.

For any $(n_1, ..., n_k)$: we have $x_{n_1,,, n_k} \in \text{int}\, F_{n_1 ... n_k}$. Since $x_{n_1,,, n_k}$ is not isolated, we can pick distinct points $x_{n_1 ... n_k 0}$ and $x_{n_1 ... n_k 1}$ (both $\neq x_{n_1 ... n_k}$) in $\text{int}\, F_{n_1 ... n_k}$ and use the Lemma 3.5 to pick disjoint closed balls $F_{n_1 ... n_k 0}$ and $F_{n_1 ... n_k 1}$ (centered at $x_{n_1,,, n_k 0}$ and $x_{n_1 ... n_k 1}$) and both $\subseteq$ int $F_{n_1 ... n_k}$. We can then shrink the balls, if necessary, so that each has diameter $< \frac{1}{2^k}$. ●

Corollary 3.7 If (X, d) is a nonempty complete separable metric space with no isolated points, then $|X| = c$.

Proof Theorem II.5.21 (using separability) tells us that $|X| \leq c$; Theorem 3.6 gives us $|X| \geq c$. ●

The following corollary gives us another variation on the basic result.

Corollary 3.8 If (X, d) is an uncountable complete separable metric space, then $|X| = c$.

(*So we might say that "the Continuum Hypothesis holds among complete separable metric spaces."*)

Proof Since (X, d) is separable, Theorem II.5.21 gives us $|X| \leq c$. If we knew that X had no isolated points, then Theorem 3.6 (or Corollary 3.7) would complete the proof. But X might have isolated points, so a little more work is needed.

Call a point $x \in X$ a condensation point if every neighborhood of x is uncountable. Let C be the set of all condensation points in X. Each point $a \in X - C$ has a countable open neighborhood O. Each point of O is also a non-condensation point, so $a \in O \subseteq X - C$. Therefore $X - C$ is open so C is closed, and therefore (C, d) is a complete metric space.

Since (X, d) is separable (and therefore second countable), $X - C$ is also second countable and therefore Lindelöf. Since $X - C$ can be covered by countable open sets, it can be covered by countably many of them, so $X - C$ is countable: therefore $C \neq \emptyset$ (in fact, C must be uncountable).

Finally, (C, d) has no isolated points in C: if $b \in C$ were isolated in C, then there would be an open set O in X with $O \cap C = \{b\}$. Since $O - \{b\} \subseteq X - C$, O would be countable $-$ which is impossible since b is a condensation point.

Therefore Corollary 3.7 therefore applies to (C, d), and therefore $|X| \geq |C| \geq c$. ∙

Why was the idea of a "condensation point" introduced in the example? Will the argument work if, throughout, we replace "condensation point" with "non-isolated point?" If not, precisely where would the proof break down?

The next corollary answers a question we raised earlier: is there a metric d' on $\mathbb{Q}$ which is equivalent to the usual metric d but for which $(\mathbb{Q}, d')$ is complete – that is, is $\mathbb{Q}$ "completely metrizable?"

Corollary 3.9 $\mathbb{Q}$ is not completely metrizable.

Proof Suppose d' is a metric on $\mathbb{Q}$ equivalent to the usual metric d, so that $\mathcal{T}_d = \mathcal{T}_{d'}$. Then $(\mathbb{Q}, d')$ is a nonempty metric space and, since $d \sim d'$, the space $(\mathbb{Q}, d')$ has no isolated points (no set $\{q\}$ is in $\mathcal{T}_d = \mathcal{T}_{d'}$: "isolated point" is a topological notion.). If $(\mathbb{Q}, d')$ were complete, Theorem 3.6 would imply that $\mathbb{Q}$ is uncountable. ∙

Exercises

E1. Prove that in a metric space (X, d), the following are equivalent:

a) every Cauchy sequence is eventually constant
b) (X, d) is complete and $\mathcal{T}_d$ is the discrete topology
c) for every $A \subseteq X$, each Cauchy sequence in A converges to a point in A
(that is, every subspace of (X, d) is complete).

E2. Suppose that X is a dense subspace of the pseudometric space (Y, d) and that every Cauchy sequence in X converges to some point in Y. Prove that (Y, d) is complete.

E3. Suppose that (X, d) is a metric space and that (x_n) is a Cauchy sequence with only finitely many distinct terms. Prove that (x_n) is eventually constant, i.e., that for some $n \in \mathbb{N}$, $x_n = x_{n+1} = \ldots$

E4. Let p be a fixed prime number. On the set $\mathbb{Q}$ of rationals, the p-adic absolute value $|\ |_p$ (or p-adic norm) is defined by:

For $0 \neq x \in \mathbb{Q}$, write $x = \frac{p^k m}{n}$ for integers k, m, n, where p does not divide m or n. (*Of course, k may be negative.*) Define $|x|_p = p^{-k} = \frac{1}{p^k}$. . We define $|0|_p = 0$.

For all $x, y \in \mathbb{Q}$,

a) $|x|_p \geq 0$ and $|x|_p = 0$ iff $x = 0$
b) $|xy|_p = |x|_p \cdot |y|_p$
c) $|x + y|_p \leq |x|_p + |y|_p$
d) $|x + y|_p \leq \max\{|x|_p, |y|_p\} \leq x|_p + |y|_p$

The p-adic metric d_p is defined on $\mathbb{Q}$ by $d_p(x, y) = |x - y|_p$

Prove or disprove that $(\mathbb{Q}, d_p)$ is complete.

E5. Suppose that (X, d) is complete, that $(Y, \mathcal{T})$ is a Hausdorff space and that $f : X \to Y$ is continuous. Suppose that $F_1 \supseteq F_2 \supseteq \ldots \supseteq F_n \supseteq \ldots$ is a sequence of closed sets in X for which $\operatorname{diam}(F_n) \to 0$. Prove that $\bigcap_{n=1}^{\infty} f[F_n] = f[\bigcap_{n=1}^{\infty} F_n]$.

E6. A metric space (X, d) is called locally complete if every point x has a neighborhood N_x

(necessarily closed) which is complete.

a) Give an example of a metric spce (X, d) that is locally complete but not complete.

b) Prove that if (D, d) is a locally complete dense subspace the complete metric space (X, d), then D is open in X.
Hint: it may be helpful to notice if O is open and D is dense, then cl $(O \cap D)$ *= cl(O). This is true in any topological space* $(X, \mathcal{T})$.

E7. Suppose A is an uncountable subset of $\mathbb{R}$ with $|A| = m < c$. (*Of course, there is no such set* A *if the Continuum Hypothesis is assumed.*) Is it possible for A to be closed? Explain.

4. The Contraction Mapping Theorem

Definition 4.1 A point $x \in X$ is called a fixed point for the function $f: X \to X$ if $f(x) = x$. We say that a topological space X has the fixed point property if every continuous $f: X \to X$ has a fixed point.

It is easy to check that the fixed point property is a topological property.

Example 4.2

1) A function $f: \mathbb{R} \to \mathbb{R}$ has a fixed point if and only if the graph of f intersects the line $y = x$.

If $A \subseteq \mathbb{R}$, then A is the set of fixed points for some function: for example $f(x) = x \cdot \chi_A(x)$, where χ_A is the characteristic function of A. *Is every set $A \subseteq \mathbb{R}$ the set of fixed points for some continuous function $f: \mathbb{R} \to \mathbb{R}$? Are there any restrictions on the cardinality of A?*

2) The interval $[a, b]$ has the fixed point property. To see this, suppose that $f: [a, b] \to [a, b]$ is continuous. If $f(a) = a$ or $f(b) = b$, then f has a fixed point so assume that $f(a) > a$ and $f(b) < b$. Define $g(x) = f(x) - x$. Then $g(a) = f(a) - a > 0$, $g(b) = f(b) - b < 0$ and g is continuous. By the Intermediate Value Theorem (from analysis) there is a $c \in (a, b)$ where $g(c) = f(c) - c = 0$, that is, $f(c) = c$.

3) The Brouwer Fixed Point Theorem states that $D^n = \{x \in \mathbb{R}^n : d(x, 0) \leq 1\}$ has the fixed point property. For $n > 1$, the proof of Brouwer's theorem is rather difficult; the usual proofs use techniques from algebraic topology. For $n = 1$, D^n is homeomorphic to $[a, b]$, so Brouwer's Theorem genralizes Part 2.

In fact, Brouwer's Theorem can be generalized as the Schauder Fixed Point Theorem: every nonempty compact convex subset K of a Banach space has the fixed point property.

Definition 4.3 $f: (X, d) \to (X, d)$ is a contraction mapping (for short, a contraction) if there is a constant $\alpha \in (0, 1)$ such that $d(f(x), f(y)) \leq \alpha d(x, y)$ for all $x, y \in X$.

Notice that a contraction f is automatically continuous: for $\epsilon > 0$, choose $\delta = \epsilon$. Then, for all $x \in X$, $d(x, y) < \delta$ implies $d(f(x), f(y)) < \alpha\delta < \epsilon$. In fact, this choice of δ depends only on ϵ and not the point x. In this case we say that f is uniformly continuous.

The definition of uniform continuity for f on X reads:

$$\forall \epsilon \ \exists \delta \ \forall x \ \forall y \ (d(x, y) < \delta \Rightarrow d(f(x), f(y)) < \epsilon)$$

Compare this to the weaker requirement for f to be continuous on X:

$$\forall x \ \forall \epsilon \ \exists \delta \ \forall y \ (d(x, y) < \delta \Rightarrow d(f(x), f(y)) < \epsilon)$$

Example 4.4

1) The function $f : [0, \frac{1}{4}] \to [0, \frac{1}{4}]$ given by $f(x) = x^2$ is a contraction because for any $x, y \in [0, \frac{1}{4}]$, we have $|f(x) - f(y)| = |x^2 - y^2| = |x + y||x - y| \le \frac{1}{2}|x - y|$.

2) But the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is not a contraction. To be a contraction would require some $\alpha \in (0, 1)$ which

$$\begin{aligned} &|x^2 - y^2| \le \alpha|x - y| \text{ holds for all choices of } x, y, \text{ which would imply that} \\ &|x + y| \quad \le \alpha \text{ for all } x \ne y \text{ (which is false).} \end{aligned}$$

We could also argue that f is not a contraction by noticing that $f : \mathbb{R} \to \mathbb{R}$ is <u>not</u> uniformly continuous. For example, if we choose $\epsilon = 2$, then no $\delta > 0$ will satisfy the condition $\forall x \ \ \forall y \ \ (d(x, y) < \delta \Rightarrow d(f(x), f(y)) < 2)$. For example, given $\delta > 0$, we can let $x = \frac{2}{\delta}$ and $y = \frac{2}{\delta} + \frac{\delta}{2}$. Then $|f(x) - f(y)| = 2 + \frac{\delta^2}{4} > 2$.

Theorem 4.5 (The Contraction Mapping Theorem) If (X, d) is a nonempty complete metric space and $f : X \to X$ is a contraction, then f has a unique fixed point. (*"Nonempty" is included in the hypothesis because the empty function $\emptyset$ from the complete space $\emptyset$ to $\emptyset$ is a contraction with no fixed point.*)

A fixed point provided by the Contraction Mapping Theorem can be useful, as we will soon see in Theorem 4.9 (Picard's Theorem) below. The <u>proof</u> of the Contraction Mapping Theorem is also useful because it shows how to make some handy numerical estimates (*see the example following Picard's Theorem.*)

Proof Suppose $0 < \alpha < 1$ and that $d(f(x), f(y)) \le \alpha d(x, y)$ for all $x, y \in X$. Pick <u>any</u> point $x_0 \in X$ and apply the function f repeatedly to define

$$\begin{aligned} &x_1 = f(x_0), \\ &x_2 = f(x_1) = f(f(x_0)) = f^2(x_0) \\ &\quad\vdots \\ &x_n = f(x_{n-1}) = \dots = f^n(x_0) \\ &\quad\vdots \end{aligned}$$

We claim that this sequence (x_n) is Cauchy and that its limit in X is a fixed point for f. (*Here you can imagine a "control system" where the initial input is x_0 and each output becomes the new input in a "feedback loop." This system approaches a "steady state" where "input $=$ output."*)

Suppose $\epsilon > 0$. We want to show that $d(x_n, x_m) < \epsilon$ for large enough m, n. Assume $m > n$. Applying the "contraction property" n times gives:

$$\begin{aligned} d(x_n, x_m) \quad &= d(f^n(x_0), f^m(x_0)) = d(f(f^{n-1}(x_0)), f(f^{m-1}(x_0))) \\ &\le \alpha\, d(f^{n-1}(x_0), f^{m-1}(x_0)) \le \alpha^2\, d(f^{n-2}(x_0), f^{m-2}(x_0)) \\ &\le \ \dots \ \le \ \alpha^n d(x_0, f^{m-n}(x_0)) = \alpha^n\, d(x_0, x_{m-n}) \end{aligned}$$

By the triangle inequality (and the "contraction property," again) we get

$$
\begin{aligned}
d(x_n, x_m) &\leq \alpha^n\, d(x_0, x_{m-n}) \\
&\leq \alpha^n\,(d(x_0,x_1) + d(x_1,x_2) + d(x_2,x_3) + \ldots + d(x_{m-n-1}, x_{m-n})) \\
&\leq \alpha^n\,(d(x_0,x_1) + \alpha d(x_0,x_1) + \alpha^2 d(x_0,x_1) + \ldots + \alpha^{m-n-1} d(x_0,x_1)) \\
&= \alpha^n\, d(x_0,x_1)\,(1+\alpha+\alpha^2+\ldots+\alpha^{m-n-1}) < \alpha^n d(x_0,x_1) \textstyle\sum_{k=0}^{\infty}\alpha^k \\
&= \frac{\alpha^n d(x_0,x_1)}{1-\alpha} \to 0 \text{ as } n \to \infty.
\end{aligned}
$$

So $\frac{\alpha^n d(x_0,x_1)}{1-\alpha} < \epsilon$ if $n >$ some N. Then $d(x_n, x_m) < \epsilon$ if $m > n > N$. Therefore (x_n) is Cauchy, and since (X,d) is complete $(x_n) \to x$ for some $x \in X$.

By continuity, $(f(x_n)) \to f(x)$. But also $(f(x_n)) \to x$ because $f(x_n) = x_{n+1}$. Therefore $f(x) = x$ because a sequence in a metric space has at most one limit. (*If d were just a pseudometric, we could conclude only that* $d(x, f(x)) = 0$.)

If y is also a fixed point for f, then $d(x,y) = d(f(x), f(y)) \leq \alpha\, d(x,y)$. Since $0 < \alpha < 1$, this implies that $d(x,y) = 0$, so $x = y$ – that is, the fixed point x is unique. (*If d were just a pseudometric, f could have several fixed points at distance* 0 *from each other.)* •

Notes about the proof

1) The proof gives us that if $m > n$, then $d(x_n, x_m) < \frac{\alpha^n d(x_0,x_1)}{1-\alpha}$ for each $n \in \mathbb{N}$. If we fix n and let $m \to \infty$, Then $(x_m) \to x$, so $d(x_n, x) \leq \frac{\alpha^n d(x_0,x_1)}{1-\alpha}$. Thus we have a computable bound on how well x_n approximates the fixed point x.

2) For large n we can think of x_n as an "approximate fixed point" – in the sense that f doesn't move the point x_n very much. To be more precise,

$$
\begin{aligned}
d(x_n, f(x_n)) &\leq d(x_n, x) + d(x, f(x_n)) = d(x_n, x) + d(f(x), f(x_n)) \\
&\leq d(x_n, x) + \alpha\, d(x, x_n) = (1+\alpha)\, d(x_n, x) \\
&\leq (1+\alpha)\frac{\alpha^n d(x_0,x_1)}{1-\alpha} \to 0 \text{ as } n \to \infty.
\end{aligned}
$$

Example 4.6 Consider the following functions that map the complete space $\mathbb{R}$ into itself:

1) $f(x) = 2x + 1$: f is (uniformly) continuous, f is not a contraction, and f has a unique fixed point: $x = -1$
2) $g(x) = x^2 + 1$ is not a contraction and has no fixed point.
3) $h(x) = x + \sin x$ has infinitely many fixed points.

We are going to use the Contraction Mapping Theorem to prove a fundamental theorem about the existence and uniqueness of a solution to a certain kind of initial value problem in differential equations. We will use the Contraction Mapping Theorem applied to the complete space in the following example.

Example 4.7 Let X be a topological space and $C^*(X)$ be the set of bounded continuous real-valued functions $f\colon X \to \mathbb{R}$, with the metric $\rho(f,g) = \sup\{|f(x) - g(x)|\colon x \in X\}$. (*Since f, g are bounded, so is $|f - g|$ and therefore this sup is always a real number. It is easy to check that ρ is a metric*.) The metric ρ is called the metric of "uniform convergence" because $\rho(f,g) < \epsilon$ implies that f is "uniformly close" to g :

$$\rho(f,g) < \epsilon \qquad \Rightarrow \quad |f(x) - g(x)| < \epsilon \text{ at every point } x \in X \quad (*)$$

In the specific case of $(C^*(\mathbb{R}), \rho)$, the statement that $(f_n) \xrightarrow{\rho} f$ is equivalent the statement (in analysis) that "(f_n) converges to f uniformly on $\mathbb{R}$."

Theorem 4.8 $(C^*(X), \rho)$ is complete.

Proof Suppose that (f_n) is a Cauchy sequence in $(C^*(X), \rho)$: given $\epsilon > 0$, there is some N so that $\rho(f_n, f_m) < \epsilon$ whenever $n, m > N$. This implies (*see* (*), *above*) that for any fixed $x \in X$, $|f_n(x), f_m(x)| < \epsilon$ whenever $n, m > N$. Therefore $(f_n(x))$ is a Cauchy sequence of real numbers, so $(f_n(x)) \to$ some $r_x \in \mathbb{R}$. We define $f(x) = r_x$. Then $f : X \to \mathbb{R}$. To complete the proof, we claim that $f \in C^*(X)$ and that $(f_n) \xrightarrow{\rho} f$

First we argue that f is bounded. Pick N so that $\rho(f_n, f_m) < 1$ whenever $n, m \geq N$.
Using $m = N$, this gives $\rho(f_n, f_N) < 1$ for $n \geq N$ and therefore $|f_n(x) - f_N(x)| < 1$ for every $x \in X$. If we now let $n \to \infty$, we conclude that

$$\begin{aligned} |f(x) - f_N(x)| \leq 1, &\quad \text{that is} \\ -1 \leq f(x) - f_N(x) \leq 1 &\quad \text{for every } x. \end{aligned}$$

Since f_N is bounded, there is a constant M such that

$$-M \leq f_N(x) \leq M \qquad\qquad \text{for every } x.$$

Adding the last two inequalities gives

$$|f(x)| \leq M + 1 \qquad\qquad \text{for every } x \text{, so } f \text{ is bounded.}$$

We now claim that $(f_n) \xrightarrow{\rho} f$. (*Technically, this is an abuse of notation – because ρ is defined on $C^*(X)$ and we don't yet know that f is in $C^*(X)$! However, the same definition for ρ makes sense on any collection of bounded real-valued functions, continuous or not. We are using this "extended definition" of ρ here.*) Let $\epsilon > 0$ and pick N so that $\rho(f_n, f_m) < \frac{\epsilon}{2}$ whenever $m, n \geq N$. Then $|f_n(x) - f_m(x)| < \frac{\epsilon}{2}$ for every x whenever $n, m \geq N$. Letting $m \to \infty$, we get that $|f_n(x) - f(x)| \leq \frac{\epsilon}{2}$ for all x and all $n \geq N$. Therefore $\rho(f_n, f) < \epsilon$ if $n \geq N$.

