Chapter V
Connected Spaces

1. Introduction

In this chapter we introduce the idea of connectedness. Connectedness is a topological property quite different from any property we considered in Chapters 1-4. A connected space X need not have any of the other topological properties we have discussed so far. Conversely, the only topological properties that imply "X is connected" are very extreme – such as "$|X| \leq 1$" or "X has the trivial topology."

2. Connectedness

Intuitively, a space is connected if it is all in one piece; equivalently a space is disconnected if it can be written as the union of two nonempty "separated" pieces. To make this precise, we need to decide what "separated" should mean. For example, we think of $\mathbb{R}$ as connected even though $\mathbb{R}$ can be written as the union of two disjoint pieces: for example, $\mathbb{R} = A \cup B$, where $A = (-\infty, 0]$ and $B = (0, \infty)$. Evidently, "separated" should mean something more than "disjoint."

On the other hand, if we remove the point 0 to "cut" $\mathbb{R}$, then we probably think of the remaining space $X = \mathbb{R} - \{0\}$ as "disconnected." Here, we can write $X = A \cup B$, where $A = (-\infty, 0)$ and $B = (0, \infty)$. A and B are disjoint, nonempty sets and (*unlike* A *and* B *in the preceding paragraph*) they satisfy the following (equivalent) conditions:

i) A and B are open in X
ii) A and B are closed in X
iii) $(B \cap \mathrm{cl}_X A) \cup (A \cap \mathrm{cl}_X B) = \emptyset$ – that is, each of A and B is disjoint from the closure of the other. (This is true, in fact, even if we use $\mathrm{cl}_{\mathbb{R}}$ instead of cl_X.)

Condition iii) is important enough to deserve a name.

Definition 2.1 Suppose A and B are subspaces of $(X, \mathcal{T})$. A and B are called separated if each is disjoint from the closure of the other – that is, if $(B \cap \mathrm{cl}_X A) \cup (A \cap \mathrm{cl}_X B) = \emptyset$.

It follows immediately from the definition that

i) separated sets must be disjoint, and

ii) subsets of separated sets are separated: if A, B are separated, $C \subseteq A$ and $D \subseteq B$, then C and D are also separated.

Example 2.2

1) In $\mathbb{R}$, the sets $A = (-\infty, 0]$ and $B = (0, \infty)$ are disjoint but not separated. Likewise in $\mathbb{R}^2$, the sets $A = \{(x, y) : x^2 + y^2 \leq 1\}$ and $B = \{(x, y) : (x - 2)^2 + y^2 < 1\}$ are disjoint but not separated.

2) The intervals $A = (-\infty, 0)$ and $B = (0, \infty)$ are separated in $\mathbb{R}$ but $\text{cl}_{\mathbb{R}} A \cap \text{cl}_{\mathbb{R}} B \neq \emptyset$. The same is true for the open balls $A = \{(x, y) : x^2 + y^2 < 1\}$ and $B = \{(x, y) : (x-2)^2 + y^2 < 1\}$ in $\mathbb{R}^2$.

The condition that two sets are separated is stronger than saying they are disjoint, but weaker than saying that the sets have disjoint closures.

Theorem 2.3 In any space $(X, \mathcal{T})$, the following statements are equivalent:

1) $\emptyset$ and X are the only clopen sets in X
2) if $A \subseteq X$ and $\text{Fr}\, A = \emptyset$, then $A = \emptyset$ or $A = X$
3) X is not the union of two disjoint nonempty open sets
4) X is not the union of two disjoint nonempty closed sets
5) X is not the union of two nonempty separated sets.

Note: Condition 2) is not frequently used. However it is fairly expressive: to say that Fr $A = \emptyset$ says that no point x in X can be "approximated arbitrarily closely" from both inside and outside A – so, in that sense, A and $B = X - A$ are pieces of X that are "separated" from each other.

Proof 1) $\Leftrightarrow$ 2) This follows because A is clopen iff $\text{Fr}\, A = \emptyset$ (*see Theorem II.4.5.3*).

1) $\Rightarrow$ 3) Suppose 3) is false and that $X = A \cup B$ where A, B are disjoint, nonempty and open. Since $X - A = B$ is open and nonempty, we have that A is a nonempty proper clopen set in X, which shows that 1) is false.

3) $\Leftrightarrow$ 4) This is clear.

4) $\Rightarrow$ 5) If 5) is false, then $X = A \cup B$, where A, B are nonempty and separated. Since $\text{cl}\, B \cap A = \emptyset$, we conclude that $\text{cl}\, B \subseteq B$, so B is closed. Similarly, A must be closed. Therefore 4) is false.

5) $\Rightarrow$ 1) Suppose 1) is false and that A is a nonempty proper clopen subset of X. Then $B = X - A$ is nonempty and clopen, so A and B are separated. Since $X = A \cup B$, 5) is false. ∙

Definition 2.4 A space $(X, \mathcal{T})$ is connected if any (therefore all) of the conditions 1) - 5) in Theorem 2.3 hold. If $C \subseteq X$, we say that C is connected if C is connected in the subspace topology.

According to the definition, a subspace $C \subseteq X$ is disconnected if we can write $C = A \cup B$, where the following (equivalent) statements are true:

1) A and B are disjoint, nonempty and open in C
2) A and B are disjoint, nonempty and closed in C
3) A and B are nonempty and separated in C.

If C is disconnected, such a pair of sets A, B will be called a disconnection or separation of C.

The following technical theorem and its corollary are very useful in working with connectedness in subspaces.

Theorem 2.5 Suppose $A, B \subseteq C \subseteq X$. Then A and B are separated in C iff A and B are separated in X.

Proof $\text{cl}_C B = C \cap \text{cl}_X B$ (*see Theorem III.7.6*), so $A \cap \text{cl}_C B = \emptyset$ iff $A \cap (\text{cl}_X B \cap C) = \emptyset$ iff $(A \cap C) \cap \text{cl}_X B = \emptyset$ iff $A \cap \text{cl}_X B = \emptyset$. Similarly, $B \cap \text{cl}_C A = \emptyset$ iff $B \cap \text{cl}_X A = \emptyset$. •

Caution: According to Theorem 2.5, C is disconnected iff $C = A \cup B$ where A and B are nonempty separated set in C iff $C = A \cup B$ where A and B are nonempty separated set in X. Theorem 2.5 is very useful because it means that we don't have to distinguish here between "separated in C" and "separated in X" – because these are equivalent. In contrast, when we say that C is disconnected if C is the union of two disjoint, nonempty open (*or closed*) sets A, B in C, then phrase "in C" cannot be omitted: the sets A, B might not be open (*or closed*) in X.

For example, suppose $X = [0, 1]$ and $C = [0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$. The sets $A = [0, \frac{1}{2})$ and $B = (\frac{1}{2}, 1]$ are open, closed and separated in C. By Theorem 2.5, they are also separated in $\mathbb{R}$ – but they are neither open nor closed in $\mathbb{R}$.

Example 2.6

1) Clearly, connectedness is a topological property. More generally, suppose $f : X \to Y$ is continuous and onto. If B is proper nonempty clopen set in Y, then $f^{-1}[B]$ is a proper nonempty clopen set in X. Therefore a continuous image of a connected space is connected.

2) A discrete space X is connected iff $|X| \leq 1$. In particular, $\mathbb{N}$ and $\mathbb{Z}$ are not connected.

3) $\mathbb{Q}$ is not connected since we can write $\mathbb{Q}$ as the union of two nonempty separated sets: $\mathbb{Q} = \{q \in \mathbb{Q} : q^2 < 2\} \cup \{q \in \mathbb{Q} : q^2 > 2\}$. Similarly, we can show $\mathbb{P}$ is not connected.

More generally suppose $C \subseteq \mathbb{R}$ and that C is not an interval. Then there are points $a < z < b$ where $a, b \in C$ but $z \notin C$. Then $\{x \in C : x < z\} = \{x \in C : x \leq z\}$ is a nonempty proper clopen set in C. Therefore C is not connected.

In fact, a subset C of $\mathbb{R}$ is connected iff C is an interval. It is not very hard, using the least upper bound property of $\mathbb{R}$, to prove that every interval in $\mathbb{R}$ is connected. (*Try it as an exercise!*) We will give a short proof soon (Corollary 2.12) using a different argument.

4) (The Intermediate Value Theorem) If X is connected and $f : X \to \mathbb{R}$ is continuous, then $\text{ran}(f)$ is connected (*by part 1*) so $\text{ran}(f)$ is an interval (*by part 3*). Therefore if $a, b \in X$ and $f(a) < z < f(b)$, there must be a point $c \in X$ for which $f(c) = z$.

5) The Cantor set C is not connected (since it is not an interval). But much more is true. Suppose $x, y \in A \subseteq C$ and that $x < y$. Since C is nowhere dense (*see IV.10*), the interval $(x, y) \not\subseteq C$, so we can choose $z \notin C$ with $x < z < y$. Then $B = (-\infty, z) \cap A = (-\infty, z] \cap A$ is clopen in A, and B contains x but not y – so A is not connected. It follows that every connected subset of C contains at most one point.

A space $(X, \mathcal{T})$ is called totally disconnected every connected subset A satisfies $|A| \leq 1$. The spaces $\mathbb{N}, \mathbb{Z}, \mathbb{P}$ and $\mathbb{Q}$ are other examples of totally disconnected spaces.

6) X is connected iff every continuous $f : X \to \{0, 1\}$ is constant: certainly, if f is continuous and not constant, then $f^{-1}[\{0\}]$ is a proper nonempty clopen set in X so X is not

connected. Conversely, if X is not connected and A is a proper nonempty clopen set, then the characteristic function $\chi_A : X \to \{0, 1\}$ is continuous but not constant.

Theorem 2.7 Suppose $f : X \to Y$. Let $\Gamma = \{(x, y) \in X \times Y : y = f(x)\}$ = "the graph of f." If f is continuous, then graph of f is homeomorphic to the domain of f; in particular, the graph of a continuous function is connected iff its domain is connected.

Proof We want to show that X is homeomorphic to Γ. Let $h : X \to \Gamma$ be defined by $h(x) = (x, f(x))$. Clearly h is a one-to-one map from X onto Γ.

Let $a \in X$ and suppose $(U \times V) \cap \Gamma$ is a basic open set containing $h(a) = (a, f(a))$. Since f is continuous and $f(a) \in V$, there exists an open set O in X containing a and such that $f[O] \subseteq V$. Then $a \in U \cap O$, and $h[U \cap O] \subseteq (U \times V) \cap \Gamma$, so h is continuous at a.

If U is open in X, then $h[U] = (U \times Y) \cap \Gamma$ is open in Γ, so h is open. Therefore h is a homeomorphism. $\bullet$

Note: It is not true that a function f with a connected graph must be continuous. See Example 2.22.

The following lemma makes a simple but very useful observation.

Lemma 2.8 Suppose M, N are separated subsets of X. If $C \subseteq M \cup N$ and C is connected, then $C \subseteq M$ or $C \subseteq N$.

Proof $C \cap M$ and $C \cap N$ are separated (since $C \cap M \subseteq M$ and $C \cap N \subseteq N$) and $C = (C \cap M) \cup (C \cap N)$. But C is connected so $(C \cap M)$ and $(C \cap N)$ cannot form a disconnection of C. Therefore either $C \cap M = \emptyset$ (so $C \subseteq N$) or $C \cap N = \emptyset$ (so $C \subseteq M$). $\bullet$

The next theorem and its corollaries are simple but powerful tools for proving that certain sets are connected. Roughly, the theorem states that if we have one "central " connected set C and other connected sets none of which is separated from C, then the union of all the sets is connected.

Theorem 2.9 Suppose C and C_α ($\alpha \in I$) are connected subsets of X and that for each α, C_α and C are not separated. Then $S = C \cup \bigcup C_\alpha$ is connected.

Proof Suppose that $S = M \cup N$ where M and N are separated. By Lemma 2.8, either $C \subseteq M$ or $C \subseteq N$. Without loss of generality, assume $C \subseteq M$. By the same reasoning we conclude that for each α, either $C_\alpha \subseteq M$ or $C_\alpha \subseteq N$. But if some $C_\alpha \subseteq N$, then C and C_α would be separated. Hence every $C_\alpha \subseteq M$. Therefore $N = \emptyset$ and the pair M, N is not a disconnection of S. $\bullet$

Corollary 2.10 Suppose that for each $\alpha \in I$, C_α is a connected subset of X and that for all $\alpha \neq \beta \in I$, $C_\alpha \cap C_\beta \neq \emptyset$. Then $\bigcup\{C_\alpha : \alpha \in I\}$ is connected.

Proof If $I = \emptyset$, then $\bigcup\{C_\alpha : \alpha \in I\} = \emptyset$ is connected. If $I \neq \emptyset$, pick an $\alpha_0 \in I$ and let C_{α_0} be the "central set" C in Theorem 2.9. For all $\alpha \in I$, $C_\alpha \cap C_{\alpha_0} \neq \emptyset$, so C_α and C_{α_0} are not separated. By Theorem 2.9, $\bigcup\{C_\alpha : \alpha \in I\}$ is connected. •

Corollary 2.11 For each $n \in \mathbb{N}$, suppose C_n is a connected subset of X and that $C_n \cap C_{n+1} \neq \emptyset$. Then $\bigcup_{n=1}^{\infty} C_n$ is connected.

Proof Let $A_n = \bigcup_{k=1}^{n} C_k$. Corollary 2.10 (and simple induction) shows that the A_n's are connected. Then $\emptyset \neq A_1 \subseteq A_2 \subseteq ... \subseteq A_n \subseteq ...$ Another application of Corollary 2.10 gives that $\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} C_n$ is connected. •

Corollary 2.12 Let $I \subseteq \mathbb{R}$. Then I is connected iff I is an interval. In particular, $\mathbb{R}$ is connected, so $\mathbb{R}$ and $\emptyset$ are the only clopen sets in $\mathbb{R}$.

Proof We have already shown that if I is not an interval, then I is not connected (*Example 2.6.3*). So suppose I is an interval. If $I = \emptyset$ or $I = \{r\}$, then I is connected.

Suppose $I = [0,1]$ and that A, B are nonempty disjoint closed sets in I. Then there are points $a_0 \in A$ and $b_0 \in B$ for which $|a_0 - b_0| = d(a_0, b_0) = d(A, B)$.

> To see this, define $f : A \times B \to [0,1]$ by $f(x,y) = |x - y|$. $A \times B$ is a closed bounded set in $\mathbb{R}^2$ so $A \times B$ is compact. Therefore f <u>has</u> a <u>minimum</u> value, occurring at some point $(a_0, b_0) \in A \times B$ (*see Exercise IV.E23.*)

Let $z = \frac{a_0 + b_0}{2} \in [0,1]$. Since $|z - b_0| = |\frac{a_0 + b_0}{2} - b_0| = |\frac{a_0 - b_0}{2}| < |a_0 - b_0|$, we conclude $z \notin A$. Similarly, $z \notin B$. Therefore $[0,1] \neq A \cup B$, so $[0,1]$ is connected.