Finally, we show that f is continuous at every point $a \in X$ and $\epsilon > 0$. For $x \in X$ and $N \in \mathbb{N}$ we have

$$|f(x) - f(a)| \le |f(x) - f_N(x)| + |f_N(x) - f_N(a)| + |f_N(a) - f(a)|$$

Since $(f_n) \xrightarrow{\rho} f$, we can choose an N large enough that $\rho(f_N, f) < \frac{\epsilon}{3}$. This implies that $|f(x) - f_N(x)| < \frac{\epsilon}{3}$ and $|f_N(a) - f(a)| < \frac{\epsilon}{3}$. Since f_N is continuous at a, we can choose a neighborhood W of a so that $|f_N(x) - f_N(a)| < \frac{\epsilon}{3}$ for all $x \in W$. So, if $x \in W$ then we have

$|f(x) - f(a)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$, so f is continuous at a. •

Theorem 4.9 (Picard's Theorem) Let $f: D \to \mathbb{R}$ be continuous, where D is a closed box in $\mathbb{R}^2$ and $(x_0, y_0) \in \operatorname{int} D$. Suppose that there exists a constant M such that for all (x_1, y_1) and (x_1, y_2) in D

$$|f(x_1, y_1) - f(x_1, y_2)| \le M|y_1 - y_2| \qquad \text{(L)}$$

Then

i) there exists an interval $I = [x_0 - a, x_0 + a] = \{x : |x - x_0| \le a\}$, and
ii) there exists a unique differentiable function $g: I \to \mathbb{R}$

such that $\begin{cases} y_0 = g(x_0) \\ g'(x) = f(x, g(x)) \end{cases}$ for $x \in I$

In other words, on some interval I centered at x_0, there is a unique solution $y = g(x)$ to the initial value problem

$$\begin{cases} y' = f(x, y) \\ y(x_0) = y_0 \end{cases} \qquad (*)$$

Before beginning the proof, we want to make some comments about the hypotheses.

1) Condition (L) says that "f satisfies a <u>Lipschitz condition in the variable y on D.</u>" This may seem a bit obscure. For our purposes, it's enough to notice that (L) is true if the partial derivative f_y exists and is continuous on the box D. In that case, we know (from analysis) that $|f_y| \le$ some constant M on D. Thinking of $f(x_1, y)$ as a single-variable function of y and using the ordinary Mean Value Theorem gives us a point z between y_1 and y_2 for which

$$|f(x_1, y_1) - f(x_1, y_2)| = |f_y(x_1, z)| \cdot |y_1 - y_2| \le M|y_1 - y_2|$$

2) As a specific example, suppose $f(x, y) = y - x$, that $D = [-1, 1] \times [-1, 1]$, and that $(x_0, y_0) = (0, 0)$. Since $f_y = 1$, the preceding comment tells us that condition (L) is true.

Consider the initial value problem $\begin{cases} y' = y - x \\ y(0) = 0 \end{cases}$. Picard's Theorem states that there is a unique differentiable function $y = g(x)$, defined on some interval $[-a, a]$ that satisfies this system: $g(0) = 0$ and $g'(x) = f(x, g(x))$ (that is, $y' = y - x$). Moreover, the proofs of Picard's Theorem and the Contraction Mapping Theorem can actually help us to actually <u>find</u> this solution. Of course, once a solution is actually found, a direct check might show that the solution is actually valid on an interval larger than the interval I that comes up in the proof.

Proof (Picard's Theorem) We begin by transforming the initial value problem $(*)$ into an equivalent problem that involves an integral equation $(**)$ rather than a differential equation.
Let $y = g(x)$ be a continuous function defined on an interval $I = [x_0 - a, x_0 + a]$:

Suppose g satisfies (*): f is continuous, so $g'(x) = f(x, g(x))$ is continuous. (*Why?*) Therefore the Fundamental Theorem of Calculus tells us for all $x \in I$,

$$g(x) - g(x_0) = \int_{x_0}^{x} g'(t)\,dt = \int_{x_0}^{x} f(t, g(t))\,dt, \text{ so}$$

$$g(x) = y_0 + \int_{x_0}^{x} f(t, g(t))\,dt \qquad (**)$$

Suppose g satisfies (**): then $g(x_0) = y_0 + \int_{x_0}^{x_0} f(t, g(t))\,dt = y_0$. Since $f(t, g(t))$ is continuous, the Fundamental Theorem gives that for all $x \in I$, $g'(x) = f(x, g(x))$. Therefore the function $g(x)$ satisfies (*).

Therefore a function continuous g on I satisfies (*) iff it satisfies (**). (*Note that the initial condition* $y_0 = g(x_0)$ *is "built into" the single equation* $(**)$.) We will find a g that satisfies (**).

Now we establish some notation:

We are given a constant M in the Lipschitz condition.

Pick a constant K so that $|f(x, y)| \leq K$ for all $(x, y) \in D$. (*We will prove later in this chapter that a continuous function on a closed box in* $\mathbb{R}^2$ *must be bounded. For now, we assume that as a fact from analysis.)*

Because $(x_0, y_0) \in \operatorname{int} D$, we can pick a constant $a > 0$ so that

i) $\{x : |x - x_0| \leq a\} \times \{y : |y - y_0| \leq Ka\} \subseteq D$, and
ii) $aM < 1$

Let $I = \{x: |x - x_0| \leq a\} = [x_0 - a, x_0 + a]$.

We consider $(C^*(I), \rho)$, where ρ is the metric of uniform convergence. By Theorem 4.8, this space is complete. Let $B = \{g \in C^*(I) : \forall t \in I,\ |g(t) - y_0| \leq Ka\}$.

For an arbitrary $g \in C^*(I)$, $g(t)$ *might be so large that* $(t, g(t)) \notin D$. *If we look only at functions in* B, *we avoid this problem: if* $g \in B$ *and* $t \in I$, *then* $(t, g(t)) \in D$ *because of how we chose* a. *So for* $g \in B$, *we know that* $f(t, g(t))$ *is defined.*)

Notice that $B \neq \emptyset$ since the constant function $g(x) \equiv y_0$ is certainly in B.

B is closed in $C^*(I)$: if $h \in \operatorname{cl} B$, then there is sequence of functions (g_n) in B such that $(g_n) \to h$ in the metric ρ. Therefore $g_n(x) \to h(x)$ for all $x \in I$, and subtracting y_0 we get

$$|g_n(x) - y_0| \to |h(x) - y_0|.$$

Since $|g_n(x) - y_0| \le Ka$ for each $x \in I$, then $|h(x) - y_0| \le Ka$ for each $x \in I$. Therefore $h \in B$, so B is closed. Therefore (B, ρ) is a nonempty complete metric space.

For $g \in B$, define a function $h = T(g)$ by

$$h(x) = T(g)(x) = y_0 + \int_{x_0}^{x} f(t, g(t))\, dt.$$

Notice that h is continuous (in fact, differentiable) and that, for every $x \in I$,

$$|h(x)| \le |y_0| + |\int_{x_0}^{x} f(t, g(t))\, dt\,| \le |\,y_0| + \int_{x_0}^{x} |f(t, g(t))|\, dt \le |\,y_0| + \int_{x_0}^{x} K\, dt \le |y_0| + Ka,$$

so h is bounded. Therefore $T : B \to C^*(I)$. But in fact, even more is true: $T\colon B \to B$, because

$$\forall x \in I,\ |T(g)(x) - y_0| = |\,h(x) - y_0| = |\int_{x_0}^{x} f(t, g(t))\, dt\,| \le Ka$$

We now claim that $T : (B, \rho) \to (B, \rho)$ is a contraction. To see this, we simply compute distances: if g_1 and $g_2 \in B$, then

$$\begin{aligned}
\rho(T(g_1), T(g_2)| &= \sup\{\,|\,T(g_1)(x) - T(g_2)(x)|\ : x \in I\} \\
&= \sup\{\,|\int_{x_0}^{x} f(t, g_1(t)) - f(t, g_2(t))\, dt|\ : x \in I\} \\
&\le \sup\{\,|\int_{x_0}^{x} |f(t, g_1(t)) - f(t, g_2(t))|\ dt| : x \in I\} \\
&\le \sup\{\,|\int_{x_0}^{x} M\,|g_1(t) - g_2(t)|\, dt\,| : x \in I\} \qquad (\textit{from Condition } L) \\
&\le \sup\{\,|\int_{x_0}^{x} M\rho(g_1, g_2)\, dt\,| : x \in I\} \\
&= \sup\{\,M\rho(g_1, g_2)\,|\,x - x_0| : x \in I\} \\
&= aM\rho(g_1, g_2) \\
&= \alpha\rho(g_1, g_2),\ \text{where } \alpha = aM < 1.
\end{aligned}$$

Then the Contraction Mapping Theorem gives us a unique function $g \in B$ for which $g = T(g)$. From the definition of T, that simply means:

$$\forall x \in I,\ \ g(x) = T(g)(x) = y_0 + \int_{x_0}^{x} f(t, g(t))dt$$

which is precisely condition (**). •

Example 4.10 The method in the proof of Picard's Theorem and numerical estimates in the proof of the Contraction Mapping Theorem can be used to get useful information about a specific initial value problem. To illustrate, we consider

$$\begin{cases} y' = y - x \\ y(0) = 0 \end{cases}$$

and find a solution that is valid on some interval containing 0.

We begin by choosing a box D with $(x_0, y_0) = (0, 0)$ in its interior. Rather arbitrarily, we select the box $D = [-1, 1] \times [-1, 1]$. Since $|f(x, y)| = |y - x| \le |x| + |y| \le 2$ on D, we can use $K = 2$ in the proof.

Because $|f_y(x, y)| = 1$ throughout D, the Lipschitz condition (L) is satisfied with $M = 1$.

Following the proof of Picard's Theorem, we now choose a constant a so that

i) $\{x : |x| \le a\} \times \{y : |y| \le Ka = 2a\} \subseteq D$ and
ii) $aM = a \cdot 1 < 1$

Again rather arbitrarily, we choose $a = \frac{1}{2}$, so that $I = [x_0 - a, x_0 + a] = [-\frac{1}{2}, \frac{1}{2}]$.
Then $B = \{g \in C^*(I) : |g(x) - 0| \le Ka\} = \{g \in C^*(I) : |g(x)| \le 1$ for all $x \in [-\frac{1}{2}, \frac{1}{2}]\}$.

Finally, we choose any function $g_0 \in B$; to make things as simple as possible, we might as well choose g_0 to be the constant function $g_0(x) = 0$ on $[-\frac{1}{2}, \frac{1}{2}]$.

According to the proof of the Contraction Mapping Theorem, the sequence of functions $(g_n) = (T^n(g_0)) \xrightarrow{\rho} g$, where g is a fixed point for T, and g is the solution to our initial value problem. We calculate:

$$g_1(x) = y_0 + \int_{x_0}^{x} f(t, g_0(t))\,dt = 0 + \int_0^x f(t, 0)\,dt = \int_0^x -t\,dt = -\frac{x^2}{2}$$

$$g_2(x) = T(g_1(x)) = 0 + \int_0^x f(t, -\frac{t^2}{2})\,dt = \int_0^x -\frac{t^2}{2} - t\,dt = -\frac{x^3}{6} - \frac{x^2}{2}$$

$$g_3(x) = T(g_2(x)) = \int_0^x f(t, -\frac{t^3}{6} - \frac{t^2}{2})\,dt = \int_0^x -\frac{t^3}{6} - \frac{t^2}{2} - t\,dt = -\frac{x^4}{24} - \frac{x^3}{6} - \frac{x^2}{2}$$

and, in general,

$$g_n(x) = T(g_{n-1}(x)) = \dots = -\frac{x^2}{2!} - \frac{x^3}{3!} - \dots - \frac{x^{n+1}}{(n+1)!}$$

The functions g_n converge (uniformly) to the solution g.

In this particular problem, we are lucky enough to recognize that the functions $g_n(x)$ are just the partial sums of the series $\sum_{n=2}^{\infty} -\frac{x^n}{n!} = 1 + x - \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x - e^x$. Therefore $g(x) = 1 + x - e^x$ is a solution of our initial value problem, and we know it is valid for all $x \in I = [-\frac{1}{2}, \frac{1}{2}]$. (*You can check by substitution that the solution is correct – and that it actually is a solution that works for all* $x \in \mathbb{R}$.)

Even if we couldn't recognize a neat formula for the limit $g(x)$, we could still make some useful approximations. From the proof of the Contraction Mapping Theorem, we know that $\rho(g_n, g) \le \frac{\alpha^n \rho(g_0, g_1)}{1 - \alpha}$. In this example $\alpha = aM = \frac{1}{2}$, so that $\rho(g_n, g) \le \frac{\frac{1}{2^n}\rho(g_0, g_1)}{1 - \frac{1}{2}}$
$= \frac{1}{2^{n-1}} \sup\{|0 - (-\frac{x^2}{2})| : x \in I\} = \frac{1}{2^{n-1}} \cdot \frac{1}{2^3} = \frac{1}{2^{n+2}}$. Therefore $g_n(x)$ is uniformly within distance $\frac{1}{2^{n+2}}$ of $g(x)$ on the interval $[-\frac{1}{2}, \frac{1}{2}]$.

Finally, recall that our initial choice of $g_0 \in B$ was arbitrary. Since $|\sin x| \le 1$, we know that $\sin \in B$, and we could just as well have chosen $g_0(x) = \sin x$. Then the functions $g_n(x)$ computed as $T(g_0) = g_1$, $T(g_2) = g_2$, ... would be quite different (*try computing* g_1 *and* g_2) but it would still be true that $g_n(x) \to g(x) = 1 + x - e^x$ uniformly on I: the same limit g because the solution g is unique. •

The Contraction Mapping Theorem can be used to prove other results – for example, the Implicit Function Theorem. (*You can see details, for example, in* Topology*, by James Dugundji.*)

Exercises

E8. $a)$ Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable and that there is a constant $K < 1$ such that $|f\,'(x)| \leq K$ for all x. Prove that f is a contraction (and therefore has a unique fixed point.)

b) Give an example of a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that

$$|\, f(x) - f(y)\,| < |x - y\,|$$

for all $x \neq y \in \mathbb{R}$ but such that f has no fixed point.

Note: The function f is not a contraction mapping. If we allow $\alpha = 1$ in the definition of contraction, then the Contraction Mapping Theorem is not be true – not even if we "compensate" by using " $<$ " instead of " $\leq$ " in the definition.

E9. Let $f\colon (X, d) \to (X, d)$, where (X, d) is a nonempty complete metric space. Let f^k denote the "k^{th} iterate of f" – that is, f composed with itself k times.

a) Suppose that $\exists k \in \mathbb{N}$ for which f^k is a contraction. Then, by the Contraction Mapping Theorem, f^k has a unique fixed point p. Prove that p is also the unique fixed point for f.

b) Prove that the function $\cos : \mathbb{R} \to \mathbb{R}$ is not a contraction.

c) Prove that $\cos^k$ is a contraction for some $k \in \mathbb{N}$.
(*Hint: the Mean Value Theorem may be helpful.*)

d) Let $k \in \mathbb{N}$ be such that $g = \cos^k$ is a contraction and let p be the unique fixed point of g. By a), p is also the unique solution of the equation $\cos x = x$. Start with 0 as a "first approximation" for p and use the technique in the proof of the Contraction Mapping Theorem to find an $n \in \mathbb{N}$ so that $|\,g^n(0) - p\,| < 0.00001$.

e) For this n, use a computer or calculator to evaluate $g^n(0)$. (*This "solves" the equation $\cos x = x$ with $|\,Error\,| < 0.00001$.*)

E10. Consider the differential equation $y' = x + y$ with the initial condition $y(0) = 1$. Choose a suitable rectangle D and suitable constants K, M and a as in the proof of Picard's Theorem. Use the technique in the proof of the contraction mapping theorem to find a solution for the initial value problem. Identify the interval I in the proof. Is the solution you found actually valid on an interval larger than I ?

5. Completions

The set of rationals $\mathbb{Q}$ (with the usual metric d) is not complete. However, the rationals are a dense subspace of the complete space $(\mathbb{R}, d)$. This is a model for the definition of a "completion" for a metric space.

Definition 5.1 $(\widetilde{X}, \widetilde{d}\,)$ is called a completion of (X, d) if: $\widetilde{d}\,|(X \times X) = d$, X is a dense subspace of $\widetilde{X}$, and $(\widetilde{X}, \widetilde{d}\,)$ is complete.

Loosely speaking, a completion of (X, d) adds the additional points needed (and no others) to provide limits for the Cauchy sequences that fail to converge in (X, d). In the case of $(\mathbb{Q}, d)$, we can think of the "additional points" as the irrational numbers, and the resulting completion is $(\mathbb{R}, d)$.

If a metric space (X, d) is complete, what would a completion look like? Since (X, d) is a complete subspace of $(\widetilde{X}, \widetilde{d}\,)$, X must be closed in $\widetilde{X}$. But X must also be dense. So $X = \widetilde{X}$ – that is, a complete metric space is its own completion. (*If* (X, d) *is a* *<u>complete pseudometric</u> space, then* X *might not be closed in* $\widetilde{X}$; *but each* $y \in \widetilde{X} - X$ *is at distance* 0 *from some* $x \in X$, *so if* $(x_n) \to y \in Y$, *then* (x_n) *already converges to* $x \in X$ *and "adding the point* y*" was unnecessary.*

Of course different spaces may have the same completion – for example, $(\mathbb{R}, d)$ is a completion for both $(\mathbb{Q}, d)$ and $(\mathbb{P}, d)$.

Notice that a completion of (X, d) depends on d, <u>not</u> just the topology $\mathcal{T}_d$. For example, $(\mathbb{N}, d)$ and $(\{\frac{1}{n} : n \in \mathbb{N}\}, d)$ are homeomorphic topological spaces (both have the discrete topology). But the completion of $(\mathbb{N}, d)$ (itself!) is <u>not</u> homeomorphic to $(\{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}, d)$, which is a completion of $(\{\frac{1}{n} : n \in \mathbb{N}\}, d)$.