Suppose $a < b$. The interval $[a,b]$ is homeomorphic to $[0,1]$, so each interval $[a,b]$ is connected. Since $(a,b) = \bigcup_{n=1}^{\infty} [a + \frac{1}{n}, b - \frac{1}{n}]$, Corollary 2.10 implies that (a,b) is connected. Similarly, Corollary 2.10 shows that each of the following unions is connected:

$$\bigcup_{n=1}^{\infty} [a, b - \tfrac{1}{n}] = [a, b)$$
$$\bigcup_{n=1}^{\infty} [a + \tfrac{1}{n}, b] = (a, b]$$
$$\bigcup_{n=1}^{\infty} [a, a+n] = [a, \infty)$$
$$\bigcup_{n=1}^{\infty} (a, a+n) = (a, \infty)$$
$$\bigcup_{n=1}^{\infty} [a-n, a] = (-\infty, a]$$
$$\bigcup_{n=1}^{\infty} [a-n, a) = (-\infty, a)$$
$$\bigcup_{n=1}^{\infty} [-n, n] = \mathbb{R} \quad \bullet$$

Corollary 2.13 For every $n \in \mathbb{N}$, $\mathbb{R}^n$ is connected.

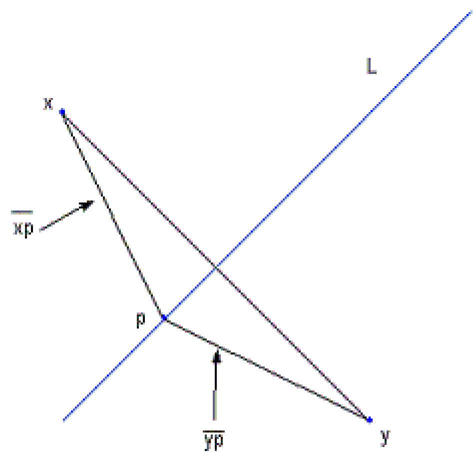
Proof By Corollary 2.12, $\mathbb{R}^1$ is connected. $\mathbb{R}^n$ can be written as a union of straight lines (each homeomorphic to $\mathbb{R}$) through the origin and Corollary 2.10 implies that $\mathbb{R}^n$ is connected. •

Corollary 2.14 Suppose that for all $x, y \in X$ there exists a connected set $C_{xy} \subseteq X$ with $x, y \in C_{xy}$. Then X is connected.

Proof Certainly $X = \emptyset$ is connected. If $X \neq \emptyset$, choose $a \in X$. By hypothesis there is, for each $y \in X$, a connected set C_{ay} containing both a and y. By Corollary 2.10, $X = \bigcup\{C_{ay} : y \in X\}$ is connected. •

Example 2.15 Suppose C is a countable subset of $\mathbb{R}^n$, where $n \geq 2$. Then $\mathbb{R}^n - C$ is connected. In particular, $\mathbb{R}^n - \mathbb{Q}^n$ is connected. To see this, suppose x, y are any two points in $\mathbb{R}^n - C$.

Choose a straight line L which is perpendicular to the line segment $\overline{xy}$ joining x and y. For each $p \in L$, let C_p be the union of the two line segments $\overline{xp} \cup \overline{py}$. C_p is the union of two intervals with a point in common, so C_p is connected.



If $p' \neq p$, then $C_{p'} \cap C_p = \{x, y\}$. So if $z \in C$, then z is in at most one C_p. Therefore (since C is countable), there is a $p^* \in L$ for which $C_{p^*} \cap C = \emptyset$. Then $x, y \in C_{p^*} \subseteq \mathbb{R}^n - C$. So Corollary 2.14 (with $C_{xy} = C_{p^*}$) shows that $\mathbb{R}^n - C$ is connected.

The definition of connectedness agrees with our intuition in the sense that every set that you think (intuitively) should be connected <u>is</u> actually connected according to Definition 2.4. But according to Definition 2.4, certain strange sets also turn out "unexpectedly" to be connected. $\mathbb{R}^n - \mathbb{Q}^n$ might fall into that category. So the official definition forces us to try to expand our intuition about what "connected" means. *Question: Is* $\mathbb{R}^n - \mathbb{P}^n$ *connected?*

This situation is analogous to what happens with the "ϵ-δ definition" of continuity. Using that definition it turns out that every function that you expect (intuitively) <u>should</u> be continuous actually <u>is</u> continuous. If you have a "problem" with the official definition of continuity, it would be that it almost seems "too generous" – it allows some "unexpected" functions also to be continuous. An example is the well-known function from elementary analysis: $f: \mathbb{R} \to \mathbb{R}$, where

$$f(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ in lowest terms} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

f is continuous at x iff x is irrational.

Definition 2.16 Suppose X is connected. If $X - \{p\}$ is not connected, then p is called a <u>cut point</u> in X.

If $f: X \to Y$ is a homeomorphism, then it is easy to check that p is a cut point in X iff $f(p)$ is a cut point in Y. Therefore <u>homeomorphic spaces have the same number of cut points</u>.

Example 2.17 $\mathbb{R}^n$ is not homeomorphic to $\mathbb{R}$ if $n \geq 2$.

Proof Every point $p \in \mathbb{R}$ is a cut point. But Example 2.15 shows that $\mathbb{R}^n$ has no cut points when $n \geq 2$.

It is also true – but much harder to prove – that $\mathbb{R}^m$ and $\mathbb{R}^n$ are not homeomorphic whenever $m \neq n$. One way to prove this is to develop theorems about a topological property called <u>dimension</u>. Then it turns out (thankfully) that dim $\mathbb{R}^m = m \neq n =$ dim $\mathbb{R}^n$ so these spaces are not homeomorphic. One can also prove this result using homology theory – a topic developed in algebraic topology.

Example 2.18 How is S^1 topologically different from $[0,1]$? Both are compact connected metric spaces with cardinality c, and there is no topological property from Chapters 1-4 that can distinguish between these spaces. The difference has to do with connectivity. The interval $[0,1]$ has cut points p (if p is not an endpoint, then p is a cut point); but S^1 has <u>no</u> cut points since $S^1 - \{p\}$ is homeomorphic to $(0,1)$ for every $p \in S^1$.

Corollary 2.19 Suppose $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$ are nonempty topological spaces. Then $X \times Y$ is connected iff X and Y are connected. (*It follows by induction that the same result holds for any finite product of spaces. When infinite products are defined in Chapter 6, it will turn out that the product of any collection of connected spaces is connected.*)

Proof $\Rightarrow$ Suppose $X \times Y$ is connected. Since $X \times Y \neq \emptyset$, we have $X = \pi_X[X \times Y]$. Therefore X is the continuous image of a connected space, so X is connected. Similarly, Y is connected.

$\Leftarrow$ Let X and Y be nonempty connected spaces, and consider any two points (a,b) and (c,d) in $X \times Y$. Then $X \times \{b\}$ and $\{c\} \times Y$ are homeomorphic to X and Y, so these "slices" of the product are connected and both contain the point (c,b). By Corollary 2.10, $C = (X \times \{b\}) \cup (\{c\} \times Y)$ is a connected set that contains both (a,b) and (c,d). By Corollary 2.14, we conclude that $X \times Y$ is connected. •

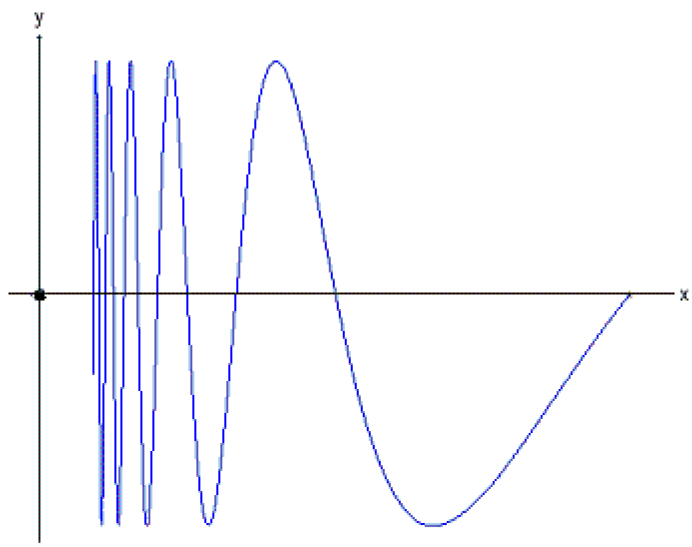
(*Corollary 2.19 gives an another reason why $\mathbb{R}^n$ is connected for $n > 1$.*)

Corollary 2.20 Suppose C is a connected subset of X. If $C \subseteq A \subseteq \operatorname{cl} C$, then A is connected. In particular, the closure of a connected set is connected.