Recall that if there is an (<u>onto</u>) isometry f between two metric spaces (X, d) and (Y, s), then (X, d) and (Y, s) can be regarded as "the same metric space": we can think of f as just "assigning new names" to the points of X. If $f : (X, d) \to (Y, s)$ is an isometry from X <u>into</u> Y, then we can <u>identify</u> (X, d) with $(f[X], s)$ – an "exact metric copy" of (X, d) inside (Y, s). So if (Y, s) happens to be complete, then $(f[X], s)$ is dense in the complete space $(\text{cl}_Y f[X], s)$. If we identify $(f[X], s)$ with (X, d), then we can call $(\text{cl}_Y f[X], s)$ a completion of (X, d) even though X is not literally a subset of $\text{cl}_Y f[X]$. Therefore, to find a completion of (X, d), it is <u>sufficient</u> to find an isometry f from (X, d) <u>into</u> some complete space (Y, d').

Example 5.2 Consider $\mathbb{N}$ with the metric $d'(n, m) = |\frac{1}{n} - \frac{1}{m}|$. $(\mathbb{N}, d')$ is isometric to $(\{\frac{1}{n} : n \in \mathbb{N}\}, d)$ where d is the usual metric on $\mathbb{N}$ ($f(n) = \frac{1}{n}$ *is an isometry*). These two spaces are "metrically identical.". Therefore $(\{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}, d)$ contains a dense isometric copy of $(\mathbb{N}, d')$. The closure of that "copy" is $\{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$, and we can view $(\{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}, d)$ as a completion of $(\mathbb{N}, d')$ – even though $\mathbb{N}$ is not literally a subspace of $\{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$.

The next theorem tells us every metric space has a completion and, as we will see later, it is "essentially" unique.

Theorem 5.3 Every metric space (X,d) has a completion.

Proof Our previous comments show that it is sufficient to produce an isometry of (X,d) into some complete metric space: the complete space we will use is $(C^*(X),\rho)$ where, as usual, ρ is the metric of uniform convergence.

The theorem is trivial if $X=\emptyset$, so we assume that we can pick a point $p\in X$. For any point $a\in X$, define $\phi_a : X\to\mathbb{R}$ by $\phi_a(x)=d(x,a)-d(x,p)$. The map ϕ_a is a difference of continuous functions so ϕ_a is continuous. For each $x\in X$, $|\phi_a(x)|=|d(x,a)-d(x,p)|\le d(a,p)$, so $\phi_a$is bounded. Therefore $\phi_a\in C^*(X)$.

Define $\Phi : (X,d)\to(C^*(X),\rho)$ by $\Phi(a)=\phi_a$. We complete the proof by showing that Φ is an isometry. This just involves computing some distances: for any $a,b\in X$,

$$\begin{aligned}\rho(\phi_a,\phi_b) &= \sup\{|\phi_a(x)-\phi_b(x)| : x\in X\}\\ &= \sup\{|(d(x,a)-d(x,p))-(d(x,b)-d(x,p))| : x\in X\}\\ &= \sup\{|d(x,a)-d(x,b)| : x\in X\}.\end{aligned}$$

For any x, $|d(x,a)-d(x,b)|\le d(a,b)$.
And letting $x=b$ gives $|d(x,a)-d(x,b)|=d(a,b)$. Therefore

$$\rho(\phi_a,\phi_b)=\sup\{|d(x,a)-d(x,b)| : x\in X\}=d(a,b). \quad \bullet$$

The completion of (X,d) given in this proof is ($\mathrm{cl}_{C^*(X)}\Phi[X]$,$\rho$). The proof is "slick" but it seems to lack any intuitive content. For example, if we applied the proof to $(\mathbb{Q},d)$, it would not at all clear that the resulting completion is isometric to $(\mathbb{R},d)$ (as we would expect). In fact, there is another more intuitive way to construct a completion of (X,d), but verifying all the details is much more tedious. We will simply sketch this alternate construction here.

> Call two Cauchy sequences (x_n) and (y_n) in (X,d) <u>equivalent</u> if $d(x_n,y_n)\to 0$. It is easy to check that $\sim$ is an equivalence relation among Cauchy sequences in (X,d). Clearly, if $(x_n)\to z\in X$, then any equivalent Cauchy sequence also converges to z. If two nonconvergent Cauchy sequences equivalent, then they are "trying to converge to the same point" but the necessary point is "missing" in X.
>
> We denote the equivalence class of a Cauchy sequence (x_n) by $[(x_n)]$ and let $\widetilde{X}$ be the set of equivalence classes. Define the distance $\widetilde{d}$ between two equivalence classes by $\widetilde{d}\,([(x_n)],[(y_n)])=\lim_{n\to\infty}d(x_n,y_n)$. (*Why must this limit exist?*) It is easy to check that $\widetilde{d}$ does not depend on the choice of representative sequences from the equivalence classes, and that $\widetilde{d}$ is a metric on $\widetilde{X}$. One then checks (this is only "tricky" part) that $(\widetilde{X}\,,\,\widetilde{d}\,)$ is complete: any $\widetilde{d}$ -Cauchy sequence of equivalence classes $(\,[(x_n)]\,)$ must converge to an equivalence class in $(\widetilde{X}\,,\,\widetilde{d}\,)$.
>
> For each $x\in X$, the sequence $(x,x,x,...)$ is Cauchy, and so $[(x,x,x,...)]\in\widetilde{X}$.

The mapping $f : (X, d) \to (\widetilde{X}, \widetilde{d})$ given by $f(x) = [(x, x, x, ...)]$ is an isometry, and it is easy to check that $f[X]$ is dense in $(\widetilde{X}, \widetilde{d})$ – so that $(\widetilde{X}, \widetilde{d})$ is a completion for (X, d).

This method is one of the standard ways to construct the real numbers from the rationals: $\mathbb{R}$ is defined as the set of these equivalence classes of Cauchy sequences of rational numbers.
The real numbers can also be constructed as a completion for the rationals by using a method called "Dedekind cuts." However that approach also uses the <u>*order*</u> *structure in* $\mathbb{Q}$ *and therefore we cannot mimic it in the general setting of metric spaces.*

The next theorem gives us the good news that, in the end, it doesn't really matter <u>how</u> we construct a completion – because the completion of (X, d) is "essentially" unique: all completions are isometric (and, in a "special" way!). We state the theorem for metric spaces. (*Are there modifications of the statement and proof to handle the case where d is merely a pseudometric?*)

Theorem 5.4 The completion of (X, d) is unique in the following sense: if (Y, s) and (Z, t) are both complete metric spaces containing X as a dense subspace, then there is an (onto) isometry $f : (Y, s) \to (Z, t)$ such that $f|X$ is the identity map on X. (*In other words, not only are two completions of (X, d) isometric, but there is an isometry between them that holds X fixed. The isometry merely "renames" the new points in the "outgrowth" $Y - X$.*)

Proof Suppose $y \in Y$. Since X is dense in Y, we can pick a sequence (x_n) in X which converges to y. Since (x_n) is convergent, it is Cauchy in (X, d), and since $t|X \times X = d$, (x_n) is also a Cauchy sequence in the complete space (Z, t). Therefore $(x_n) \to$ some point $z \in Z$. We define $f(y) = z$. Then $f : (Y, s) \to (Z, t)$. (*It is easy to check that f is well-defined – that is, we get the same z no matter which of the possibly many sequences (x_n) we first choose converging to y.*)

In particular, if $x \in X$, we can choose (x_n) to be the constant sequence $x_n = x$; then $f(x) = x$, so that $f|X$ is the identity map on X.

We need to verify that f is an isometry, Suppose $y' \in Y$ and we choose $(x'_n) \to y'$. We then have that

$$d(x_n, x'_n) = s(x_n, x'_n) \to s(y, y') \text{ and}$$
$$d(x_n, x'_n) = t(x_n, x'_n) \to t(z, z') = t(f(y), f(y')), \text{ so}$$
$$s(y, y') = t(f(y), f(y')) \quad \bullet$$

According to Theorem 5.4, the space $(\mathbb{R}, d)$ – <u>no matter how we construct it</u> – is <u>the</u> completion of the space $(\mathbb{Q}, d)$.

6. Category

In mathematics there are many different ways to compare the "size" of sets. These different methods are useful for different purposes. One of the simplest ways is to say that one set is "bigger" if it has "more points" than another set $-$ that is, by comparing their cardinal numbers.

In a totally different spirit, we might call one subset of $\mathbb{R}^2$ "bigger" than another if it has a larger area. In analysis, there is a generalization of area. A certain collection $\mathcal{M}$ of subsets of $\mathbb{R}^2$ contains sets that are called measurable, and for each set $S \in \mathcal{M}$ a nonnegative real number $\mu(S)$ is assigned. $\mu(S)$ is called the "measure of S" and we can think of $\mu(S)$ as a kind of "generalized area." A set with a larger measure is "bigger."

In this section, we will look at a third completely different and useful topological idea for comparing the "size" of certain sets. The most interesting results in this section are be about complete metric spaces, but the basic definitions make sense in any topological space $(X, \mathcal{T})$.

Definition 6.1 A subset A of the topological space $(X, \mathcal{T})$ is called nowhere dense in X if $\text{int}_X(\text{cl}_X\, A) = \emptyset$. (*In some books, a nowhere dense set is called rare.*)

A set has empty interior iff its complement is dense. Therefore A is nowhere dense in X iff $\text{int}_X(\text{cl}_X A) = \emptyset$ iff $X - \text{cl}_X A$ is dense.

Intuitively, we can think of an open set O as including some "elbow room" around each of its points $-$ if $x \in O$, then all sufficiently nearby points are also in O. Then we think of a nowhere dense set as being "skinny" $-$ so skinny that not only does it contain no "elbow room" around any of its points, but even its closure contains no "elbow room" around any of its points.

Example 6.2

1) A closed set F is nowhere dense in X iff $\text{int}\, F = \emptyset$ iff $X - F$ is dense in X. In particular, if a singleton set $\{p\}$ is a closed set in X, then $\{p\}$ is nowhere dense unless p is isolated in X.

2) Suppose $A \subseteq \mathbb{R}$. A is nowhere dense iff cl A contains no interval of positive length. For example, each singleton $\{r\}$ is nowhere dense in $\mathbb{R}$. In particular, for $n \in \mathbb{N}$, $\{n\}$ is nowhere dense in $\mathbb{R}$. But notice that $\{n\}$ is not nowhere dense in $\mathbb{N}$, since n is isolated in $\mathbb{N}$. The property of "nowhere dense" is relative to the space in which a set "lives."

If $B \subseteq A \subseteq X$, then $\text{int}_X(\text{cl}_X B)) \subseteq \text{int}_X(\text{cl}_X A))$. Therefore if A is nowhere dense in X, then B is also nowhere dense in X. However, the set B might not be nowhere dense in A. For example, consider $\{1\} \subseteq \mathbb{N} \subseteq \mathbb{R}$.

$\{\frac{1}{n} : n \in \mathbb{N}\}$ is nowhere dense in $\mathbb{R}$.

$\mathbb{Q}$ and $\mathbb{P}$ are not nowhere dense in $\mathbb{R}$ (*the awkward "double negative" in English is one reason why some authors prefer to use the term "rare" for "nowhere dense."*

4) Since $\text{cl}_X\, A = \text{cl}_X(\text{cl}_X A)$, the set on the left side has empty interior iff the set on the right side has empty interior $-$ that is, A is nowhere dense in X iff $\text{cl}_X\, A$ is nowhere dense in X.

Theorem 6.3 Suppose $B \subseteq A \subseteq (X, \mathcal{T})$. If B is nowhere dense in A, then B is nowhere dense in X.

Proof If not, then there is a point $x \in \text{int}_X(\text{cl}_X B) \subseteq \text{cl}_X\, B$. Since $\text{int}_X(\text{cl}_X B)$ is an open set containing x and $x \in \text{cl}_X B$, then $\emptyset \neq \text{int}_X(\text{cl}_X B) \cap B$.

So $\emptyset \neq \text{int}_X(\text{cl}_X B) \cap B \subseteq \text{int}_X(\text{cl}_X B) \cap A \subseteq \text{cl}_X B \cap A = \text{cl}_A B$.

Since $\text{int}_X(\text{cl}_X B) \cap A$ is a nonempty open set in A, we conclude that $\text{int}_A(\text{cl}_A B) \neq \emptyset$, which contradicts the hypothesis that B is nowhere dense in A. $\bullet$

The following technical results are sometimes useful for handling manipulations involving open sets and dense sets.

Lemma 6.4 In any space $(X, \mathcal{T})$

1) If O open in X and D is dense in X, then $\text{cl}\, O = \text{cl}\,(O \cap D)$.
2) If O is open and D is dense in X, then $O \cap D$ is dense in O
3) If O is open and dense, and D is dense in X, then $O \cap D$ is dense in X. In particular, the intersection of two $-$ and therefore finitely many $-$ dense open sets is dense. (*This result is* <u>*not*</u> *true for countable intersections; can you provide an example?*)

Proof 1) We need to show that $\text{cl}\, O \subseteq \text{cl}\,(O \cap D)$. Suppose $x \in \text{cl}\, O$ and let U be an open set containing x. Then $U \cap O \neq \emptyset$, so, since D is dense, we must have $\emptyset \neq (U \cap O) \cap D = U \cap (O \cap D)$. Therefore $x \in \text{cl}\,(O \cap D)$.

2) Using part 1), we have $\text{cl}_O(O \cap D) = \text{cl}_X\,(O \cap D) \cap O = (\text{cl}_X\, O) \cap O = \text{cl}_O O = O$.

3) If O is dense, then part i) gives $\text{cl}(O \cap D) = \text{cl}\, O = X$.

Theorem 6.5 A <u>finite</u> union of nowhere dense sets in $(X, \mathcal{T})$ is nowhere dense in X.

Proof We will prove the result for the union of two nowhere dense sets. The general case follows using a simple induction. If A_1 and A_2 are nowhere dense in X, then

$$X - \text{cl}\,(A_1 \cup A_2) = X - (\text{cl}\, A_1 \cup \text{cl}\, A_2) = (X - \text{cl}\, A_1) \cap (X - \text{cl}\,_2).$$

Since the last two sets are open and dense, Lemma 6.4(3) gives that $X - \text{cl}\,(A_1 \cup A_2)$ is dense. Therefore $A_1 \cup A_2$ is nowhere dense. $\bullet$

Theorem 6.5 is <u>false for infinite unions</u>: for each $q \in \mathbb{Q}$, $\{q\}$ is nowhere dense in $\mathbb{R}$, but $\bigcup_{q \in \mathbb{Q}} \{q\} = \mathbb{Q}$ is not nowhere dense in $\mathbb{R}$.

Definition 6.6 Suppose $A \subseteq (X\,, \mathcal{T})$. A is called a <u>first category set in X</u> if A can be written as a countable union of sets that are nowhere dense in X. If A is not first category in X, we say that A is <u>second</u> <u>category</u> <u>in</u> <u>X</u>. (*In some books, a first category set is called* <u>*meager*</u>.)

If we think of a nowhere dense set in X as "skinny," then a first category set is a bit larger $-$ merely "thin."

Example 6.7

1) If A is nowhere dense in X, then A is first category in X.

2) $\mathbb{Q}$ is a first category set in $\mathbb{R}$.

3) If $B \subseteq A \subseteq X$ and A is first category in X, then B is first category in X. However B may not be first category in A, as the example $\{0\} \subseteq \{0, 1\} \subseteq \mathbb{R}$ shows. However, using Theorem 6.3, we can easily prove that if B is first category in A, then B is also first category in X.

4) A countable union of first category sets in X is first category in X.

5) $A = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ is second category in A $-$ since any subset of A containing 1 is not nowhere dense in A. But A is first category (in fact, nowhere dense) in $\mathbb{R}$.

6) Cardinality and category are totally independent ways to talk about the "size" of a set.

a) Consider the right-ray topology $\mathcal{T} = \{\emptyset, \mathbb{R}\} \cup \{(a, \infty) : a \in \mathbb{R}\}$ on $\mathbb{R}$. $\mathbb{R}$ is uncountable, but $\mathbb{R} = \bigcup_{n=1}^{\infty} F_n$, where $F_n = (-\infty, n]$. Each F_n is nowhere dense in $(\mathbb{R}, \mathcal{T})$, so $\mathbb{R}$ is first category in $(\mathbb{R}, \mathcal{T})$.

b) On the other hand, $\mathbb{N}$ is countable but $\mathbb{N}$ (with the usual topology) is second category in $\mathbb{N}$.

7) For $n > 1$, a straight line is nowhere dense in $\mathbb{R}^n$, so a countable union of straight lines in $\mathbb{R}^n$ is first category in $\mathbb{R}^n$. (*Aside: it follows from the Baire Category Theorem, below, that the union of countably many straight lines cannot equal $\mathbb{R}^n$. But nothing that fancy is needed: can you give an argument that is based just on countable and uncountable sets to show why a countable union of lines cannot $= \mathbb{R}^n$? Can you extend your argument to show why $\mathbb{R}^3$ is not a countable union of planes?*)

More generally, if $k < n$, a countable union of k-dimensional linear subspaces of $\mathbb{R}^n$ is first category in $\mathbb{R}^n$. (*Can a countable union of k-dimensional linear subspaces $= \mathbb{R}^n$?*)

It is not always easy to see what the "category" of a set is. Is $\mathbb{P}$ a first or second category set in $\mathbb{R}$? For that matter, is $\mathbb{R}$ first or second category in $\mathbb{R}$? (*If you know the answer to either of these questions, you also know the answer to the other: why?*)

As we observed in Lemma 6.4, the intersection of two (and therefore, finitely many) dense open sets is dense. However, in certain cases, the intersection of a countable collection of dense open sets is dense. The following theorem discusses this condition and leads us to a definition.

Theorem 6.8 The following are equivalent in any space $(X, \mathcal{T})$:

1) If A is first category in X, then $X - A$ is dense in X
2) If $G_1, G_2, \ldots, G_k, \ldots$ is a sequence of dense open sets, then $\bigcap_{k=1}^{\infty} G_k$ is dense.

Proof $(1 \Rightarrow 2)$ Let $G_1, G_2, ..., G_k, ...$ be a sequence of dense open sets. Each $X - G_k$ is closed and nowhere dense, so $\bigcup_{k=1}^{\infty}(X - G_k)$ is first category. By 1), $X - \bigcup_{k=1}^{\infty}(X - G_k) = \bigcap_{k=1}^{\infty} G_k$ is dense.

$(2 \Rightarrow 1)$ Let A be a first category set, say $A = \bigcup_{n=1}^{\infty} N_k$, where each N_k is nowhere dense. Then each $X - \mathrm{cl}\, N_k = G_k$ is dense so, by 2), $\bigcap_{k=1}^{\infty}(X - \mathrm{cl}\, N_k)$ is dense. Since $\bigcap_{k=1}^{\infty}(X - \mathrm{cl}\, N_k) = X - \bigcup_{k=1}^{\infty} \mathrm{cl}\, N_k \subseteq X - \bigcup_{k=1}^{\infty} N_k = X - A$, we see that $X - A$ is dense. •

Definition 6.9 A space $(X, \mathcal{T})$ is called a Baire space if one (therefore both) of the conditions in Theorem 6.8 hold.