Proof For each $a \in A$, $\{a\}$ and C are not separated. By Theorem 2.9, $A = C \cup \bigcup\{\{a\} : a \in A\}$ is connected. ∙

Example 2.21 By Corollary 2.20, the completion of a connected pseudometric space (X, d) must be connected.

Example 2.22 Let $f(x) = \begin{cases} \sin \frac{\pi}{x} & 0 < x \leq 1 \\ 0 & x = 0 \end{cases}$. Then $f(\frac{1}{n}) = 0$ for every $n \in \mathbb{N}$ and the graph oscillates more and more rapidly between -1 and 1 as $x \to 0^+$. Part of the graph is pictured below. Of course, f is not continuous at $x = 0$. Let Γ be the graph of the restricted function $g = f|(0, 1]$. Since g is continuous, Theorem 2.7 shows that Γ is homeomorphic to $(0, 1]$ so Γ is connected.



Γ is sometimes called the "topologist's sine curve."

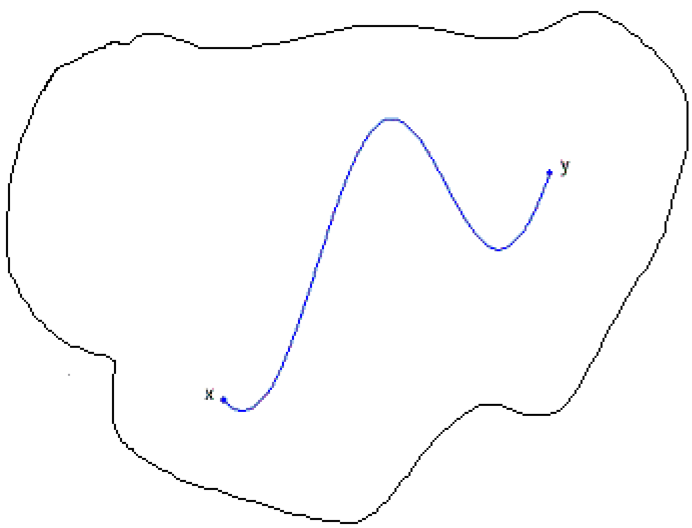
Because $\operatorname{cl}\Gamma = \Gamma \cup (\{0\} \times [-1, 1])$, Corollary 2.20 gives that $\Gamma \cup A$ is connected for any set $A \subseteq \{0\} \times [-1, 1]$. In particular, $\Gamma_f = \Gamma \cup \{(0, 0)\}$ (the graph of f) is connected.

Therefore, a function f with a connected graph need not be continuous. However, it is true that if the graph of a function $f : \mathbb{R} \to \mathbb{R}$ is a closed connected subset of $\mathbb{R}^2$, then f is continuous. (The proof is easy enough to read: see C.E. Burgess, *Continuous Functions and Connected Graphs*, The American Mathematical Monthly, April 1990, 337-339.)

3. Path Connectedness and Local Path Connectedness

In some spaces X, every pair of points can be joined by a path in X. This seems like a very intuitive way to describe "connectedness". However, this property is actually stronger than our definition for a connected space. .

Definition 3.1 A path in X is a continuous map $f : [0,1] \to X$. The path starts at its initial point $f(0)$ and ends at its terminal point $f(1)$. We say f is a path from $f(0)$ to $f(1)$.



Sometimes it helps to visualize a path by thinking of a point moving in X from $f(0)$ to $f(1)$ with $f(t)$ representing its position at "time" $t \in [0,1]$. Remember, however, that the path, by definition, is the function f, not the set ran$(f) \subseteq X$. To illustrate the distinction: suppose f is a path from x to y. Then the function $g : [0,1] \to X$ defined by $g(t) = f(1-t)$ is a different path (running in the "opposite direction," from y to x), even though ran$(f) =$ ran(g).

Definition 3.2 A topological space X is called path connected if, for every pair of points $x, y \in X$, there is a path from x to y in X.

Note: X is called arcwise connected if, for every pair of points $x, y \in X$, there exists a homeomorphism $f : [0,1] \to X$ with $f(0) = x$ and $f(1) = y$. Such a path f is called an arc from x to y. If a path f in a Hausdorff space X is not an arc, the reason must be that f is not one-to-one (why?). It can be proven that a Hausdorff space is path connected iff X is arcwise connected. Therefore some books use "arcwise connected" to mean the same thing as "path connected."

Theorem 3.3 A path connected space X is connected.

Proof $\emptyset$ is connected, so assume $X \neq \emptyset$ and choose a point $a \in X$. For each $x \in X$, there is a path f_x from a to x. Let $C_x =$ ran(f_x). Each C_x is connected and contains a. By Corollary 2.10, $X = \bigcup\{C_x : x \in X\}$ is connected. $\bullet$

Sometimes path connectedness and connectedness are equivalent. For example, a subset $I \subseteq \mathbb{R}$ is connected iff I is an interval iff I is path connected. But in general, the converse to Theorem 3.,3 is false as the next example shows.

Example 3.4 Consider $f(x) = \begin{cases} \sin\frac{\pi}{x} & 0 < x \le 1 \\ 0 & x = 0 \end{cases}$. In Example 2.22, we showed that the graph Γ_f is connected. However, we claim that there is no path in Γ_f from $(0,0)$ to $(1,0)$ and therefore Γ_f is not path connected.

Suppose, on the contrary, that $h : [0,1] \rightarrow \Gamma_f$ is a path from $(0,0)$ to $(1,0)$. For $t \in [0,1]$, write $h(t) = (h_1(t), h_2(t)) \in \Gamma_f$. h_1 and h_2 are continuous *(why?)*. Since $[0,1]$ is compact, h is uniformly continuous (*Theorem IV.9.6*) so we can choose $\delta_1 > 0$ for which $|u - v| < \delta_1 \Rightarrow d(h(u), h(v)) < 1 \Rightarrow |h_2(u) - h_2(v)| < 1$.

We have $0 \in h^{-1}((0,0))$. Let $t^* = \sup h^{-1}(0,0)$. Then $0 \le t^* < 1$. Since $h^{-1}(0,0)$ is a closed set, $t^* \in h^{-1}(0,0)$ so $h(t^*) = (0,0)$. (*We can think of t^* as the last "time" that the path h goes through the origin*).

Choose a positive $\delta < \delta_1$ so that $0 \le t^* < t^* + \delta < 1$. Since $h_1(t^*) = 0$ and $h_1(t^* + \delta) > 0$, we can choose a positive integer N for which

$$0 = h_1(t^*) \le \tfrac{2}{N+1} < \tfrac{2}{N} < h_1(t^* + \delta).$$

By the Intermediate Value Theorem, there exist points $u, v \in (t^*, t^* + \delta)$ where $h_1(u) = \frac{2}{N+1}$ and $h_1(v) = \frac{2}{N}$. Then $h_2(u) = \sin\frac{(N+1)\pi}{2}$ and $h_2(v) = \sin\frac{N\pi}{2}$, so $|h_2(u) - h_2(v)| = 1$. But this is impossible since $|u - v| < \delta < \delta_1$ and therefore $|h_2(u) - h_2(v)| < 1$. •

Note: Let Γ be the graph of the restriction $g = f|(0\,1]$. For any set $A \subseteq \{0\} \times [-1,1]$, a similar argument shows that $\Gamma \cup A$ is not path connected. In particular, cl Γ $= \Gamma \cup (\{0\} \times [-1,1])$ is not path connected. But Γ is homeomorphic to $(0,1]$, so Γ is path connected. So <u>the closure of a path connected space need not be path connected</u>.

Definition 3.5 A space $(X, \mathcal{T})$ is called

a) <u>locally connected</u> if for each point $x \in X$ and for each neighborhood N of x, there is a connected open set U such that $x \in U \subseteq N$.

b) <u>locally path connected</u> if for each point $x \in X$ and for each neighborhood N of x, there is a path connected open set U such that $x \in U \subseteq N$.

Note: to say U is path connected means that any two points in U can be joined by a path <u>in U</u>. Roughly, "locally path connected" means that "nearby points can be joined by short paths."

Example 3.6

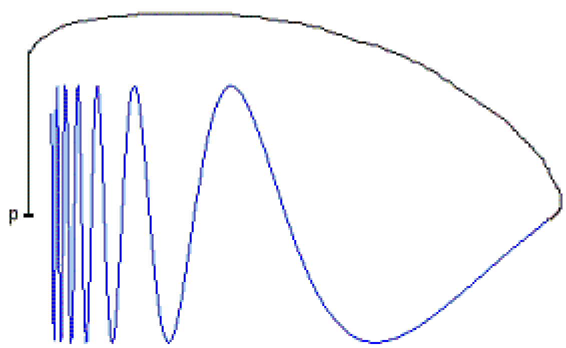
1) $\mathbb{R}^n$ is connected, locally connected, path connected and locally path connected.

2) A locally path connected space is locally connected.

3) Connectedness and path connected are "global" properties of a space X: they are statements about X "as a whole." Local connectedness and local path connectedness are statement about what happens "locally" (in arbitrarily small neighborhoods of points) in X In general, global properties do not imply local properties, nor vice-versa.

a) Let $X = (0,1) \cup (1,2]$. X is not connected (and therefore not path connected) but X is locally path connected (and therefore locally connected). The same relations hold in a discrete space X with more than one point.

b) Let X be the subset of $\mathbb{R}^2$ pictured below. Note that X contains the "topologist's sine curve" as a subspace $-$ you need to imagine it continuing to oscillate faster and faster as it approaches the vertical line segment in the picture:

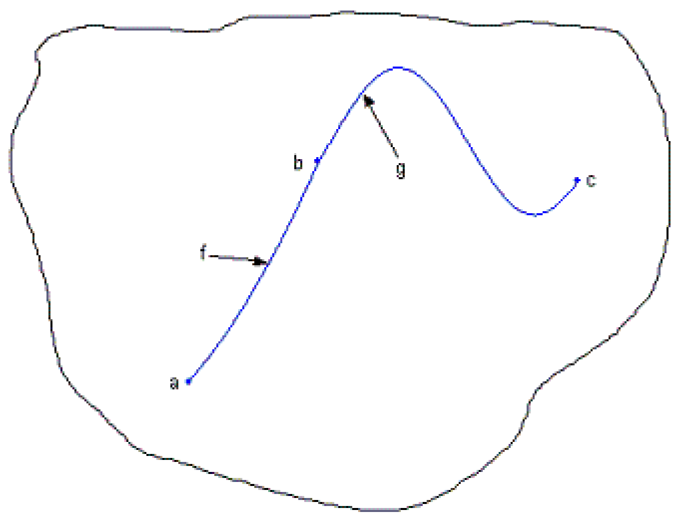


The X is path connected (therefore connected, but X is not locally connected: if $p = (0,0)$, there is no open connected set containing p inside the neighborhood $N = B_{\frac{1}{2}}(p) \cap X$. Therefore X is also not locally path connected.

Notice that Examples a) and b) also show that neither "(path) connected" nor "locally (path) connected" implies the other.

Lemma 3.7 Suppose that f is a path in X from a to b and g is a path from b to c. Then there exists a path h in X from a to c.

Proof f ends where g begins, so we feel intuitively that we can "join" the two paths "end-to-end" to get a path h from a to c. The only technical detail to handle is that, by definition, a path h must be a function with domain $[0,1]$. To get h we simply "join and reparametrize:"



Define $h : [0,1] \to X$ by $h(t) = \begin{cases} f(2t) & 0 \le t \le \frac{1}{2} \\ g(2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$. (*You can imagine a point moving twice as fast as before: first along the path f and then continuing along the path g.*)
The function h is continuous by the Pasting Lemma (*see Exercise III.E22*). •

Theorem 3.8 If X is connected and locally path connected, then X is path connected.

Proof If $X = \emptyset$, then X is path connected. So assume $X \neq \emptyset$. For $a \in X$, let $C = \{x \in X : \text{there exists a path in } X \text{ from } a \text{ to } x\}$. Then $C \neq \emptyset$ since $a \in C$ (*why?*). We want to show that $C = X$.

Suppose $x \in C$. Let f be a path in X from a to x. Choose a path connected open set U containing x. For any point $y \in U$, there is a path g in U from x to y. By Lemma 3.7, there is a path h in X from a to y, so $y \in C$. Therefore $x \in U \subseteq C$, so C is open.

Suppose $x \notin C$ and choose a path connected open set U containing x. If $y \in U$, there is a path g in U from y to x. Therefore there cannot exist a path in X from a to y – or else, by Lemma 3.7, there would be a path h from a to x and x would be in C. Therefore $y \notin C$, so $x \in U \subseteq X - C$, so C is closed – and therefore clopen.

X is connected and C is a nonempty clopen set, so $C = X$. Therefore X is path connected. •

Here is another situation (particularly useful in complex analysis) where connectedness and path connected coincide:

Corollary 3.9 An <u>open</u> connected set O <u>in $\mathbb{R}^n$</u> is path connected.

Proof Suppose $x \in O$. If N is any neighborhood of x <u>in O</u>, then $x \in \text{int}_O N = U \subseteq O$. Since O is open in $\mathbb{R}^n$, and U is open in O, U is also open in $\mathbb{R}^n$. Therefore there is an $\epsilon > 0$ such that $B_\epsilon(x) \subseteq U \subseteq N$. Since $B_\epsilon(x)$ is an ordinary ball in $\mathbb{R}^n$, $B_\epsilon(x)$ is path connected. (*Of course,* *this might not be true for a ball in an arbitrary metric space.*) We conclude that O is locally path connected so, by Theorem 3.8, O is path connected. •

4. Components

Informally, the "components" of a space X are its largest connected subspaces. A connected space X has exactly one component – X itself. In a totally disconnected space, for example $\mathbb{N}$, the components are the singletons $\{x\}$. In very simple examples, the components "look like" just what you imagine. In more complicated situations, some mild surprises can occur.

Definition 4.1 A component C of a space X is a maximal connected subspace. (Here, "maximal connected" means: C is connected and if $C \subseteq D \subseteq X$ where D is connected, , then $C = D$)

For any $p \in X$, let $C_p = \bigcup\{A : p \in A \subseteq X$ and A is connected$\}$. Then $\{p\} \in C_p$; since C_p is a union of connected sets each containing p, C_p is connected (*Corollary 2.10*). If $C_p \subseteq D$ and D is connected, then D was one of the sets A in the collection whose union defines C_p – so $D \subseteq C_p$ and therefore $C_p = D$. Therefore C_p is a component of X that contains p, so can be written as the union of components: $X = \bigcup_{p \in X} C_p$.

Of course it can happen that $C_p = C_q$ when $p \neq q$: for example, in a connected space X, $C_p = X$ for every $p \in X$. But if $C_p \neq C_q$, then $C_p \cap C_q = \emptyset$: if $x \in C_p \cap C_q$, then $C_p \cup C_q$ would be a connected set strictly larger than C_p.

The preceding paragraphs show that the distinct components of X form a partition of X : a pairwise disjoint collection whose union is X. If we define $p \sim q$ to mean that p and q are in the same component of X, then it is easy to see that " $\sim$ " is an equivalence relation on X and that C_p is the equivalence class of p.

Theorem 4.2 X is the union of its components. Distinct components of X are disjoint and each component is a closed connected set.