Intuitively we can think of a Baire space as one which is "thick" – a "thin" first category set A can't "fill up" X. In fact $X - A$ is not merely nonempty but actually dense in X. In fact, this is often the way a Baire space is used in "applications": if you want to prove that there is an element in a Baire space having a certain property P, you can consider $A = \{x \in X : x$ does <u>not</u> have property $P\}$. If you can then show that A is first category in X, it follows that $X - A \neq \emptyset$.

The following theorem tells us a couple of important things.

Theorem 6.10 Suppose O is an open set in a Baire Space $(X, \mathcal{T})$.

1) If $O \neq \emptyset$, then O is second category in X.
(*In particular, a <u>nonempty</u> Baire space X is second category in itself.*)

2) O is a Baire space.

Proof 1) If O were first category in X, then (by definition of a Baire space) $X - O$ would be dense in X; since $X - O$ is closed, it would follow that $X - O = X$, and therefore $O = \emptyset$.

2) Suppose O is open in the Baire space X, and let A be a first category set <u>in O</u>. We must show that $O - A$ is dense in O. Suppose $A = \bigcup_{k=1}^{\infty} N_k$, where N_k is nowhere dense in O. By Theorem 6.3, N_k is also nowhere dense in X. Therefore A is first category in X so $X - A$ is dense in X. Since O is open, $O \cap (X - A) = O - A$ is dense in O by part 2) of the Lemma 6.4. •

Example 6.11 By Theorem 6.10.1, a nonempty Baire space is second category in itself. The converse is false. To see this, let $X = (\mathbb{Q} \cap [0,1]) \cup \{2\}$, with its usual topology. If we write $X = \bigcup_{k=1}^{\infty} N_k$, then the isolated point "2" must be in some N_k, so N_k is not nowhere dense. Therefore X is second category in itself. If X were Baire, then, by Theorem 6.10, the open subspace $\mathbb{Q} \cap [0,1]$ would also be Baire and therefore second category in itself. However this is false since $\mathbb{Q} \cap [0,1]$ is a countable union of (nowhere dense) singleton sets.

To make use of properties of Baire spaces, it would be helpful to have a "large supply" of Baire spaces. The next theorem provides us what we want.

Theorem 6.12 (The Baire Category Theorem) A complete metric space (X, d) is a Baire space. (*and therefore, by 6.10, a nonempty complete metric space is second category in itself.*)

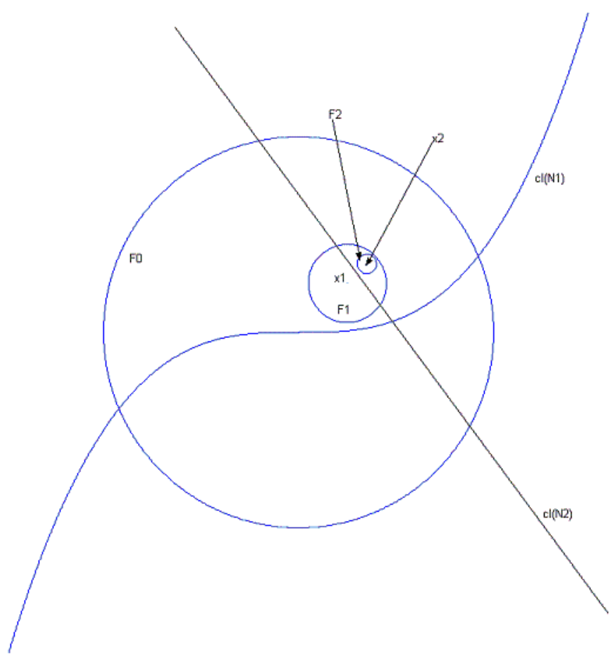
Proof If $X = \emptyset$, then X is Baire, so we assume $X \neq \emptyset$.

Let $A = \bigcup_{k=1}^{\infty} N_k$, where N_k is nowhere dense. We must show that $X - A$ is dense in X. Suppose $x_0 \in X$. The closed balls of the form $\{x : d(x_0, x) \leq \epsilon\}$ form a neighborhood base at x_0. Let F_0 be such a closed ball, centered at x_0 with radius $\epsilon > 0$. We will be done if we can show $F_0 \cap (X - A) \neq \emptyset$. We do that by using the Cantor Intersection Theorem.

Since int $F_0 \neq \emptyset$ (it contains x_0) and since N_1 is nowhere dense, we know that int F_0 is not a subset of cl N_1. Therefore we can choose a point x_1 in the open set int F_0 $-$ cl N_1. Pick a closed ball F_1, centered at x_1, so that $x_1 \in F_1 \subseteq$ int $F_0 -$ cl $N_1 \subseteq F_0$. If necessary, choose F_1 even smaller so that diam$(F_1) < \frac{1}{2}$.

(*This is the first step of an inductive construction. We could now move to the induction step, but actually include "step two" to be sure the process is clear.*)

Since int $F_1 \neq \emptyset$ (it contains x_1) and since N_2 is nowhere dense, we know that int F_1 is not a subset of cl N_2. Therefore we can choose a point x_2 in the open set int F_1 $-$ cl N_2. Pick a closed ball F_2, centered at x_2, so that $x_2 \in F_2 \subseteq$ int $F_1 -$ cl $N_2 \subseteq F_1$. If necessary, choose F_2 even smaller so that diam$(F_2) < \frac{1}{3}$.



For the induction step: suppose we have defined closed balls $F_1, F_2, ..., F_k$ centered at points $x_1, x_2, ..., x_k$, with diam$(F_i) < \frac{1}{i+1}$ and so that $x_i \in F_i \subseteq$ int $F_{i-1} -$ cl $N_i \subseteq F_{i-1}$.
Since int $F_k \neq \emptyset$ (it contains x_k) and since N_{k+1} is nowhere dense, we know that int F_k is not a subset of cl N_{k+1}. Therefore we can choose a point x_{k+1} in the open set int F_k $-$ cl N_{k+1}. Pick a closed ball F_{k+1}, centered at x_{k+1}, so that $x_{k+1} \in F_{k+1} \subseteq$ int $F_k -$ cl $N_{k+1} \subseteq F_k$. If necessary, choose F_{k+1} even smaller so that diam$(F_{k+1}) < \frac{1}{k+2}$.

By induction, the closed sets F_k are defined for all k, $\text{diam}(F_k) \to 0$ and $F_0 \supseteq F_1 \supseteq ... \supseteq F_k \supseteq ...$. Since (X, d) is complete, the Cantor Intersection Theorem says that there is a point $x \in \bigcap_{k=1}^{\infty} F_k \subseteq F_0$. For each $k \geq 1$, $x \in F_k \subseteq \text{int}\, F_{k-1} - \text{cl}\, N_k$, so $x \notin \text{cl}\, N_k$, so $x \notin N_k$. Therefore $x \in X - \bigcup_{k=1}^{\infty} N_k = X - A$, so $x \in F_0 \cap (X - A)$ and we are done. •

Example 6.13

1) Now we can see another reason why $\mathbb{Q}$ is not completely metrizable. Suppose d' is a metric on $\mathbb{Q}$ that is equivalent to the usual metric d and for which $(\mathbb{Q}, d')$ is complete. By the Baire Category Theorem, $(\mathbb{Q}, d')$ must be second category in itself. But this is false since the space is topologically the same as the usual space of rationals.

2) Suppose (X, d) is a nonempty complete metric space without isolated points. We proved in Theorem 3.6 that $|X| \geq c$. We can now that, in addition, that each point of X must be a condensation point.

Otherwise, there would be a non-condensation point $x_0 \in X$, and there would be some countable open set $O = \{x_0, x_1, x_2, ...x_n, ...\}$. Since each x_n is non-isolated, the singleton sets $\{x_n\}$ are nowhere dense in (X, d), so O is first category in (X, d). Since (X, d) is Baire, this would mean that the closed set $X - O$ is dense $-$ which is impossible.

Notice that this example illustrates <u>again</u> that $\mathbb{Q}$ cannot be completely metrizable: if its topology were produced by a some complete metric, then each point in $\mathbb{Q}$ would have to be a condensation point.

3) The set $\mathbb{P}$ is a second category set in $\mathbb{R}$: if not, we could write $\mathbb{R} = \mathbb{P} \cup \mathbb{Q}$, so that $\mathbb{R}$ would be a first category set in $\mathbb{R}$ $-$ which contradicts the Baire Category Theorem. (*Is* $\mathbb{P}$ *second category <u>in $\mathbb{P}$</u>?*)

4) Recall that $A \subseteq (X, \mathcal{T})$ is called an F_σ set if A can be written as a countable union of closed sets, and that A is a G_δ set if it can be written countable intersection of open sets. The complement of a G_δ set is an F_σ set and vice-versa.

$\mathbb{P}$ is not an F_σ set in $\mathbb{R}$. To see this, suppose that $\mathbb{P} = \bigcup_{n=1}^{\infty} F_n$, where the F_n's are closed in $\mathbb{R}$. Since $\mathbb{P}$ is second category in $\mathbb{R}$, one of these F_n's must be not nowhere dense in $\mathbb{R}$. This F_n must therefore contain an open interval (a, b). But then $(a, b) \subseteq F_n \subseteq \mathbb{P}$, which is impossible because there are also rational numbers in (a, b).

Taking complements, we see that $\mathbb{Q}$ is not a G_δ-set in $\mathbb{R}$.

Example 6.14 Suppose $f : [0,1] \to \mathbb{R}$ is continuous. By the Fundamental Theorem of Calculus, f has an antiderivative f_1. Since f_1 is differentiable (therefore continuous), it has an antiderivative f_2. Continuing in this way, let f_n denote an antiderivative of f_{n-1}. It is a trivial observation that:

$$(\exists k\, \forall x\, f_k(x) = 0) \;\Rightarrow\; f(x) = 0 \text{ for all } x \in [0,1]$$

We will use the Baire Category Theorem to prove a less obvious fact:

$$(\forall x\; \exists k\; f_k(x) = 0) \Rightarrow f(x) = 0 \text{ for all } x \text{ in } [0,1] \quad (*)$$

In other words, if $f(x)$ is not identically 0 on $[0,1]$, then there must exist a point $x_0 \in [0,1]$ such that every antiderivative $f_k(x_0) \neq 0$.

By hypothesis, we can write $[0,1] = \bigcup_{k=1}^{\infty} F_k$, where $F_k = \{x \in [0,1] : f_k(x) = 0\}$ $= f_k^{-1}[\{0\}]$. Since each f_k is continuous, F_k is closed in $[0,1]$. Since $[0,1]$ is second category in itself, one of the sets F_{k_0} is not nowhere dense and therefore must contain an interval (a,b). Since f_{k_0} is identically 0 on (a,b), f is identically 0 on (a,b).

We can repeat the same argument on any closed subinterval $J = [c,d] \subseteq [0,1]$: letting $g = f|J$ and $g_k = f_k|J$, we can conclude that J contains an open interval on which f is identically 0. <u>In particular</u>, each closed subinterval $J = [c,d]$ contains a point x at which $f(x) = 0$. Therefore $\{x : f(x) = 0\}$ is dense in $[0,1]$. Since f is continuous and is 0 on a dense set, f must be 0 everywhere in $[0,1]$. (*See Theorem II.5.12 and Exercise III.E.16.*)

Example 6.15 (The Banach-Mazur Game) The equipment for the game consists of two disjoint sets A, B for which $A \cup B = [0,1]$. Set A belongs to Andy and set B belongs to Beth. Andy gets the first "move" and selects a closed interval $I_1 \subseteq [0,1]$. Beth then chooses a closed interval $I_2 \subseteq I_1$, where I_2 has length $\leq \frac{1}{2}$. Andy then selects a closed interval $I_3 \subseteq I_2$, where I_3 has length $\leq \frac{1}{3}$. They continue back-and-forth in this way forever. (*Of course, to finish in a finite time they must make their choices faster and faster.*) When all is done, they look at $\bigcap_{n=1}^{\infty} I_n = \{x\}$. If $x \in A$, Andy wins; if $x \in B$, Beth wins. <u>We claim that if A is first category, then Beth can always win</u>.

Suppose $A = \bigcup_{n=1}^{\infty} N_k$, where A_k is nowhere dense in $[0,1]$. Let Andy's k^{th} choice be J $(= I_{2k-1})$. Of course, $J \cap N_k$ is nowhere dense in $[0,1]$; but actually $J \cap N_k$ is also nowhere dense <u>in the interval J</u>:

> If $\text{int}_J \text{cl}_J(J \cap N_k) \neq \emptyset$, then $\text{cl}_J(J \cap N_k)$ must contain an nonempty open interval (a,b). But then
> $$(a,b) \subseteq \text{cl}_J(J \cap N_k) \subseteq \text{cl}_J N_k = \text{cl}_{[0,1]} N_k \cap J \subseteq \text{cl}_{[0,1]} N_k,$$
> which contradicts the fact that N_k is nowhere dense in $[0,1]$.

Then the open set $\text{int}\, J - \text{cl}_J(J \cap N_k)$ is nonempty and Beth can make her k^{th} choice I_{2k} to be a closed interval $[a,b] \subseteq \text{int}\, J - \text{cl}_J(J \cap N_k)$. This implies that $I_{2k} \cap N_k = \emptyset$, since

$$I_{2k} \subseteq \text{int}\, J - \text{cl}_J(J \cap N_k) \subseteq \text{int}\, J - (J \cap N_k) \subseteq J - (J \cap N_k) = J - N_k$$

Therefore $\bigcap_{k=1}^{\infty} I_k \cap \bigcup_{k=1}^{\infty} N_k = \emptyset$ and Beth wins!

Example 6.16 The Baire Category Theorem can also be used to prove the existence of a continuous function on $[0,1]$ that is nowhere differentiable. The details can be found, for example, in Willard's *General Topology*. In outline, one looks at the space $C^*([0,1])$ with the uniform metric ρ, and argues that the set N of functions which have a derivative <u>at one or more points</u> is a ("thin") first category set in $C^*([0,1])$. But the complete space $(C^*([0,1]), \rho)$ is a ("thick") Baire space. Therefore it is second category in itself so $C^*([0,1]) - N \neq \emptyset$. In fact, $C^*([0,1]) - N$ is <u>dense</u> in $C^*([0,1])$. Any function in $C^*([0,1]) - N$ does the trick.

7. Complete Metrizability

Which metrizable spaces (X,d) are completely metrizable – that is, when does there exist a metric d' on X with $d \sim d'$ and with (X,d') complete? We have already seen that $\mathbb{Q}$ is not completely metrizable (using the Baire Category Theorem), and that certain familiar spaces like $\{\frac{1}{n} : n \in \mathbb{N}\}$ are completely metrizable. In this section, we will give an answer to this question.

Lemma 7.1 If (X,d) is a pseudometric space and $g : X \to \mathbb{R}$ continuous and $g(a) \neq 0$. Then $\frac{1}{g} : X \to \mathbb{R}$ is continuous at a.

Proof The proof is a perfect "mimic" of the standard proof in advanced calculus of the same result for $X = \mathbb{R}$.

Our first theorem tells us that certain subspaces of complete spaces are completely metrizable.

Theorem 7.2 If (X,d) is complete and O is open in X, then there is a metric $d' \sim d$ on O such that (O,d') is complete – that is, O is completely metrizable.

If O is not already complete with metric d, it is because there are some nonconvergent Cauchy sequences in O. However those Cauchy sequences <u>do</u> have limits in X – they converge to points outside O but in $\mathrm{Fr}\, O$. The idea of the proof is to create a new metric d' on O equivalent of d but which "blows up" near the boundary of O: in other words, the new metric destroys the "Cauchyness" of the nonconvergent Cauchy sequences.

Here is a concrete (but slightly simpler) example to illustrate the idea.

> Let d be the usual metric on the interval $J = (-\frac{\pi}{2}, \frac{\pi}{2})$. The sequence $(x_n) = (\frac{\pi}{2} - \frac{1}{n})$ is a nonconvergent Cauchy sequence in J. The function $\tan : J \to \mathbb{R}$ is a homeomorphism for which $(f(x_n)) \to \infty$.
>
> $d'(x,y) = |\tan x - \tan y|$ defines a new metric on J, and $\tan : (J,d') \to (\mathbb{R}, d)$ is an isometry because $d'(x,y) = = |\tan x - \tan y| = d(\tan x, \tan y)$. Since $(\mathbb{R}, d)$ is complete, so in (J,d'). (*The sequence* (x_n) *is still nonconvergent in* (J,d') *because* $d \sim d'$; *but* (x_n) *is no longer a Cauchy sequence in* (J,d'))
>
> In the proof of Theorem 7.2, we will not have a homeomorphism like "tan" to use. Instead, we define a continuous $f : (O,d) \to \mathbb{R}$ and use f to create a new metric d' on O that destroys Cauchy sequences (x_n) when they approach Fr O.

Proof of 7.2 Define $f(x) = \frac{1}{d(x, X-O)}$. Since O is open, the denominator is never 0 for $x \in O$ so $f: O \to \mathbb{R}$ is continuous by Lemma 7.1. On O, define $d'(x,y) = d(x,y) + |f(x) - f(y)|$. It is easy to check that d' is a metric on O.

Since sequences are sufficient to determine the topology in metric spaces, we show that $d' \sim d$ on O by showing that they produce the same convergent sequences in O. Suppose (x_n) is a sequence in O and $x \in O$:

If $d'(x_n, x) \to 0$, then since $d' \geq d$ we get $d\,(x_n, x) \to 0$.

. If $d\,(x_n, x) \to 0$, then $|f(x_n) - f(x)| \to 0$ (by continuity) so $d'\,(x_n, x) \to 0$.

(O, d') is complete : suppose (x_n) is a d'-Cauchy sequence in O. Since $d' \geq d$, (x_n) is also d-Cauchy so $(x_n) \xrightarrow{d}$ some $x \in X$. But we claim that this x must be <u>in</u> O.

If $x \in X - O$, then $d(x_n, X-O) \to 0$ and so $f(x_n) \to \infty$. In particular, this means that for every n we can find $m > n$ for which $f(x_m) > f(x_n) + 1$. Then $|f(x_m) - f(x_n)| > 1$, so $d'(x_m, x_n) > 1$. This contradicts the assumption that (x_n) is d'-Cauchy.

Therefore $(x_n) \xrightarrow{d} x \in O$. Since $d \sim d'$ on O, $(x_n) \xrightarrow{d'} x \in O$ and so (O, d') is complete. •

The next theorem generalizes this result. In fact, the interesting "idea" is in Theorem 7.2. The proof of Theorem 7.3 just uses Theorem 7.2 repeatedly to "patch together" a more general result.

Theorem 7.3 (Alexandroff) If (X, d) is complete and A is a G_δ in X, then there is a metric ρ *on* A, with $\rho \sim d$ on A and such that (A, ρ) is complete. In other words: a G_δ subset of a complete space is completely metrizable.

Proof Let $A = \bigcap_{i=1}^{\infty} O_i$, where O_i is open in X. Let d_i' be a metric on O_i, equivalent to d on O_i, such that $(O_i,\, d_i')$ is complete. Let $d_i(x,y) = \min\{d_i'(x,y), 1\}$. Then $d_i \sim d_i' \sim d$ on O, and (O_i, d_i) is complete. (*Note that d_i and d_i' are identical for distances smaller than one: this is why they produce the same convergent sequences and the same Cauchy sequences.*)

For $x, y \in A$, define $\rho(x,y) = \sum_{i=1}^{\infty} \frac{d_i(x,y)}{2^i}$ (the series converges since all $d_i(x,y) \leq 1$).

ρ is a metric on A:

Exercise

$\rho \sim d$ on A:

We show that ρ and d produce the same convergent sequences in A.

Suppose $(a_n) \xrightarrow{d} a \in A \subseteq O_i$. Let $\epsilon > 0$.

Choose k so that $\sum_{j=k+1}^{\infty} \frac{1}{2^j} < \frac{\epsilon}{2}$. For every i, $d \sim d_i$ on O_i, so

$(a_n) \overset{d_i}{\to} a \in O_i$. Therefore, for $i = 1, ..., k$, we can pick N_i so that $n > N_i$ $\Rightarrow \frac{d_i(a_n,a)}{2^i} < \frac{\epsilon}{2k}$. Then if $n > N = \max\{N_1, ..., N_k\}$, we get $\rho(a_n,a) = \sum_{i=1}^{k} \frac{d_i(a_n,a)}{2^i} + \sum_{i=k+1}^{\infty} \frac{d_i(a_n,a)}{2^i} \leq k \cdot \frac{\epsilon}{2k} + \frac{\epsilon}{2} = \epsilon$, so $(a_n) \overset{\rho}{\to} a \in A$.

Conversely, if $(a_n) \overset{\rho}{\to} a \in A$, then for $\epsilon > 0$, we can pick N so that $n > N \Rightarrow \rho(a_n, a) = \sum_{i=1}^{\infty} \frac{d_i(a_n,a)}{2^i} < \frac{\epsilon}{2} \quad \Rightarrow d_1(a_n, a) < \epsilon$, so $(a_n) \overset{d_1}{\to} a \in O_1$. But $d_1 \sim d$ on O_1, so $(a_n) \overset{d}{\to} a$.

(A, ρ) is complete:

Let (a_n) be ρ-Cauchy in A. Let $\epsilon > 0$ and let $i \in \mathbb{N}$. Choose N_i so that

if $m, n > N_i$, then $\rho(a_n, a_m) < \frac{\epsilon}{2^i}$, and therefore $d_i(a_n,a_m) < \epsilon$.

Therefore (a_n) is Cauchy in the complete space (O_i, d_i) and there is a point $x_i \in O_i$ such that $(a_n) \overset{d_i}{\to} x_i \in O_i$. But $d_i \sim d$ on O_i, so $(a_n) \overset{d}{\to} x_i \in O_i \subseteq X$.

This is true for each i. But (a_n) can have only one limit $a \in (X, d)$, so we conclude that $x_1 = x_2 = ... = x_i = ... = a$. Since $a \in O_i$ for each i, we have $a \in A$, so $(a_n) \overset{d}{\to} a \in A$. But $d \sim \rho$ on A, so $(a_n) \overset{\rho}{\to} a \in A$. Therefore (A, ρ) is complete. •

Corollary 7.4 $\mathbb{P}$ is completely metrizable (and therefore $\mathbb{P}$ is a Baire space, so $\mathbb{P}$ is second category in $\mathbb{P}$).

Proof $\mathbb{Q} = \bigcup_{q \in \mathbb{Q}} \{q\}$, so $\mathbb{Q}$ is an F_σ set in $\mathbb{R}$. Taking complements, we get that $\mathbb{P} = \bigcap_{q \in \mathbb{Q}} (\mathbb{R} - \{q\})$ is a G_δ set in $\mathbb{R}$. •

In fact, a sort of "converse" to Alexandroff's Theorem is also true.

Theorem 7.5 For any metric space (X, d), the following are equivalent:

i) (X, d) is completely metrizable
ii) (X, d) is homeomorphic to a G_δ set in some complete metric space (Y, d')
iii) If (Z, d'') is any metric space and $f : (X, d) \to (f[X], d'') \subseteq (Z, d'')$ is a homeomorphism, then $f[X]$ is a G_δ set in (Z, d'')
(*so we say that "X is an absolute G_δ set among metric spaces"*)
iv) X is a G_δ set in the completion $(\widetilde{X}, \widetilde{d})$.

Various parts of Theorem 7.5 are due to Mazurkiewicz (1916) & Alexandroff (1924).

The proof of one of the implications in Theorem 7.5 requires a technical result whose proof we will omit. (See, for example Willard, *General Topology*)

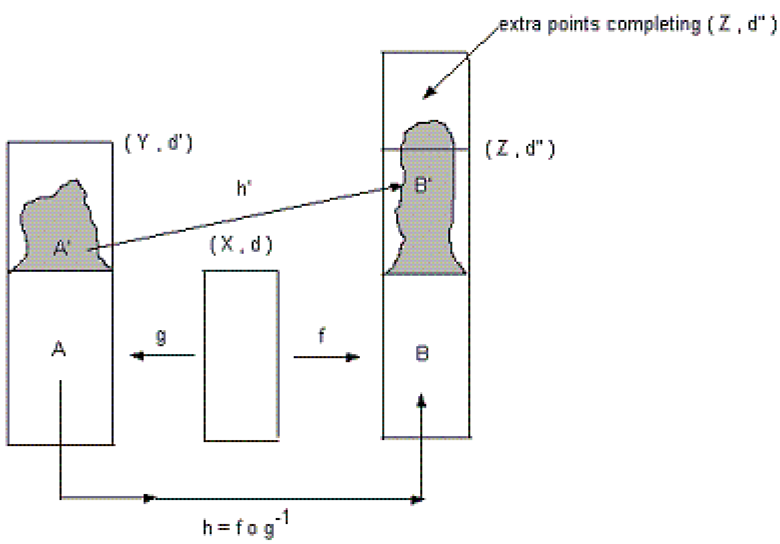
Theorem 7.6 (Lavrentiev) Suppose (X, d) and (Y, d') are complete metric spaces with $A \subseteq X$ and $B \subseteq Y$. Let h be a homeomorphism from A onto B. Then there exist G_δ sets A' in X and B' in Y with $A \subseteq A' \subseteq \mathrm{cl}\, A$ and $B \subseteq B' \subseteq \mathrm{cl}\, B$ and there exists a homeomorphism h' from A' onto B' such that $h'|A = h$. (*Loosely stated: a homeomorphism between subsets of two complete metric spaces can always be "extended" to a homeomorphism between G_δ sets.*)

Assuming Lavrentiev's Theorem, we now prove Theorem 7.5.

Proof i) $\Rightarrow$ ii) Suppose (X, d) is completely metrizable and let d' be a complete metric on X equivalent to d. Then the identity map $i : (X, d) \to (X, d')$ is a homeomorphism and $i[X] = X$ is certainly a G_δ set in (X, d').

ii) $\Rightarrow$ iii) Suppose we have a homeomorphism $g : (X, d) \to g[X] = A$, where A is a G_δ set in a complete space (Y, d'). Let $f : (X, d) \to (Z, d'') \subseteq$ the completion $(\widetilde{Z}, \widetilde{d}'')$ be a homeomorphism (into Z). We want to prove that $B = f[X]$ must be a G_δ set in Z.

We have that $h = fg^{-1} : A \to B$ is a homeomorphism. Using Lavrentiev's Theorem, we get an extension of h to a homeomorphism $h' : A' \to B'$ where A', B' are G_δ sets with $A \subseteq A' \subseteq Y$ and $B \subseteq B' \subseteq \widetilde{Z}$ (*see the figure*).



Since A is a G_δ in Y, there are open sets O_n in Y such that $A = \bigcap_{n=1}^{\infty} O_n$ $= A' \cap \bigcap_{n=1}^{\infty} O_n = \bigcap_{n=1}^{\infty} (O_n \cap A')$. Therefore A is also a G_δ in A'. But $h' : A' \to B'$ is a homeomorphism, so $h'[A] = h[A] = B$ is a G_δ set in B'. Therefore $B = \bigcap_{n=1}^{\infty} V_n$, where V_n is open in B' and, in turn, $V_n = B' \cap W_n$ where W_n is open in $\widetilde{Z}$.

Also, B' is a G_δ in $\widetilde{Z}$, so $B' = \bigcap_{n=1}^{\infty} U_n$, where the U_n's are open in $\widetilde{Z}$.

Putting all this together, $B = \bigcap_{n=1}^{\infty} V_n = \bigcap_{n=1}^{\infty}(W_n \cap B')$
$= (\bigcap_{n=1}^{\infty} W_n) \cap B' = (\bigcap_{n=1}^{\infty} W_n) \cap (\bigcap_{n=1}^{\infty} U_n)$. The last expression shows that B is a G_δ set in $\widetilde{Z}$. But then $B = Z \cap B = Z \cap (\bigcap_{n=1}^{\infty} W_n) \cap (\bigcap_{n=1}^{\infty} U_n) = (\bigcap_{n=1}^{\infty} W_n \cap Z) \cap (\bigcap_{n=1}^{\infty} U_n \cap Z)$ is a G_δ set in Z.

iii) $\Rightarrow$ iv) iii) states that any "homeomorphic copy" of X in a metric space (Z, d'') must be a G_δ in Z. Letting $(Z, d'') = (\widetilde{X}, \widetilde{d}\ )$, it follows that X must be a G_δ set in the completion $(\widetilde{X}, \widetilde{d}\ )$.

iv) $\Rightarrow$ i) This follows from Alexandroff's Theorem 7.3: a G_δ set in a complete space is completely metrizable. ●

Theorem 7.5 raises the questions: what spaces are "absolutely open"? what spaces are "absolutely closed?" Of course in each case a satisfactory answer might involve some kind of qualification. For example, "among Hausdorff spaces, a space X is absolutely closed iff ... "

Example 7.7 Since $\mathbb{Q}$ is not a G_δ in $\mathbb{R}$ (an earlier consequence of the Baire Category Theorem), Theorem 7.5 gives us yet another reason why $\mathbb{Q}$ is not completely metrizable.

Example 7.8 Since $\mathbb{P}$ is completely metrizable, it follows that $\mathbb{P}$ is a Baire space. If d is the usual metric on $\mathbb{P}$, then $(\mathbb{P}, d)$ is an example of a Baire metric space that isn't complete.

To finish this section on category, we mention, without proof and just as a curiosity, a generalization of an earlier result (*Blumberg's Theorem: see Example II.5.8*).

Theorem 7.9 Suppose (X, d) is a Baire metric space. For every $f : X \to \mathbb{R}$, there exists a dense subset D of X such that $f|D : D \to \mathbb{R}$ is continuous,

The original theorem of this type was proved for $X = \mathbb{R}$ or $\mathbb{R}^2$. $X = \mathbb{R}^2$. (Blumberg, *New properties of all real-valued functions*, Transactions of the American Mathematical Society, 24(1922) 113-128).

The more general result stated in Theorem 7.9 is (essentially) due to Bradford and Goffman (*Metric spaces in which Blumberg's Theorem holds*, Proceedings of the American Mathematical Society 11(1960), 667-670)

Exercises

E11. In a pseudometric space (X,d), every closed set is a G_δ set and every open set is an F_σ set (*see Exercise II.E.15*).

a) Find a space $(X,\mathcal{T})$ which contains a closed set F that is not a G_δ set.

b) Recall that the "scattered line" is the space $(\mathbb{R},\mathcal{T})$ where

$$\mathcal{T} = \{\, U \cup V : U \text{ is a usual open set in } \mathbb{R} \text{ and } V \subseteq \mathbb{P}\,\}$$

Prove that the scattered line is not metrizable. (*Hint: G_δ or F_σ sets are relevant.*)

E12. a) Prove that if O is open in $(X,\mathcal{T})$, then Fr (O) is nowhere dense.

b) Suppose that D is a discrete subspace of the Hausdorff space $(X,\mathcal{T})$ and that X has no isolated points. Prove that D is nowhere dense in X.

E13. Let $\mathcal{T}$ be the confinite topology. Prove that $(X,\mathcal{T})$ is a Baire space if and only if X is either finite or uncountable.

E14. Suppose d is any metric on $\mathbb{Q}$ which is equivalent to the usual metric on $\mathbb{Q}$. Prove that the completion $(\widetilde{\mathbb{Q}},\widetilde{d})$ is uncountable.

E15. Suppose that (X,d) is a nonempty complete metric space and that $\mathcal{F}$ is a family of continuous functions from X to $\mathbb{R}$ with the following property:

$$\forall x \in X,\ \exists \text{ a constant } M_x \text{ such that } |f(x)| \le M_x \text{ for all } f \in \mathcal{F}$$

Prove that there exists a nonempty open set U and a constant M (independent of x) such that $|f(x)| \le M$ for all $x \in U$ and all $f \in \mathcal{F}$.

This result is called the Uniform Boundedness Principle.

Hint: Let $E_k = \{x \in X : |f(x)| \le k$ for all $f \in \mathcal{F}\}$. Use the Baire Category Theorem.

E16. Suppose $f\colon (X, \mathcal{T}) \to (Y, d)$, where (Y, d) is a metric space where d is a metric. For each $x \in X$, define

$$\omega_f(x) \;=\; \text{“the oscillation of } f \text{ at } x\text{”} = \inf\{\text{diam}(f[N]) : N \text{ is a neighborhood of } x\}$$

a) Prove that f is continuous at a if and only if $\omega_f(a) = 0$.

b) Prove that for $n \in \mathbb{N}$, $\{x \in X : \omega_f(x) < \frac{1}{n}\}$ is open in X.

c) Prove that $\{x \in X : f \text{ is continuous at } x\}$ is a G_δ-set in X.

Note: Since $\mathbb{Q}$ is not a G_δ-set in $\mathbb{R}$, a continuous function $f : \mathbb{R} \to \mathbb{R}$ cannot have $\mathbb{Q}$ as its set of points of continuity. On the other hand, $\mathbb{P}$ is a G_δ-set in $\mathbb{R}$ and, from analysis, you should probably know an example of a continuous function $f : \mathbb{R} \to \mathbb{R}$ that is continuous at each $p \in \mathbb{P}$ and discontinuous at each point $q \in \mathbb{Q}$.

d) Prove that there cannot exist a function $f : \mathbb{R} \to (0, \infty)$ such that for all $x \in \mathbb{Q}$ and all $y \in \mathbb{P}$, $f(x)f(y) \leq |\, x - y \,|$. *Hint for d): Suppose f exists.*

> *i) First prove that if (x_n) is a sequence of rationals converging to an irrational, then $(f(x_n)) \to 0$ and, likewise, if (y_n) is a sequence of irrationals converging to a rational, then $(f(y_n)) \to 0$.*
>
> *ii) Define a new function $g : \mathbb{R} \to \mathbb{R}$ by $g \mid \mathbb{Q} = 0$ and $g(y) = f(y)$ for y irrational. Examine the set of points where g is continuous.*)

E17. There is no continuous function $f : \mathbb{R} \to \mathbb{R}$ for which the set of points of continuity is $\mathbb{Q}$. The reason (*see problem E16*) ultimately depends on the Baire Category Theorem. Find the error in the following more elementary “proof”:

> Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous and that f is continuous at x iff $x \in \mathbb{Q}$.
> Then $\mathbb{Q} = \{x \in \mathbb{R} : \omega_f(x) = 0\} = \bigcap_{n=1}^{\infty} U_n$, where $U_n = \{x \in \mathbb{R} : \omega_f(x) < \frac{1}{n}\}$.
>
> Each $U_n \supseteq \mathbb{Q}$, and U_n can be written as a countable union of disjoint open intervals. If (a, b) and (c, d) are consecutive intervals in U_n, then $b = c$ – or else there would be a rational in (b, c). Therefore $\mathbb{R} - U_n$ consists only of the endpoints of some disjoint open intervals. Therefore $\mathbb{R} - U_n$ is countable.
>
> It follows that $\{x \in \mathbb{R} : f \text{ is not continuous at } x\} = \mathbb{R} - \bigcap_{n=1}^{\infty} U_n = \bigcup_{n=1}^{\infty} \mathbb{R} - U_n$ is countable. Therefore f is continuous on more than just the points in $\mathbb{Q}$.

E18. Suppose that for each irrational p, we construct an equilateral triangle T_p (including its interior) in $\mathbb{R}^2$ with one vertex at $(p, 0)$ and its opposite side above and parallel to the x-axis. Prove that $\bigcup\{T_p : p \in \mathbb{P}\}$ must contain an “open box” of the form $(a, b) \times (0, \frac{1}{k})$ for some $a, b \in \mathbb{R}$ and some $k \in \mathbb{N}$. In the hypothesis, we could weaken “equilateral” to read “ ... ” ?
(*Hint: Consider $N_k = \{p \in \mathbb{P} : T_p$ has height $\geq \frac{1}{k}\}$.*)

E19. Suppose $f : \mathbb{R} \to \mathbb{R}$. Prove that if f is discontinuous at every irrational $p \in \mathbb{P}$, then there must exist an interval $I = (a, b)$ such that f is discontinuous at every point in I.

E20. $(X, \mathcal{T})$ is "absolutely open" if whenever $f : (X, \mathcal{T}) \to (Y, \mathcal{T}\,')$ is a homeomorphism (into), then $f[X]$ must be open in Y. Find all absolutely open spaces.

8. Compactness

Compactness is one of the most important topological properties. Formally, it is a strengthening of the Lindelöf property.

Compact spaces are often particularly simple to work with because of the "rule of thumb" that "compact spaces often act like finite spaces."

Definition 8.1 A topological space $(X, \mathcal{T})$ is called <u>compact</u> if every open cover of X has a finite subcover.

Example 8.2

1) A finite space is compact.
2) Any set X with the cofinite topology is compact – because any nonempty open set, alone, covers X except for perhaps finitely many points.
3) An infinite discrete space is not compact – because the cover consisting of all singleton sets has no finite subcover. In particular, $\{\frac{1}{n} : n \in \mathbb{N}\}$ is not compact.
4) The space $X = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ is compact: if $\mathcal{U}$ is any open cover, and $0 \in U \in \mathcal{U}$, then the set U covers X except for perhaps finitely many points.
5) $\mathbb{R}$ is not compact since $\mathcal{U} = \{(-n, n) : n \in \mathbb{N}\}$ has no finite subcover.

Definition 8.3 A family $\mathcal{F}$ of sets has the <u>finite intersection property (FIP)</u> if every finite subfamily of $\mathcal{F}$ has nonempty intersection.

Sometimes it is useful to have a characterization of compact spaces in terms of closed sets, and we can do this using FIP. As you read the proof, you should see that Theorem 8.4 is hardly any more than a reformulation, using complements, of the definition of compactness in terms of closed sets.

Theorem 8.4 $(X, \mathcal{T})$ is compact iff whenever $\mathcal{F}$ is a family of closed sets with the finite intersection property, then $\bigcap \mathcal{F} \neq \emptyset$.