Proof In light of the preceding comments, we only need to show that each component C_p is closed. But this is clear: $C_p \subseteq \operatorname{cl} C_p$ and $\operatorname{cl} C_p$ is connected (*Corollary 2.20*). By maximality, we conclude that $C = \operatorname{cl} C_p$. •

It should be clear that a homeomorphism maps components to components. Therefore homeomorphic spaces have the same number of components.

Example 4.3

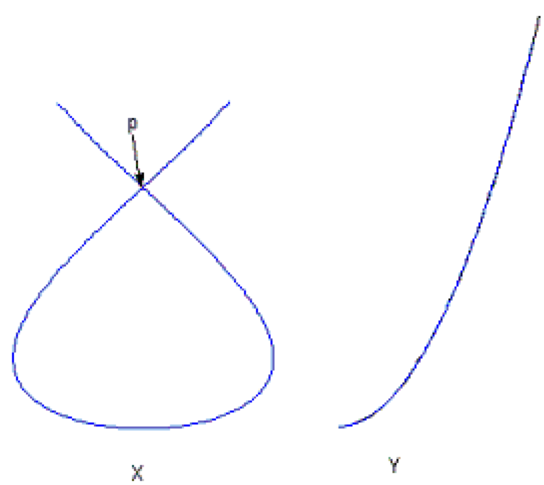
1) Let $X = [1,2] \cup [3,4] \cup [5,6] \subseteq \mathbb{R}$. X has three components: $[1,2]$, $[3,4]$, and $[5,6]$. For each $0 \leq p \leq 1$, we have $C_p = [0,1]$. If C is a component in a space X that has only finitely many components, then $X - C$ is the union of the other finitely many (closed) components. Therefore C is clopen.

However, a space can have infinitely many components and in general they need not be open. For example, if $X = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\} \subseteq \mathbb{R}$, then the components are the singleton sets $\{x\}$ (*why?*). The component $\{0\}$ is not open in X.

2) In $\mathbb{R}^2$, $X = \bigcup_{n=1}^{2} \mathrm{B}_{1/4}((n,0))$ is not homeomorphic to $Y = \bigcup_{n=1}^{3} \mathrm{B}_{1/4}((n,0))$ because X has two components but Y has three.

3) If $C \subseteq X$ and C is nonempty connected and clopen, then C is a component of X: for if $\emptyset \neq C \subseteq D \subseteq X$, then C is clopen in D so if D is connected, then $C = D$.

4) The sets X and Y in $\mathbb{R}^2$ pictured below are not homeomorphic since X contains a cut point p for which $X - \{p\}$ has three components. Y contains no such cut point.



Example 4.4 The following examples are meant to help "fine-tune" your intuition about components by pointing out some false assumptions that you need to avoid. (*Take a look back at Definition 2.4 to be sure you understand what is meant by a "disconnection."*)

1) Let $X = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$. One of the components of X is $\{0\}$, but $\{0\}$ is not clopen in X. Therefore the sets $A = \{0\}$ and $B = \{\frac{1}{n} : n \in \mathbb{N}\}$ do not form a disconnection of X. A component and its complement may not form a disconnection of X.

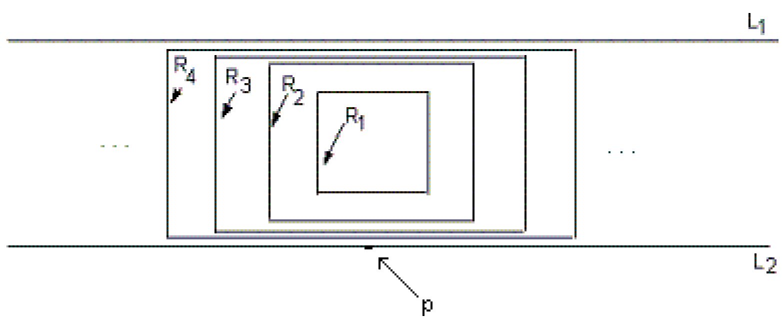
2) If a space X is the union of disjoint closed connected sets, these sets need not be components. For example, $[0,1] = \bigcup_{r \in [0,1]} \{r\}$

.

3) <u>If x and y are in different components of X, then there might not exist a disconnection $A \cup B$ of X for which $x \in A$ and $y \in B$.</u> For example, consider $X = L_1 \cup L_2 \cup \bigcup_{n=1}^{\infty} R_n \subseteq \mathbb{R}^2$, where:

L_1 and L_2 are the straight lines with equations $y = 2$ and $y = -2$.

For each $n \in \mathbb{N}$, R_n is the rectangle $\{(x, y) : |x| = n \text{ and } |y| = 2 - \frac{1}{n}\}$. (The top and bottom edges of the R_n's approach the lines L_1 and L_2.) The first four R_n's are pictured:



Each R_n is connected and clopen in X. Therefore each R_n is a component of X. (*See Example 4.3.3.*)

The other components of X are L_1 and L_2. For example:

L_1 is connected. Let C be the component that contains L_1. C must be disjoint from each component R_n, so if $L_1 \neq C$, then the additional points in C are from L_2 – that is, $C = L_1 \cup D$ where $D \subseteq L_2$. But in that case, L_1 would be a nontrivial clopen set in C and C would not be connected. Therefore $L_1 = C$.

Suppose that A and B are disjoint clopen sets in X for which $X = A \cup B$. A and B are separated so, by Lemma 2.8, L_2 is either a subset of A or a subset of B: without loss of generality, assume that $L_2 \subseteq A$ and let $p \in L_2$. Since A is open, p has a neighborhood $N \subseteq A$. But N intersects infinitely many connected R_n's, each of which, therefore, must also be a subset of A. Since the top edges of these R_n's approach L_1, there are points on L_1 in cl $A = A$. Therefore L_1 intersects A, so $L_1 \subseteq A$. So $L_1 \cup L_2 \subseteq A$.

In particular: $(0, 2) \in L_1$ and $(0, -2) \in L_2$ are in different components of X, but both are in the same piece A of a disconnection.

Conversely, however: suppose $A \cup B$ is a disconnection of some space Y, with $x \in A$ and $y \in B$. Then x and y must be in different components of Y. (*Why?*)

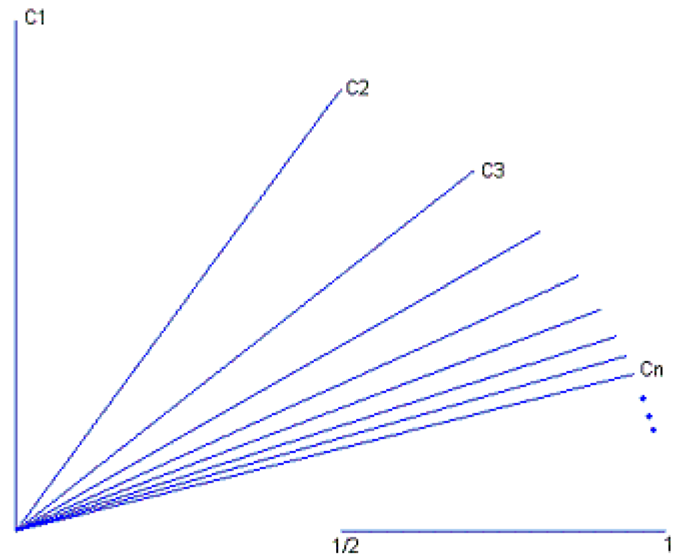
4) Suppose X is a connected space with a cut point v. Let C be a component of $X - \{v\}$. (*Draw a few simple pictures before reading on.*)

It can happen that $v \notin \mathrm{cl}_X\, C$ (*Would you have guessed that v must be in cl$_X C$?*)

For example, consider the following subspace $X = C \cup B$ of $\mathbb{R}^2$ where

C is the interval $[\frac{1}{2}, 1]$ on the x-axis and

$B = \{(0,0)\} \cup \bigcup_{n=1}^{\infty} C_n$ is a "broom" made up of disjoint "straws" C_n (each a copy of $(0,1]$) extending out from the origin and arranged so that slope$(C_n) \to 0$.