Proof Suppose X is compact and that $\mathcal{F}$ is a family of closed sets with FIP. Let $\mathcal{U} = \{X - F : F \in \mathcal{F}\}$. For any $F_1, F_2, ..., F_n \in \mathcal{F}$, the FIP tells us that $X \neq X - \bigcap_{i=1}^{n} F_n$ $= (X - F_1) \cup ... \cup (X - F_n)$. In other words, no finite subcollection of $\mathcal{U}$ covers X. Therefore $\mathcal{U}$ itself cannot cover X, that is, $X - \bigcup \mathcal{U} = \bigcap\{X - U : U \in \mathcal{U}\} = \bigcap \mathcal{F} \neq \emptyset$.

The proof of the converse is similar. ∙

Theorem 8.5 For any space $(X, \mathcal{T})$

a) If X is compact and F is closed in X, then F is compact.
b) If X is a Hausdorff space and K is a compact subset, then K is closed in X.

Proof a) *The proof is like the proof that a closed subspace of a Lindelöf space is Lindelöf. (Theorem III.3.10)* Suppose F is a closed set in a compact space X, and let $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ be a cover of F by sets open <u>in F</u>. For each α, pick an open set V_α in X such that $U_\alpha = V_\alpha \cap F$. Since F is closed, the collection $\mathcal{V} = \{X - F\} \cup \{V_\alpha : \alpha \in A\}$ is an open cover of X.

Therefore we can find $V_{\alpha_1}, ..., V_{\alpha_n}$ so that $\{X - F, V_{\alpha_1}, ..., V_{\alpha_n}\}$ covers X. Then $\{U_{\alpha_1}, ...U_{\alpha_n}\}$ covers F, so F is compact.

Note: the definition of compactness requires that we look at a cover of F by sets open in F, but it should be clear that it is equivalent to look at a cover of F using sets open in X.

b) Suppose K is a compact set in a Hausdorff space X. Let $y \in X - K$. For each $x \in K$, we can choose disjoint open sets U_x and V_x in X with $x \in U_x$ and $y \in V_x$. Since $\mathcal{U} = \{U_x : x \in K\}$ covers K, there are finitely many points $x_1, ..., x_n \in K$ so that $U_{x_1}, ..., U_{x_n}$ covers K. Then $y \in V = V_{x_1} \cap ... \cap V_{x_n} \subseteq X - K$. Therefore $X - K$ is open, so K is closed. ●

Notes 1) Reread the proof of part b) assuming K is a finite set. This highlights how "compactness" has been used in place of "finiteness" and illustrates the rule of thumb that compact spaces often behave like finite spaces.

2) The "Hausdorff" hypothesis cannot be omitted in part b). For example, let X be a set with $|X| > 1$ and let d be the trivial pseudometric on X. Then each singleton set $\{x\}$ is compact but not closed.

3) Compactness is a clearly a topological property, so part b) implies that if a compact space X is homeomorphic to a subspace K in a Hausdorff space Y, then K is closed in Y. So we can say that a compact space is "absolutely closed" among Hausdorff spaces – "it's closed wherever you put it." In fact, the converse is also true among Hausdorff spaces – a space which is absolutely closed is compact – but we do not have the machinery to prove that now.

Corollary 8.6 A compact metric space (X, d) is complete.

Proof (X, d) is a dense subspace of its completion $(\widetilde{X}, \widetilde{d})$. But (X, d) is compact, so X must be closed in $\widetilde{X}$. Therefore $(X, d) = (\widetilde{X}, \widetilde{d})$, so (X, d) is complete. ●
(*Note: this proof doesn't work for pseudometric spaces (why?). But is Corollary 8.6 still true for pseudometric spaces? Would a proof using the Cantor Intersection Theorem work in that case?*)

You may have seen a different definition of compactness in some other book. In fact there are several different "kinds of compactness" and, in general they are not equivalent. But we will see that they are equivalent in certain spaces – for example, in $\mathbb{R}^n$.

Definition 8.7 A topological space $(X, \mathcal{T})$ is called

sequentially compact	if every sequence in X has a convergent subsequence in X
countably compact	if every countable open cover of X has a finite subcover (*therefore "Lindelöf + countably compact = compact"*)
pseudocompact	if every continuous $f : X \to \mathbb{R}$ is bounded (*You should check that this is equivalent to saying that every continuous real -valued function on X assumes both a maximum and a minimum value*).

We want to look at the relations between these "compactness-like" properties.

Lemma 8.8 If every sequence in $(X, \mathcal{T})$ has a cluster point, then every infinite set in X has a limit point. The converse is true if $(X, \mathcal{T})$ is a T_1-space – that is, if all singleton sets $\{x\}$ are closed in X.

Proof Suppose A is an infinite set in X. Choose a sequence (a_n) of distinct terms in A and let x be a cluster point of (a_n). Then every neighborhood N of x contains infinitely many a_n's, so $N \cap (A - \{x\}) \neq \emptyset$. Therefore x is a limit point of A.

Suppose X is a T_1-space in which every infinite set has a limit point. Let (x_n) be a sequence in X. We want to show that (x_n) has a cluster point. Without loss of generality, we may assume that the terms of the sequence are distinct (why?), so that $A = \{x_n : n \in \mathbb{N}\}$ is infinite. Let x be a limit point of the set A. We claim x is a cluster point for (x_n).

Suppose U is an open set containing x and $n \in \mathbb{N}$. Let $V = U - \{x_1, ..., x_n\} \cup \{x\}$. Since $\{x_1, ..., x_n\}$ is closed, V is open and $x \in V \subseteq U$. Then $V \cap (A - \{x\}) \neq \emptyset$, so $U \cap (A - \{x\}) \neq \emptyset$. Therefore U contains a term x_k for some $k > n$, so x is a cluster point of (x_n). ●

Example 8.9 The "T_1" hypothesis in the second part of Lemma 8.8 cannot be dropped. Let $X = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{p_n : n \in \mathbb{N}\}$ where the p_n's are distinct points and $p_n \neq \frac{1}{k}$ for all $n, k \in \mathbb{N}$. The idea is to make $d(p_n, x_n) = 0$ for each n so that p_n is a sort of "double" for $\frac{1}{n}$. So we define

$$\begin{cases} d(p_n, x_n) = 0 \\ d(\frac{1}{n}, \frac{1}{m}) = |\frac{1}{n} - \frac{1}{m}| = d(p_n, p_m) \text{ if } m \neq n \\ d(p_n, \frac{1}{m}) = |\frac{1}{n} - \frac{1}{m}| \text{ if } m \neq n \end{cases}$$

It is easy to check that d is a pseudometric on X. In (X, d), every nonempty set A (finite or infinite) has a limit point – if $x \in A$, then its "double" is a limit point of A. But the sequence $(\frac{1}{n})$ has no cluster point in (X, d).

Theorem 8.10 $(X, \mathcal{T})$ is countably compact iff every sequence in X has a cluster point (Then by Lemma 8.8, a T_1-space $(X, \mathcal{T})$ is countably compact iff every infinite set in X has a limit point).

Proof Suppose $(X, \mathcal{T})$ is countably compact and consider any sequence (x_n). Let T_n = "the n^{th} tail of (x_n)" $= \{x_k : k \geq n\}$. We claim that $\bigcap_{n=1}^{\infty} \text{cl}\, T_n \neq \emptyset$.

> Assume not. Then every x is in $(X - \text{cl}\, T_n)$ for some n, so $\{X - \text{cl}\, T_n : n \in \mathbb{N}\}$ is an open cover of X. Since X is countably compact, there exists an N such that $X = (X - \text{cl}\, T_1) \cup ... \cup (X - \text{cl}\, T_N)$. Taking complements gives $\emptyset = \bigcap_{n=1}^{N} \text{cl}\, T_n = \text{cl}\, T_N$, which is impossible.

Let $x \in \bigcap_{n=1}^{\infty} \text{cl}\, T_n$ and let N be a neighborhood of x. Then $N \cap T_n \neq \emptyset$ for every n, so N contains an x_k for arbitrarily large values of n. Therefore x is a cluster point of (x_n).

Conversely, suppose X is <u>not</u> countably compact. Then X has a countable open cover $\{U_1, ..., U_n, ...\}$ with no finite subcover. For each n, pick a point $x_n \in X - \bigcup_{i=1}^{n} U_i$. Then the sequence (x_n) has no cluster point: for any $x \in X$, we know x is in some U_n. U_n is a neighborhood of x and $x_m \notin U_n$ for $m > n$. ●

The next theorem tells us some connections between the different types of compactness.

Theorem 8.11 In any space $(X, \mathcal{T})$, the following implications hold:

$$\left\{\begin{array}{l} X \text{ is compact} \\ \qquad \text{or} \\ X \text{ is sequentially compact} \end{array}\right. \quad \Rightarrow \quad X \text{ is countably compact} \quad \Rightarrow \quad X \text{ is pseudocompact}$$

Proof It is clear from the definitions that a compact space is countably compact.

Suppose X is sequentially compact. Then each sequence (x_n) in X has a subsequence that converges to some point $x \in X$. Then x is a cluster point of (x_n). From Lemma 8.8 we conclude that X is countably compact.

Suppose X is countably compact and that $f : X \to \mathbb{R}$ is continuous. The sets $U_n = f^{-1}[(-n, n)] = \{x \in X : -n < f(x) < n\}$ form a countable open cover of X. So $U_1, ..., U_N$ cover X for some N. Since $U_1 \subseteq U_2 \subseteq ... \subseteq U_N$, this implies that $U_N = X$. So $-N < f(x) < N$ for all $x \in X$ – that is, f is bounded. Therefore X is pseudocompact. •

In general, no other implications hold in among these four types of compactness, but we do not have the machinery to provide counterexamples now. (*See Corollary 8.5 in Chapter VIII and Example 6.5 in Chapter X*). We will prove, however, that they are <u>all equivalent in a pseudometric space</u> (X, d). Much of the proof is developed in the following sequence of lemmas – some of which have intrinsic interest of their own.

Lemma 8.12 If X is first countable, then X is sequentially compact iff X is countably compact. (So, in particular, sequential and countable compactness are equivalent in pseudometric spaces.)

Proof We know from Theorem 8.11 that if X is sequentially compact, then X is countably compact. So assume X is countably compact and that (x_n) is a sequence in X. By Theorem 8.10, we know (x_n) has a cluster point, x. Since X is first countable, there is a subsequence$(x_{n_k}) \to x$ (*see Theorem III.10.6*). Therefore X is sequentially compact. •

Definition 8.13 A pseudometric space (X, d) is called <u>totally bounded</u> if, for each $\epsilon > 0$, X can be covered by a finite number of ϵ-balls. If $X \neq \emptyset$, this is the same as saying that for $\epsilon > 0$, there exist $x_1, x_2, ..., x_n \in X$ such that $X = B_\epsilon(x_1) \cup ... \cup B_\epsilon(x_n)$.

More informally, a totally bounded pseudometric space is one that can be kept under full surveillance using a finite number of policemen with an arbitrary degree of nearsightedness.

Example 8.14

1) Neither $\mathbb{R}$ nor $\mathbb{N}$ (with the usual metric) is totally bounded since neither can be covered by a finite number of 1-balls.

2) A compact pseudometric space (X, d) is totally bounded: for any $\epsilon > 0$, we can pick a finite subcover from the open cover $\mathcal{U} = \{B_\epsilon(x) : x \in X\}$.

Lemma 8.15 If (X, d) is countably compact, then (X, d) is totally bounded.

Proof If $X = \emptyset$, X is totally bounded, so assume $X \neq \emptyset$. If (X, d) is not totally bounded, then for some $\epsilon > 0$, no finite collection of ϵ-balls can cover X. Choose any point $x_1 \in X$. Then we can choose a point x_2 so that $d(x_1, x_2) \geq \epsilon$ (or else $B_\epsilon(x_1)$ would cover X).

We continue inductively. Suppose we have chosen points $x_1, x_2, ..., x_n$ in X such that $d(x_i, x_j) \geq \epsilon$ for each $i \neq j$, $i, j = 1, ..., n$. Then we can choose a point x_{n+1} at distance $\geq \epsilon$ from each of $x_1, ..., x_n$ – because otherwise the ϵ-balls centered at $x_1, ..., x_n$ would cover X.

The sequence (x_n) chosen in this way cannot have a cluster point because, for any x, $B_{\frac{\epsilon}{2}}(x)$ contains at most one x_n. Therefore X is not countably compact. •

Lemma 8.16 A totally bounded pseudometric space (X, d) is separable.

Proof For each n, choose a finite number of $\frac{1}{n}$-balls that cover X and let D_n be the (finite) set consisting of the centers of these balls. For any $y \in X$, $d(x, y) < \frac{1}{n}$ for some $x \in D_n$. This means that $D = \bigcup_{n=1}^{\infty} D_n$ is dense. Since D is countable, X is separable. •

Theorem 8.17 In a pseudometric space (X, d), the properties of compactness, countable compactness, sequential compactness and pseudocompactness are equivalent.

Proof We already have the implications from Theorem 8.11.

If (X, d) is countably compact, then (X, d) is totally bounded (*see Lemma 8.15*) and therefore separable (*see Lemma 8.16*). But a separable pseudometric space is Lindelöf *(see Theorem III.6.5*) and a countably compact Lindelöf space is compact. Therefore compactness and countable compactness are equivalent in (X, d).

We observed in Lemma 8.12 that countable compactness and sequential compactness are equivalent in (X, d).

Since a countably compact space is pseudocompact, we complete the proof by showing that if (X, d) is not countably compact, then (X, d) is not pseudocompact. This is the only part of the proof that takes some work. (*A bit of the maneuvering in the proof is necessary because d is a pseudometric. If d is actually a metric, some minor simplifications are possible.*)

If (X, d) is not countably compact, we can choose a sequence (x_n) with no cluster point (*see Theorem 8.10*). In fact, we can choose this (x_n) so that all the x_n's are distinct, and $d(x_m, x_n) > 0$ if $m \neq n$ (*why?*). We can then find open sets U_n such that i) $x_n \in U_n$, ii) $U_n \cap U_m = \emptyset$ for $m \neq n$, and so that iii) diam $U_n \to 0$ as $n \to \infty$.

> Here is a sketch for finding the U_n's; the details are left to check as an exercise. First check that for any $a, b \in X$ and positive reals r_1, r_2 : if $r_1 + r_2 < d(a, b)$ then $B_{r_1}(a)$ and $B_{r_2}(b)$ are disjoint – in fact, they have disjoint closures.
>
> Since (x_n) has no cluster point we can find, for each n, a ball $B_{\delta_n'}(x_n)$ that contains no other x_m – that is, $d(x_n, x_m) \geq \delta_n'$ for all $m \neq n$. Let $\delta_n = \delta_n'/2$. Then for $m \neq n$, we have $x_m \notin B_{\delta_n}(x_n)$ and $\delta_n < d(x_n, x_m)$. Of course, two of these balls

might overlap. To get the U_n's we want, we shrink these balls (choose smaller radii ϵ_n to replace the δ_n) to eliminate any overlap. We define the ϵ_n's inductively:

Let $\quad \epsilon_1 = \delta_1 < d(x_1, x_2)$.
Pick $\quad \epsilon_2 > 0$ so that

$$\begin{aligned} \epsilon_1 + \epsilon_2 &< d(x_1, x_2) \text{ and} \\ \epsilon_2 &< \delta_2 \end{aligned}$$

Since
$\epsilon_1 = \delta_1 < d(x_1, x_3)$ and
$\epsilon_2 < \delta_2 < d(x_2, x_3)$, we can

pick $\epsilon_3 > 0$ so that

$$\begin{aligned} \epsilon_1 + \epsilon_3 &< d(x_1, x_3) \text{ and} \\ \epsilon_2 + \epsilon_3 &< d(x_2, x_3) \text{ and} \\ \epsilon_3 &< \delta_3 \end{aligned}$$

Continue in this way. At the n^{th} step, we get a new ball $B_{\epsilon_n}(x_n)$ for which

if $m \neq n,\ x_m \notin B_{\epsilon_n}(x_n)$ (because $\epsilon_n < \delta_n$) and
$B_{\epsilon_n}(x_n) \cap B_{\epsilon_j}(x_j) = \emptyset$ for all $j < n$ (because $\epsilon_j + \epsilon_n < d(x_j, x_n)$)

Finally, when we choose ϵ_n at each step, we can also add the condition that $\epsilon_n < \frac{1}{n}$, so that $\operatorname{diam}(B_{\epsilon_n}(x_n)) \to 0$.

Then we can let $U_n = B_{\epsilon_n}(x_n)$.

For each n, define $f_n : X \to \mathbb{R}$ by $f_n(x) = \frac{nd(x, X - U_n)}{d(x_n, X - U_n)}$. Since $x_n \in U_n$ and $X - U_n$ is closed, the denominator of $f_n(x)$ is not 0. Therefore f_n is continuous, and $f_n(x_n) = n$. Define $f(x) = \sum_{n=1}^{\infty} f_n(x)$. (*For any $x \in X$, this series converges because x is in at most one U_n and therefore at most one term $f_n(x) \neq 0$.*) Then f is unbounded on X because $f(x_n) = f_n(x_n)$ $= n$. So we are done if we can show that f is continuous at each in X.

Let $a \in X$. We claim that there is an open set V_a containing a such that $V_a \cap U_n \neq \emptyset$ for at most finitely many n's.

If $d(a, x_n) = 0$ for some x_n, we can simply let $V_a = U_n$.

Suppose $d(a, x_n) > 0$ for all n. Since a is not a cluster point of (x_n), we can choose $\epsilon > 0$ so that $B_\epsilon(a)$ contains none of the x_n's. For this ϵ, choose N so that if $n > N$, then diam $U_n < \frac{\epsilon}{2}$. Then $B_{\frac{\epsilon}{2}}(a) \cap U_n = \emptyset$ for $n > N$ (*if $n > N$ and $z \in B_{\frac{\epsilon}{2}}(a) \cap U_n$, then* $d(a, x_n) \leq d(a, z) + d(z, x_n) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$). Then let $V_a = B_{\frac{\epsilon}{2}}(a)$.

Then $f_n | V_a$ is identically 0 for all but finitely many n. Therefore $f | V_a$ is really only a finite sum of continuous functions, so $f | V_a : V_a \to \mathbb{R}$ is continuous. Suppose W is any neighborhood of $f(a)$. Then there is an open set U <u>in V_a</u> containing a for which $(f|V_a)[U] \subseteq W$. But V_a is open in X, so U is also open in X, and $f[U] = (f|V_a)[U] \subseteq W$. So f is continuous at a. $\bullet$

We now explore some properties of compact metric spaces.

Theorem 8.18 Suppose A is a compact subset of a metric space (X, d). Then A is closed and bounded (that is, A has finite diameter).

Proof A compact subset of the Hausdorff space must be closed (*see Theorem 8.5b*).

If $A = \emptyset$, then A is certainly bounded. If $A \neq \emptyset$, choose a point $a_0 \in A$ and let $f : A \to \mathbb{R}$ by $f(a) = d(a, a_0)$. Then f is continuous and f is bounded (since X is pseudocompact): say $|f(x)| < M$ for all $x \in A$. But that means that $A \subseteq B_M(a_0)$, so A is a bounded set . ∙

<u>Caution: the converse of Theorem 8.18 is false</u>. For example, suppose d is the discrete unit metric on an infinite set X. Every subset of (X, d) is both closed and bounded, but an infinite subspace of X is not compact. However, the converse to Theorem 8.18 <u>is true in $\mathbb{R}^n$</u>, as you may remember from analysis.

Theorem 8.19 Suppose $A \subseteq \mathbb{R}^n$. The A is compact iff A is closed and bounded.

Proof Because of Theorem 8.18, we only need to prove that if A is closed and bounded, then A is compact. First consider the case $n = 1$. Then A is a closed subspace of some interval $I = [a, b]$ and it is sufficient to show that I is compact. To do this, let (x_n) be a sequence in I and choose (*see Lemma 2.10*) a monotone subsequence (x_{n_k}). Since (x_{n_k}) is bounded, we know that $(x_{n_k}) \to$ some point $r \in I$ (*see Lemma 2.9*). Since (x_n) has a convergent subsequence, I is sequentially compact and therefore compact.

When $n > 1$, the proof is similar. We illustrate for $n = 2$. If A is a closed bounded set in $\mathbb{R}^2$, then A is a closed subspace of some closed box of the form $I = [a, b] \times [c, d]$. Therefore it is sufficient to prove that I is compact. To do this, choose a sequence (x_n, y_n) in I. Since $[a, b]$ is compact, the sequence (x_n) has a subsequence (x_{n_k}) that converges to some point $x \in [a, b]$. Now consider the subsequence (x_{n_k}, y_{n_k}) in I. Since $[c, d]$ is compact, the sequence (y_{n_k}) has a subsequence $(y_{n_{k_l}})$ that converges to some point $y \in [c, d]$. But in $\mathbb{R}^2$, $(u_n, v_n) \to (u, v)$ iff $(u_n) \to u$ and $(v_n) \to v$ in $\mathbb{R}$. So $(x_{n_{k_l}}, y_{n_{k_l}}) \to (x, y) \in I$. This shows that I is sequentially compact and therefore compact.

For $n > 2$, a similar argument clearly works – it just involves "taking subsequences" n times. ∙

9. Compactness and Completeness

We already know that a compact metric space (X, d) is complete. Therefore all the "big" theorems that we proved for complete spaces are true in a compact metric space (X, d) – for example Cantor's Intersection Theorem, the Contraction Mapping Theorem and the Baire Category Theorem (which implies that a nonempty compact metric space is second category in itself).

Of course, a complete space (X, d) need not be compact: $\mathbb{R}$, for example. We want to see the exact relationship between compactness and completeness.

We begin with a couple of preliminary results.

Theorem 9.1 If (X, d) is totally bounded, then (X, d) is bounded.

Proof For $\epsilon = 1$, pick points $x_1, ..., x_n$ so that $X = B_1(x_1) \cup ... \cup B_1(x_n)$. Let $M =$ max $\{d(x_i, x_j) : i, j = 1, ..., n\}$. If $x, y \in X$, then we have $x \in B_1(x_i)$ and $y \in B_1(x_j)$ for some i, j , so that

$$d(x, y) \leq d(x, x_i) + d(x_i, y_j) + d(y_j, y) < 1 + M + 1 = M + 2.$$

Therefore (X, d) has finite diameter – that is, (X, d) is bounded. •

Notice that the converse to the theorem is false: "total boundedness" is stronger than "boundedness." For example, let $d'(x, y) = \min\{1, |x - y|\}$ on $\mathbb{R}$. Then diam $(\mathbb{R}, d') = 1$ so $(\mathbb{R}, d')$ is bounded. Because d' and the usual metric d agree for distances smaller that 1, $B_1^{d'}(x) = (x - 1, x + 1)$. Since $\mathbb{R}$ cannot be covered by a finite number of these 1-balls, $(\mathbb{R}, d')$ is not totally bounded.

Theorem 9.2 If (X, d) is totally bounded and $A \subseteq X$, then (A, d) totally bounded: that is, a subspace of a totally bounded space is totally bounded.

Proof Let $\epsilon > 0$ and choose $x_1, ..., x_n$ so that $X = B_{\frac{\epsilon}{2}}(x_1) \cup ... \cup B_{\frac{\epsilon}{2}}(x_n)$. Of course these balls also cover A. But to show that (A, d) is totally bounded, we need to show that we can cover A with a finite number of ϵ-balls <u>centered at points in A</u>.

Let $J = \{j : B_{\frac{\epsilon}{2}}(x_j) \cap A \neq \emptyset\}$ and, for each $j \in J$, pick $a_j \in B_{\frac{\epsilon}{2}}(x_j) \cap A$. We claim that the balls $B_\epsilon^A(a_j)$ cover A. In fact, if $a \in A$, then $a \in B_{\frac{\epsilon}{2}}(x_j)$ for some j; and for this j, we also have $a_j \in B_{\frac{\epsilon}{2}}(x_j)$. Then $d(a, a_j) \leq d(a, x_j) + d(x_j, a_j) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$, so $a \in B_\epsilon^A(a_j)$. •

The next theorem gives the exact connection between compactness and completeness. It is curious because it states that "compactness" (a topological property) is the "sum" of two nontopological properties.

Theorem 9.3 (X,d) is compact iff (X,d) is complete <u>and</u> totally bounded.

Proof $(\Rightarrow)$ We have already seen that the compact space (X,d) is totally bounded and complete. (*For completeness, here is a fresh argument: let* (x_n) *be a Cauchy sequence in* (X,d). *Since* X *is countably compact,* (x_n) *has a cluster point* x *in* X. *But a Cauchy sequence must converge to a cluster point. So* (X,d) *is complete.*)

$(\Leftarrow)$ Let (x_n) be a sequence in (X,d). We will show that (x_n) has a Cauchy subsequence which (by completeness) must converge. That means (X,d) is sequentially compact and therefore compact. Without loss of generality, we can assume that the terms of the sequence (x_n) are distinct (*why?*).

Since (X,d) is totally bounded, we can cover X with a finite number of 1-balls, and one of them – call it B_1 – must contain infinitely many x_n's. Since (B_1,d) is totally bounded (*see Theorem 9.2*) we can cover B_1 with a finite number of $\frac{1}{2}$-balls and one of them – call it $B_2 \subseteq B_1$ – must contain infinitely many x_n's.

Continue inductively in this way. At the induction step, suppose we have $\frac{1}{k}$-balls $B_1 \supseteq B_2 \supseteq \ldots \supseteq B_k$ where each ball contains infinitely many x_n's. Since (B_k,d) is totally bounded, we can cover it with a finite number of $\frac{1}{k+1}$-balls, one of which – call it $B_{k+1} \subseteq B_k$ – must contain infinitely many x_n's. This inductively defines an infinite descending sequence of balls $B_1 \supseteq B_2 \supseteq \ldots \supseteq B_k \supseteq \ldots$ where each B_k contains x_n for infinitely many values of n.

Then we can choose

$$\begin{aligned} &x_{n_1} \in B_1 \\ &x_{n_2} \in B_2, \text{ with } n_2 > n_1 \\ &\quad\vdots \\ &x_{n_k} \in B_k, \text{ with } n_k > n_{k-1} > \ldots > n_1 \\ &\quad\vdots \end{aligned}$$

The subsequence (x_{n_k}) is Cauchy since $x_{n_l} \in B_k$ for $l \geq k$ and diam $B_k \to 0$. •

Example 9.4 With the usual metric d:

i) $\mathbb{N}$ is complete and not compact (and therefore not totally bounded).
ii) $\{\frac{1}{n} : n \in \mathbb{N}\}$ is totally bounded, because it is a subspace of the totally bounded space $[0,1]$. But it is not compact (and therefore not complete).
iii) $\mathbb{N}$ and $\{\frac{1}{n} : n \in \mathbb{N}\}$ are homeomorphic because both are countable discrete spaces. Total boundedness is not a topological property.
iv) $\{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ is compact and is therefore both complete and totally bounded.

During the earlier discussion of the Contraction Mapping Theorem, we defined uniform continuity. For convenience, the definition is repeated here.

Definition 9.5 A function $f : (X,d) \to (Y,d')$ is uniformly continuous if

$$\forall \epsilon\, \exists \delta\, \forall x\, \forall y \in X \;\; (d(x,y) < \delta \Rightarrow d'(f(x),f(y)) < \epsilon)$$

Clearly, if $X = \{x_1, ..., x_n\}$ is finite, then every continuous function $f : (X, d) \to (Y, s)$ is uniformly continuous. (*For $\epsilon > 0$ and each $i = 1, ..., n$, we pick the the δ_i that works at x_i in the definition of continuity. Then $\delta = min\,\{\delta_1, ...\delta_n\}$ works "uniformly" across the space X.*)

The next theorem generalizes this observation to compact metric spaces (X, d) – and illustrates once more "rule of thumb" that "compact spaces act like finite spaces."

Theorem 9.6 If (X, d) is compact and $f : (X, d) \to (Y, d')$ is continuous, then f is uniformly continuous.

Proof Suppose f is not uniformly continuous. Then for some $\epsilon > 0$, "no δ works". In particular, for that ϵ, $\delta = \frac{1}{n}$ "doesn't work" – therefore we can find, for each n, a pair of points u_n and v_n with $d(u_n, v_n) < \frac{1}{n}$ but $d'(f(u_n), f(v_n)) \geq \epsilon$. Since X is compact, there is a subsequence $(u_{n_k}) \to x \in X$. It follows that the corresponding subsequence $(v_{n_k}) \to x$ also because $d(v_{n_k}, x) \leq d(v_{n_k}, u_{n_k}) + d(u_{n_k}, x) < \frac{1}{n_k} + d(x_{n_k}, x) \to 0$. Therefore, by continuity, we should have $(f(u_{n_k})) \to f(x)$ and also $(f(v_{n_k})) \to f(x)$. But this is impossible since $d'(f(u_{n_k}), f(v_{n_k})) \geq \epsilon$ for all k. ∙

Uniform continuity is a strong and useful condition. For example, the preceding theorem implies that a continuous function $f : [a, b] \to \mathbb{R}$ must be uniformly continuous. This is one of the reasons why in calculus "continuous functions on closed intervals" are so nice to work with. The following example is a simple illustration of how uniform continuity might be used in a simple analysis setting.

Example 9.7 Suppose $f : [a, b] \to \mathbb{R}$ be bounded. Choose any partition of $[a, b]$

$$x_0 = a < x_1 < ... < x_{i-1} < x_i < ... < x_n = b$$

and let M_i and m_i denote the sup and inf of the set $f[\,[x_{i-1}, x_i]\,]$. Let $\Delta_i x = x_i - x_{i-1}$. The sums

$$\sum_{i=1}^{n} m_i \Delta_i x \;\leq\; \sum_{i=1}^{n} M_i \Delta_i x$$

are called the lower and upper sums for f associated with this partition.

It is easy to see that the sup of the lower sums (over all possible partitions) is finite: it is called the <u>lower integral of</u> f <u>on</u> $[a, b]$ and denoted $\underline{\int_a^b} f(x)\,dx$. Similarly, the inf of all the upper sums is finite: it is called the <u>upper integral of</u> f <u>on</u> $[a, b]$ and denoted $\overline{\int_a^b} f(x)\,dx$. It is easy to verify that $\underline{\int_a^b} f(x)\,dx \leq \overline{\int_a^b} f(x)\,dx$. If the two are actually <u>equal</u>, we say f is (Riemann) integrable on $[a, b]$ and we write $\int_a^b f(x)\,dx = \underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx$.

Uniform continuity is just what we need to prove that a continuous f on $[a, b]$ is integrable. If f is continuous on $[a, b]$, then f is uniformly continuous. Therefore, for $\epsilon > 0$, we can choose $\delta > 0$ such that $|f(x) - f(y)| < \frac{\epsilon}{b-a}$ whenever $|x - y| < \delta$. Pick a partition of $[a, b]$ for which each $\Delta_i x < \delta$. Then for each i, $M_i - m_i \leq \frac{\epsilon}{b-a}$. Therefore

$$\overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx \leq \sum_{i=1}^{n} M_i \Delta_i x - \sum_{i=1}^{n} m_i \Delta_i x$$
$$= \sum_{i=1}^{n} (M_i - m_i)\,\Delta_i x < \epsilon \sum_{i=1}^{n} \Delta_i x = \frac{\epsilon}{b-a}(b-a) = \epsilon$$

Since $\epsilon > 0$ was arbitrary, we conclude that $\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx$.

10. The Cantor Set

The Cantor set is an example of a compact subspace of $\mathbb{R}$ with a surprising combination of properties. Informally, we can construct the Cantor set C as follows. Begin with the closed interval $[0, 1]$ and delete the open "middle third." The remainder is the union of 2 disjoint closed intervals: $[0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. At the second stage of the construction, we then delete the open middle thirds of each interval – leaving a remainder which is the union of 2^2 disjoint closed intervals: $[0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$. We repeat this process of deleting the middle thirds "forever." The Cantor set C is the set of survivors – the points that are never discarded.

Clearly there <u>are</u> some survivors: for example, the endpoint of a deleted middle third – for example, $\frac{1}{9}$ – clearly survives forever and ends up in C.

To make this whole process precise, we name the closed subintervals that remain after each stage. Each interval contains the 2 new remaining intervals below it:

$[0, \frac{1}{3}]$ (F_0)				$[\frac{2}{3}, 1]$ (F_2)			
$[0, \frac{1}{9}]$ (F_{00})		$[\frac{2}{9}, \frac{1}{3}]$ (F_{02})		$[\frac{2}{3}, \frac{7}{9}]$ (F_{20})		$[\frac{8}{9}, 1]$ (F_{22})	
$[0, \frac{1}{27}]$ (F_{000})	$[\frac{2}{27}, \frac{1}{9}]$ (F_{002})	$[\frac{2}{9}, \frac{7}{27}]$ (F_{020})	$[\frac{8}{27}, \frac{1}{3}]$ (F_{022})	$[\frac{2}{3}, \frac{19}{27}]$ (F_{200})	$[\frac{20}{27}, \frac{7}{9}]$ (F_{202})	$[\frac{8}{9}, \frac{25}{27}]$ (F_{220})	$[\frac{26}{27}, 1]$ (F_{222})
etc.				etc.			

At the k^{th} stage the set remaining is the union 2^k closed subintervals, each with length $\frac{1}{3^k}$. We label them $F_{n_1 n_2 \ldots n_k}$ where $(n_1, n_2, \ldots n_k) \in \{0, 2\}^k$.

Notice that, for example, $F_0 \supseteq F_{02} \supseteq F_{020} \supseteq F_{0200} \supseteq F_{02002} \supseteq \ldots$. As we go down the chain "toward" the Cantor set, each new 0 or 2 in the subscript indicates whether the next set down is "the remaining left interval" or "the remaining right subinterval."

For each sequence $s = (n_1, n_2, \ldots, n_k, \ldots) \in \{0, 2\}^{\mathbb{N}}$, we have

$$F_{n_1} \supseteq F_{n_1 n_2} \supseteq \ldots \supseteq F_{n_1 n_2 \ldots n_k} \supseteq \ldots$$

Since $[0, 1]$ is complete, the Cantor Intersection Theorem gives that $\bigcap_{k=1}^{\infty} F_{n_1 n_2, \ldots n_k}$ contains a <u>single point</u> of $[0, 1]$, and we call that point x_s. The Cantor set C is then defined by

$C = \{x_s : s \in \{0,2\}^{\mathbb{N}}\} \subseteq [0,1]$.

If $s \neq s' = (n'_1, n'_2, ..., n'_k, ...) \in \{0,2\}^{\mathbb{N}}$, we can look at the <u>first</u> k for which $n_k \neq n'_k$. Then $x_s \in F_{n_1...n_{k-1}n_k}$ and $x_{s'} \in F_{n_1...n_{k-1},n'_k}$. Since these closed intervals are disjoint, we conclude $x_s \neq x_{s'}$. Therefore each sequence $s \in \{0,2\}^{\mathbb{N}}$ corresponds to a different point in C, so $|C| \geq |\{0,2\}^{\mathbb{N}}| = 2^{\aleph_0} = c$. Since $C \subseteq [0,1]$, we conclude $|C| = c$.

Each $x \in [0,1]$ can be written using a "ternary decimal" expansion $x = \sum_{n=1}^{\infty} \frac{t_n}{3^n} = 0.t_1t_2...t_k..._{\text{three}}$, where each $t_k \in \{0,1,2\}$. Just as with base 10 decimals, the expansion for a particular x may not be unique: for example, $\frac{1}{3} = 0.1000..._{\text{three}}$ but also $\frac{1}{3} = 0.022222..._{\text{three}}$. It's easy to check that an $x \in [0,1]$ has two different ternary expansions iff x is the endpoint of one of the deleted "open middle thirds" in the construction. Therefore it is easy to see that a point is in C iff it <u>has</u> a ternary expansion involving <u>only</u> 0's and 2's. For example, $\frac{1}{3} \in C$ and $\frac{1}{2} \notin C$.

What are some of the properties of C ?

i) $[0,1] - C$ is the union of the deleted open "middle third" intervals, so $[0,1] - C$ is open in $[0,1]$. Therefore C is closed in $[0,1]$, so C is a compact metric space (and therefore complete in the usual metric).

ii) Every point in C is a limit point of C – that is, C has no isolated points. (*Note: such a space is sometimes called <u>dense-in-itself</u>. This is an awkward but well-established term; it is awkward because it means something different from the obvious fact that every space X is a dense subset of itself.*)

Suppose $x_s \in C$, where $s = (n_1, n_2, ..., n_k, n_{k+1}, n_{k+2}...) \in \{0,2\}^{\mathbb{N}}$. Given $\epsilon > 0$, pick k so $\frac{1}{3^k} < \epsilon$. Define n'_{k+1} ($= 0$ or 2) so that $n'_{k+1} \neq n_{k+1}$, and let

$$s' = (n_1, n_2, ..., n_k, n'_{k+1}, n_{k+2}...) \in \{0,2\}^{\aleph_0}.$$

Then x_s and $x_{s'}$ are distinct points in C, but both are in $F_{n_1,...n_k}$ and diam $F_{n_1,...n_k} = \frac{1}{3^k} < \epsilon$. Therefore $|x_s - x_{s'}| < \epsilon$, so x_s is not isolated in C.

iii) With the usual metric d, C is a nonempty complete metric space with no isolated points. It follows from Theorem 3.6 that $|C| \geq c$ and, since $C \subseteq [0,1]$, then $|C| = c$. However, we can also see this from the discussion in ii): there are $2^{\aleph_0} = c$ different ways we could redefine the infinite "tail" $(n_{k+1}, ...n_{k+l}, ...)$ of s – and each version produces a different point in C at distance $< \epsilon$ from x_s.

iv) In some ways, C is "big." For example, the Cantor set has as many points as $[0,1]$. Also, since C is complete with the usual metric, C is second category in itself. But C is "small" in other ways. For one thing, C is nowhere dense in $[0,1]$ (and therefore in $\mathbb{R}$). Since C is closed, this simply means that C contains no nonempty open interval.