The broom B is connected (*because it's path connected*), so cl$_X B = X$ is connected.

Let $v = (0,0)$. Each straw C_n is connected and clopen in $X - \{v\}$. Therefore each C_n is a component of $X - \{v\}$, and the remaining connected subset, C is the remaining component of $X - \{v\}$.

So v is a cut point of X and $v \notin \mathrm{cl}_X C$.

Note: for this example, $v \notin \mathrm{cl}_X C$, but v is in the closure of the each of the other components C_n of $X - \{v\}$.

There is a much more complicated example, due to Knaster and Kuratowski (*Fundamenta Mathematicae, v. 2, 1921*). There, X is a connected set in $\mathbb{R}^2$ with a cut point v such that $X - \{v\}$ is totally disconnected. Intuitively, all the singleton sets $\{x\}$ are "tied together" at the point v to create the connected space X; removing v causes X to "explode" into "one-element fragments." In contrast to the "broom space", all components in $X - \{v\}$ are singletons, so v is not in the closure (in X) of any of them.

Here is a description of the Knaster-Kuratowski space X (sometimes called "Cantor's teepee"). The proof that is has the properties mentioned is omitted. (*You can find it on p. 145 of the book* <u>*Counterexamples in Topology (Steen & Seebach)*</u>. Define

$C =$ the Cantor set $(\subseteq [0,1])$ on the x-axis in $\mathbb{R}^2$, and let $v = (\frac{1}{2}, 1)$.

$D = \{p \in C :\ p$ is the endpoint of one of the "deleted middle thirds" in the construction of $C\}$
$= \{p \in C :\ p = 0.a_1a_2a_3...a_n..._{\text{base three}}$, where the a_n's are eventually equal to 0 or eventually equal to $2\}$. Of course, D is countable.

$E = C - D$ ($=$ "the points in C that are not isolated on either side in C")

Then $C = D \cup E,\ D \cap E = \emptyset$ and both D and E are dense in C.

For each $p \in C$, let $\overline{vp}$ $=$ the line segment from v to p and define a subset of $\overline{vp}$ by

$$C_p = \begin{cases} \{(x,y) \in \overline{vp} : y \text{ is rational}\} & \text{if } p \in D \\ \\ \{(x,y) \in \overline{vp} : y \text{ is irrational}\} & \text{if } p \in E. \end{cases}$$

Cantor's teepee is the space $X = \bigcup_{p \in C} C_p$. One can show that X is connected and that $X - \{v\}$ is totally disconnected.

5. Sierpinski's Theorem

Let $\mathcal{T}$ be the cofinite topology on $\mathbb{N}$. Clearly, $(\mathbb{N}, \mathcal{T})$ is connected. But is it path connected? (*Try to prove or disprove it.*) This innocent sounding question turns out to be harder than you might expect.

If $f : [0,1] \rightarrow (\mathbb{N}, \mathcal{T})$ is a path from (say) 1 to 2 in $(\mathbb{N}, \mathcal{T})$. Then ran$(f)$ is a connected set containing at least two points. But every <u>finite</u> subspace of $(\mathbb{N}, \mathcal{T})$ is discrete, so ran(f) must be infinite. Therefore $[0,1] = \bigcup_{n=1}^{\infty} f^{-1}(n)$, where infinitely many of the sets $f^{-1}(n)$ are nonempty. Is this possible? The question is not particularly easy. In fact, the question of whether $(\mathbb{N}, \mathcal{T})$ is path connected is equivalent to the question of whether $[0,1]$ can be written as a countable union of pairwise disjoint nonempty closed sets.

The answer lies in a famous old theorem of Sierpinski which states that a compact connected Hausdorff space X cannot be written as a countable union of two or more nonempty pairwise disjoint closed sets. (*Of course, "countable"* <u>*includes*</u> *"finite." But the"finite union" case is trivial: X is not a union of n nonempty disjoint closed sets $(n \geq 2)$ since each set would be clopen $-$ an impossibility since X is connected.*)

We will prove Sierpinski's result after a series of several lemmas. The line of argument used is due to R. Engelking. (*It is possible to prove Sierpinski's theorem just for the special case $X = [0,1]$. That proof is a little easier but still nontrivial.*)

Lemma 5.1 If A and B are disjoint closed sets in a compact Hausdorff space X, then there exist disjoint open sets U and V with $A \subseteq U$ and $B \subseteq V$.

Proof Consider first the case where $A = \{x\}$, a singleton set. For each $y \in B$, choose disjoint open sets U_y and V_y with $x \in U_y$ and $y \in V_y$. The open sets V_y cover the compact set B so a finite number of them cover B, say $B \subseteq V_{y_1} \cup \ldots \cup V_{y_n} = V$. Let $U = U_{y_1} \cap \ldots \cap U_{y_n}$. Then $A \subseteq U$, $B \subseteq V$ and U, V are disjoint open sets.

Suppose now that A and B are any pair of disjoint closed sets in X. For each $x \in A$, pick disjoint open sets U_x and V_x such that $\{x\} \subseteq U_x$ and $B \subseteq V_x$. The open U_x's cover the compact set A, so a finite number of them cover A, say $A \subseteq U_{x_1} \cup \ldots \cup U_{x_n} = U$. Let $V = V_{x_1} \cap \ldots \cap V_{x_n}$. Then $A \subseteq U$, $B \subseteq V$ and U, V are disjoint open sets. ∙

Note: If A and B were both finite, an argument analogous to the proof given above would work in <u>any</u> Hausdorff space X. The proof of Lemma 5.1 illustrates the rule of thumb that "compact sets act like finite sets."

Lemma 5.2 Suppose O is an open set in the compact space $(X, \mathcal{T})$. If $\mathcal{F} = \{F_\alpha : \alpha \in I\}$ is a family of closed sets in X for which $\bigcap \mathcal{F} \subseteq O$, then there exist $\alpha_1, \ldots, \alpha_n \in I$ such that $F_{\alpha_1} \cap F_{\alpha_2} \cap \ldots \cap F_{\alpha_n} \subseteq O$.

Proof For each $y \in X - O$, there is an α such that $y \notin F_\alpha$. Therefore $\{X - F_\alpha : \alpha \in I\}$ is an open cover of the compact set $X - O$. There exist $\alpha_1, \ldots \alpha_n \in I$ such that $(X - F_{\alpha_1}) \cup (X - F_{\alpha_2}) \cup \ldots \cup (X - F_{\alpha_n}) \supseteq X - O$. Taking complements gives that $F_{\alpha_1} \cap F_{\alpha_2} \cap \ldots \cap F_{\alpha_n} \subseteq O$. ∙

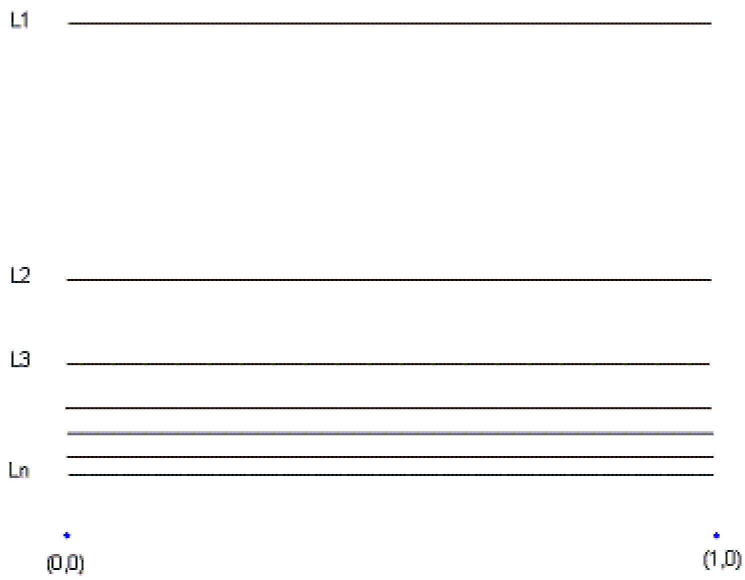
Definition 5.3 Suppose $p \in X$. The set $Q_p = \bigcap\{C \subseteq X : p \in C \text{ and } C \text{ is clopen in } X\}$ is called the <u>quasicomponent</u> of X containing p.