To see this, suppose $I = (a,b) \subseteq C$. At the k^{th} stage of the construction, we must have $I \subseteq$ <u>exactly one</u> of the $F_{n_1,...n_k}$. (This follows immediately from the definition of an interval.) Therefore $b - a < \frac{1}{3^k}$ for every k, so $a = b$ and $I = \emptyset$.

v) C is also "small" in another sense. In the construction of C, the open intervals we delete have total length $\frac{1}{3} + 2\left(\frac{1}{9}\right) + 4\left(\frac{1}{27}\right) + ... = \frac{\frac{1}{3}}{1-\frac{2}{3}} = 1$. In some sense, the "length" of what remains is 0 ! This is made precise in measure theory, where we say that $[0,1] - C$ has <u>measure 1</u> and that C has <u>measure 0</u>. Measure is yet another way, differing from both cardinality and category, to measure the "size" of a set.

The following theorem states that the topological properties of C which we have discussed actually <u>characterize</u> the Cantor set. The theorem can be used to prove that the "generalized Cantor set" resulting from any of these modified constructions is topologically the same as C.

Theorem 10.1 Suppose A is a nonempty compact subset of $\mathbb{R}$ which is dense-in-itself and contains no nonempty open interval. Then A is homeomorphic to the Cantor set C.

Proof The proof is omitted. (See Willard, *General Topology*.)

It is possible to modify the construction of C by changing "middle thirds" to, say, "middle fifths," or even by deleting "middle thirds" at stage one, "middle fifths" at stage two, etc. The resulting "generalized Cantor set" is homeomorphic to C (using Theorem 10.1) but the total lengths of the deleted open intervals may be different: that is, the result is can be a topological Cantor set with positive measure. "Measure" is not a topological property.

As we noted earlier, C consists of those points $x \in [0,1]$ for which we can write

$$x = \sum_{n=1}^{\infty} \frac{t_n}{3^n} = 0.t_1t_2...t_k..._{\text{three}}, \text{ with each } t_n \in \{0,2\}.$$

We can obviously rewrite this as $x = \sum_{n=1}^{\infty} \frac{2b_n}{3^n}$, where each $b_n \in \{0,1\}$ and therefore we can define a function $g : C \to [0,1]$ as follows:

$$\text{for } x \in C,\ g(x) = g\left(\sum_{n=1}^{\infty} \frac{2b_n}{3^n}\right) = \sum_{n=1}^{\infty} \frac{b_n}{2^n}.$$

More informally,:

$$g(0.\,2b_1\,2b_2\,2b_3\,..._{\text{three}}) = 0.b_1b_2b_3..._{\text{two}}$$

This mapping g is <u>not</u> one-to-one . For example,

$$g(\tfrac{7}{9}) = g(0.202222...._{\text{three}}) = 0.101111..._{\text{two}} = \tfrac{3}{4} \text{ and}$$
$$g(\tfrac{8}{9}) = g(0.220000..._{\text{three}}) = 0.11000..._{\text{two}} = \tfrac{3}{4}$$

In fact, for $a, b \in C$, we have $g(a) = g(b)$ iff the interval (a,b) is one of the "deleted middle thirds." It should be clear that g is continuous and that if $a < b$ in C, then $g(a) \leq g(b)$ – that is, g is weakly increasing. (*Is g onto?*)

The function g is not defined on $[0,1] - C$. But $[0,1] - C$ is, by construction, a union of disjoint open intervals. We can therefore extend the definition of g to a mapping $G : [0,1] \to [0,1]$ in the following simple-minded way:

$$G(x) = \begin{cases} g(x) & \text{if } x \in C \\ g(a) & \text{if } x \in (a,b), \text{ where } (a,b) \text{ is a "deleted middle third"} \end{cases}$$

Since we know $g(a) = g(b)$, this amounts to extending the graph of g over each "deleted middle third" (a,b) by using a horizontal line segment from $(a, g(a))$ to $(b, g(b))$. The result is the graph of the continuous function G.

G is sometimes called the Cantor-Lebesgue function. It satisfies:

i) $G(0) = 0,\ G(1) = 1$
ii) G is continuous
iii) G is (weakly) increasing
iv) at any point x in a "deleted middle third" (a,b), G is differentiable and $G'(x) = 0$.

Recall that C has "measure 0." Therefore we could say "$G'(x) = 0$ almost everywhere" $-$ even though G rises monotonically from 0 to 1!

Note: The technique that we used to extend g works in a similar way for any closed set $F \subseteq \mathbb{R}$ and any continuous function $g : F \to \mathbb{R}$. Since $\mathbb{R} - F$ is open in $\mathbb{R}$, we know that we can write $\mathbb{R} - F$ as the union of a countable collection of disjoint open intervals I_n which have for (a,b), $(-\infty, b)$ or (a, ∞) We can extend f to a continuous function $F : \mathbb{R} \to \mathbb{R}$ simply by extending the graph of f over linearly over each I_n :

if $I_n = (a,b)$, then let the graph F over I_n be the straight line segment joining $(a, f(a),\ to\ (b, f(b))$.
if $I_n = (-\infty, b)$, then let F have the constant value $f(b\ on\ I_n$
if $I_n = (a, \infty)$, then let F have the constant value $f(a)$ on I_n,

There is a much more general theorem that implies that whenever F is a closed subset of (X,d), then each $f \in C(F)$ and be extended to a function $F \in C(X)$. In the particular case $X = \mathbb{R}$, proving this was easy because $\mathbb{R}$ is ordered and we completely understand the structure of the open sets in $\mathbb{R}$..

Another curious property of C, mentioned without proof, is that its "difference set" $\{x - y : x, y \in C\} = [-1, 1]$. Although C has measure 0, the difference set in this case has measure 2!

Exercises

E21. Suppose X is compact and that $f: X \to Y$ is continuous and onto. Prove that Y is compact. ("*A continuous image of a compact space is compact.*")

E22. a) Suppose that X is compact and Y is Hausdorff. Let $f: X \to Y$ be a continuous bijection. Prove that f is a homeomorphism.

b) Let $\mathcal{T}$ be the usual topology on $[0,1]$ and suppose $\mathcal{T}_1$ and $\mathcal{T}_2$ are two other topologies on $[0,1]$ such that $\mathcal{T}_1 \underset{\neq}{\subset} \mathcal{T} \underset{\neq}{\subset} \mathcal{T}_2$ Prove that $([0,1], \mathcal{T}_1)$ is not Hausdorff and that $([0,1], \mathcal{T}_2)$ is not compact. (*Hint: Consider the identity map* $i: [0,1] \to [0,1]$.)

c) Is part b) true if $[0,1]$ is replaced by an arbitrary compact Hausdorff space $(X, \mathcal{T})$?

E23. Prove that a nonempty space $(X, \mathcal{T})$ is pseudocompact iff every continuous $f: \mathbb{R} \to \mathbb{R}$ achieves both a maximum and a minimum value.

E24. Suppose $f: (X,d) \to (Y,d')$. Prove that f is continuous iff $f|K$ is continuous for every compact set $K \subseteq X$.

E25. Suppose that A and B are nonempty disjoint closed sets in (X,d) and that A is compact. Prove that $d(A,B) > 0$.

E26. Let X and Y be topological spaces.

a) Prove that X is compact iff every open cover by <u>basic</u> open sets has a finite subcover.

b) Suppose $X \times Y$ is compact. Prove that if $X, Y \neq \emptyset$, then X and Y are compact. (*By induction, a similar statement applies to any finite product.*)

c) Prove that if X and Y are compact, then $X \times Y$ is compact. (*By induction, a similar statement applies to any finite product.*)
(*Hint: for any* $x \in X$, $\{x\} \times Y$ *is homeomorphic to* Y. *Part a) is also relevant.*)

d) Point out explicitly why the proof in c) cannot be altered to prove that a product of two countably compact spaces is countably compact. (*An example of a countably compact space* X *for which* $X \times X$ *is not even pseudocompact is given in Chapter X, Example 6.8.*)

Note: In fact, an arbitrary product of compact spaces is compact. This is the "Tychonoff Product Theorem" which we will prove later.

It <u>is</u> true that the product of a countably compact space and a compact space is countably compact. You might trying proving this fact. A very similar proof shows that a product of a *compact space and a Lindelöf space is Lindelöf. That proof does* <u>*not*</u> *generalize, however, to show that a product of two Lindelöf spaces is Lindelöf. Can you see why?*

E27. Prove that if (X, d) is a compact metric space and $|X| = \aleph_0$, then X has infinitely many isolated points.

E28. Suppose X is any topological space and that Y is compact. Prove that the projection map $\pi_X : X \times Y \to X$ is closed. (*A projection "parallel to a compact factor" is closed.*)

E29. Let X be an uncountable set with the discrete topology $\mathcal{T}$. Prove that there does not exist a totally bounded metric *d* on X such that $\mathcal{T}_d = \mathcal{T}$.

E30. a) Give an example of a metric space (X, d) which is totally bounded and an isometry from (X, d) into (X, d) which is not onto.
Hint: on the circle S^1, start with a point p and keep rotating it around the boundary by increments of some angle β.

b) Prove that if (X, d) is totally bounded and if f is an isometry from (X, d) into (X, d), then $f[X]$ must be dense in (X, d).
Hint: Given x, $\epsilon > 0$, cover X with finitely many $\frac{\epsilon}{2}$ spheres; if the sequence $x, f(x), f(f(x))$,... consists of distinct terms, there must be infinitely many of them in one sphere.

c) Show that a compact metric space cannot be isometric to a proper subspace of itself.
Hint: you might use part b).

d) Prove that if each of two compact metric spaces is isometric to a subspace of the other, then the two spaces are isometric to each other.
Note: Part d) is an analogue of the Cantor-Schroeder-Bernstein Theorem for compact metric spaces.

E31. Suppose (X, d) is a metric space and that (X, d') is totally bounded for every metric $d' \sim d$. Must (X, d) be compact?

E32. Let $\mathcal{U}$ be an open cover of the compact metric space (X, d). Prove that there exists a constant $\delta > 0$ such that for all x: $B_\delta(x) \subseteq U$ for some $U \in \mathcal{U}$. (*The number δ is called a* <u>*Lebesgue number*</u> *for $\mathcal{U}$.*)

E33. Suppose (X, d) is a metric space with no isolated points. Prove that X is compact iff $d(A, B) > 0$ for every pair of disjoint closed sets $A, B \subseteq X$.

Chapter IV Review

Explain why each statement is true, or provide a counterexample.

1. Let A denote the set of fixed points of a continuous function $f : X \to X,$ where X is a Hausdorff space. A is closed in X.

2. Let S denote the set of all Cauchy sequences in $\mathbb{Q}$ which converge to a point in $\mathbb{R}$. Then $|\, S \,| = c\,.$

3. For a metric space (X, d), if every continuous function $f : X \to \mathbb{R}$ assumes a minimum value, then every infinite set in X has a limit point.

4. There exists a continuous function $f : \mathbb{R} \to \mathbb{R}$ for which the Cantor set is the set of fixed points of f.

5. If X has the cofinite topology, then every subspace is pseudocompact.

6. Let $[0, 1]$ have the subspace topology from the Sorgenfrey line. Then $[0, 1]$ is countably compact.

7. Let $\mathfrak{S} = \{A \subseteq \mathbb{R} : A$ is first category in $\mathbb{R}\}$, and let $\mathcal{T}$ be the topology on $\mathbb{R}$ for which $\mathfrak{S}$ is a subbase. Then $(\mathbb{R}, \mathcal{T})$ is a Baire space.

8. Every sequence in $[0, 1]^2$ has a Cauchy subsequence.

9. If A is a dense subspace of (X, d) and every Cauchy sequence in A converges to some point in X, then (X, d) is complete.

10. If $\mathcal{T}$ is the cofinite topology on $\mathbb{N}$, then $(\mathbb{N}, \mathcal{T})$ has the fixed point property.

11. Every sequence in $[0, 1)$ has a Cauchy subsequence.

12. Let $Z = (0, \frac{1}{3}) \cap C$, where C is the Cantor set. Then Z is completely metrizable.

13. The Sorgenfrey plane is countably compact.

14. If every sequence in the metric space (X, d) has a convergent subsequence, then every continuous real valued function on X must have a minimum value.

15. Suppose (X, d) is complete and $f : X \to X$. The set $C = \{x \in \mathbb{R} : f$ is continuous at $x\}$ is second category in itself.

16. In the space $C([0, 1])$, with the metric ρ of uniform convergence, the subset of all polynomials is first category.

17. A nonempty open set in $(X, \mathcal{T})$ cannot be nowhere dense.

18. Every nonempty nowhere dense subset of $\mathbb{R}$ contains an isolated point.

19. There are exactly c nowhere dense subsets of $\mathbb{R}$.

20. Suppose B is nonempty subset of $\mathbb{Q}$ with no isolated points. B cannot be completely metrizable.

21. Suppose (X, d) is a countable complete metric space. If $A \subseteq X$, then (A, d) may not be complete, but A is completely metrizable.

22. If A is first category in X and $B \subseteq A$, then B is first category in A.

23. There are c different metrics d on $\mathbb{Q}$, each equivalent to the usual metric, for which $(\mathbb{Q}, d)$ is complete.

24. In $\mathbb{N}$, with the cofinite topology, every infinite subset is sequentially compact.

25. Suppose $f\colon [0,1]^2 \to \mathbb{R}$ and $A = \{p \in (0,1)^2\colon\ \frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ both exist at $p\}$. Then A, with its usual metric, is totally bounded.

26. A space $(X, \mathcal{T})$ is a Baire space iff X is second category in itself.

27. Let E denote the subset of the Cantor set C consisting of the endpoints of the open intervals deleted from $[0,1]$ in the construction of C. Suppose $f\colon [0,1] \to \mathbb{R}$ is continuous and that $f[E] \subseteq [0, 0.1]$. Then $f[C] \subseteq [0, 0.1]$.

28. Let d be a metric on the irrationals $\mathbb{P}$ which is equivalent to the usual metric and such that $(\mathbb{P}, d)$ is complete. Then $(\mathbb{P}, d)$ must be totally bounded.

29. Suppose (X, d) is a nonempty complete metric space and A is a closed subspace such that (A, d) is totally bounded. Then every sequence in A has a cluster point.

30. There is a metric d on $[0,1]$, equivalent to the usual metric, such that $([0,1], d)$ is not complete.

31. Let $Z = [0,1]^2$ have the usual topology. Every nonempty closed subset of Z is second category in itself.

32. Suppose $f : [0,1]^n \to \mathbb{R}^m$ and $C = \{x \in [0,1]^n :\ f$ is continuous at $x\}$. There is a metric d equivalent to the usual metric on C for which (C, d) is complete.

33. If the continuum hypothesis (CH) is true, then $\mathbb{R}$ cannot be written as the union of fewer than c nowhere dense sets.

34. Suppose A is a complete subspace of $[0,1]$. Then A is a G_δ set in $\mathbb{R}$.

35. There are 2^c different subsets of $\mathbb{R}$ none of which contains an interval of positive length.

36. Suppose (X, d) is a nonempty complete metric space and A is a closed subspace such that

(A, d) is totally bounded. Then every sequence in A has a cluster point.

37. Let d' be a metric on the irrationals $\mathbb{P}$ which is equivalent to the usual metric and for which $(\mathbb{P}, d')$ is totally bounded. Then $(\mathbb{P}, d')$ cannot be complete.

38. The closure of a discrete subspace of $\mathbb{R}$ may be uncountable.

39. Suppose $f : [0,1]^n \to \mathbb{R}^m$ and $C = \{x \in [0,1]^n : f \text{ is continuous at } x\}$. Then there is a metric d on C, equivalent to the usual metric, such that (C, d) is totally bounded.

40. Suppose $(X, \mathcal{T})$ is compact and $f : X \to \mathbb{R}$ is a continuous function such that $f(x) > 0$ for all $x \in X$. Then there is an $\epsilon > 0$ such that $f(x) > \epsilon$ for all $x \in X$.

41. If every point of (X, d) is isolated, then (X, d) is complete.

42. If (X, d) has no isolated points, then its completion has no isolated points.

43. Suppose that for each point $x \in (X, d)$, there is an open set U_x such that (U_x, d) is complete. Then there is a metric $d' \sim d$ on X such that (X, d') is complete.

44. For a metric space (X, d) with completion $(\widetilde{X}, \widetilde{d})$, it can happen that $|\widetilde{X} - X| = \aleph_0$.

45. Let d be the usual metric on $\mathbb{Q}$. There is a completion $(\widetilde{\mathbb{Q}}, \widetilde{d})$ of $(\mathbb{Q}, d)$ for which $|\widetilde{\mathbb{Q}} - \mathbb{Q}| = \aleph_0$.

46. A subspace of $\mathbb{R}$ is bounded iff it is totally bounded.

47. A countable metric space must have at least one isolated point.

48. The intersection of a sequence of dense open subsets in $\mathbb{P}$ must be dense in $\mathbb{P}$.

49. If C is the Cantor set, then $\mathbb{R} - C$ is an F_σ set in $\mathbb{R}$.

50. A subspace A of a metric space (X, d) is compact iff A is closed and bounded.

51. Let $\mathcal{B} = \{ [a, b) \cap [0,1] : a, b \in \mathbb{R},\ a < b \}$ be the base for a topology $\mathcal{T}$ on $[0,1]$. The space $([0,1], \mathcal{T})$ is compact.

52. (ℓ_2, d), where d is the usual metric on ℓ_2, is sequentially compact.

53. A subspace of a pseudocompact space is pseudocompact.

54. (ℓ_2, d), where d is the usual metric on ℓ_2, is countably compact.

55. An uncountable closed set in $\mathbb{R}$ must contain an interval of positive length.

56. Suppose C is the Cantor set and $f : C \to C$ is continuous. Then the graph of f (a subspace of $C \times C$, with its usual metric) is totally bounded.

57. Let C denote the Cantor set $\subseteq [0,1]$, with the metric $d(x,y) = \frac{|x-y|}{1+|x-y|}$. Then (C,d) is totally bounded.

58. A discrete subspace of $\mathbb{R}$ must be closed in $\mathbb{R}$

59. A subspace of $\mathbb{R}$ which is discrete in its relative topology must be countable.

60. If A and B are subspaces of $(X, \mathcal{T})$ and each is discrete in its subspace topology, then $A \cup B$ is discrete in the subspace topology.

61. Let $\cos^n$ denote the composition of cos with itself n times. Then for each $x \in [-1,1]$, there exists an n (perhaps depending on x) such that $\cos^n x = x$.

62. Suppose d is any metric on equivalent to the usual metric. Then $([0,1], d)$ is totally bounded.

63. Let S^1 denote the unit circle in $\mathbb{R}^2$. S^1 is homeomorphic to a subspace of the Cantor set C.

64. There is a metric space (X,d) satisfying the following condition: for every metric $d' \sim d$, (X, d') is totally bounded.

65. $[0,1]^2$, with the subspace topology from the Sorgenfrey plane, is compact.