Q_p is always a closed set in X. The next two lemmas give some relationships between the component C_p containing p and the quasicomponent Q_p.

Lemma 5.4 If $p \in X$, then $C_p \subseteq Q_p$.

Proof If C is any clopen set containing p, then $C_p \subseteq C$ – for otherwise $C_p \cap C$ and $C_p \cap (X - C)$ disconnect C_p. Therefore $C_p \subseteq \bigcap\{C : p \in C \text{ and } C \text{ is clopen}\} = Q_p$. ∙

Example 5.5 <u>An example where</u> $C_p \neq Q_p$. In $\mathbb{R}^2$, let L_n be the horizontal line segment $[0,1] \times \{\frac{1}{n}\}$ and define $X = \bigcup_{n=1}^{\infty} L_n \cup \{(0,0),(1,0)\}$.



The components of X are the sets L_n and the singleton sets $\{(0,0)\}$ and $\{(1,0)\}$.

If C is any clopen set in X containing $(0,0)$, then C <u>intersects</u> infinitely many L_n's so (since the L_n's are connected) C <u>contains</u> those L_n's. Hence the closed set C contains points arbitrarily close to $(1,0)$ – so $(1,0)$ is also in C. Therefore $(0,0)$ and $(1,0)$ are both in $Q_{(0,0)}$, so $C_{(0,0)} \neq Q_{(0,0)}$. (*In fact, it is easy to check that* $Q_{(0,0)} = \{(0,0),\ (1,0)\}$.)

Lemma 5.6 If X is a compact Hausdorff space and $p \in X$, then $C_p = Q_p$.

Proof C_p is a <u>maximal</u> connected set and $C_p \subseteq Q_p$; therefore $C_p = Q_p$ <u>if</u> we can prove that Q_p is connected. Suppose $Q_p = A \cup B$, where A, B are disjoint closed sets in Q_p. We can assume that $p \in A$. We will show that $B = \emptyset$ – in other words, that there is no disconnection of Q_p.

Q_p is closed in X, so A and B are also closed in X. By Lemma 5.1, we can choose disjoint open sets U and V in X with $p \in A \subseteq U$ and $B \subseteq V$. Then $Q_p = A \cup B \subseteq U \cup V$. Since X is compact and Q_p is an intersection of clopen sets in X, Lemma 5.2 lets us pick finitely many clopen sets $C_1, ..., C_n$ such that $Q_p \subseteq C_1 \cap ... \cap C_n \subseteq U \cup V$. Let $C = C_1 \cap ... \cap C_n$. C is clopen in X and $Q_p \subseteq C \subseteq U \cup V$.

$U \cap C$ is open in X and, in fact, $U \cap C$ is also closed in X: since $\mathrm{cl}\, U \subseteq X - V$, we have that $U \cap C = \mathrm{cl}\, U \cap C = \mathrm{cl}\, U \cap \mathrm{cl}\, C \supseteq \mathrm{cl}\,(U \cap C)$. Therefore $U \cap C$ is one of the clopen sets containing p whose intersection defines Q_p, so $Q_p \subseteq U \cap C \subseteq U$. Therefore $Q_p \cap B = \emptyset$, so $B = \emptyset$. •

Definition 5.7 A <u>continuum</u> is a compact connected Hausdorff space.

Lemma 5.8 Suppose A is a closed subspace of a continuum X and that $\emptyset \neq A \neq X$. If C is a component of A, then $C \cap \mathrm{Fr}\, A \neq \emptyset$.

Proof Let C be a component of A and let $p \in C$. Since $C \subseteq A = \mathrm{cl}\, A$, we have that $C \cap \mathrm{Fr}\, A = C \cap \mathrm{cl}\, A \cap \mathrm{cl}\,(X - A)) = C \cap \mathrm{cl}\,(X - A)$, so we need to show that $C \cap \mathrm{cl}\,(X - A) \neq \emptyset$. We do this by contraposition: assuming $C \cap \mathrm{cl}\,(X - A) = \emptyset$, we will prove $A = X$.

A is compact so Lemma 5.6 gives $C = C_p = Q_p = \bigcap\{C_\alpha \subseteq A :\ p \in C_\alpha$ and C_α is clopen in $A\}$. By assumption, $C \subseteq X - \mathrm{cl}(X - A)$, so by Lemma 5.2 there exist indices α_1, α_2,..., α_n for which $C \subseteq C_{\alpha_0} = C_{\alpha_1} \cap C_{\alpha_2} \ldots \cap C_{\alpha_n} \subseteq X - \mathrm{cl}(X - A)$. Since $C_{\alpha_0} \cap \mathrm{cl}\,(X - A) = \emptyset$, we have $C_{\alpha_0} \cap \mathrm{Fr}\, A = \emptyset$.

C_{α_0} is clopen in A and $C_{\alpha_0} \subseteq A - \mathrm{Fr}\, A = \mathrm{int}\, A$. Since C_{α_0} is open in $\mathrm{int}\, A$ which is open in X, C_{α_0} is open in X. But C_{α_0} is also closed in the closed set A, so C_{α_0} is closed in X. Since X is connected, we conclude that $A = X$. •

Lemma 5.9 Suppose X is a continuum and that $X = \bigcup\{F_n : n \in \mathbb{N}\}$ where the F_n's are pairwise disjoint closed sets and $F_n \neq \emptyset$ for <u>at least</u> two values of n. Then for <u>each</u> n there exists a continuum $C_n \subseteq X$ such that $C_n \cap F_n = \emptyset$ and $C_n \cap F_i \neq \emptyset$ for <u>at least two</u> $i \in \mathbb{N}$.

Before proving Lemma 5.9, consider the formal statement of Sierpinski's Theorem..

Theorem 5.10 (Sierpinski) Let X be a continuum. If $X = \bigcup\{F_n : n \in \mathbb{N}\}$ where the F_n's are pairwise disjoint closed sets, then <u>at most one</u> F_n is nonempty.

(*Of course, the statement of the theorem includes the easy "finite union" case.*) *In proving Sierpinski's theorem we will assume that* $X = \bigcup\{F_n : n \in \mathbb{N}\}$ *where the* F_n*'s are pairwise disjoint closed sets and* $F_n \neq \emptyset$ *for <u>at least</u> two values of* n. *Then we will apply Lemma 5.9 to arrive at a contradiction. When all the smoke clears we see that, in fact, <u>there are no continua which satisfy the hypotheses of Lemma 5.9</u>. Lemma 5.9 is really the first part of the proof (by contradiction) of Sierpinski's theorem* $-$ *set off as a preliminary lemma to break the argument into more manageable pieces.*

Proof of Lemma 5.9

If $F_n = \emptyset$, let $C_n = X$.

Assume $F_n \neq \emptyset$. Choose $m \neq n$ with $F_m \neq \emptyset$ and pick a point $p \in F_m$. By Lemma 5.1, we can choose disjoint open sets U, V in X with $F_n \subseteq U$ and $p \in F_m \subseteq V$. Let C_n be the component of p in $\mathrm{cl}\, V$. Certainly C_n is a continuum, and we prove that this choice of C_n works. We have that $C_n \cap F_n = \emptyset$ (since $C_n \subseteq \mathrm{cl}\, V \subseteq X - U$) and that $C_n \cap F_m \neq \emptyset$ (since $p \in C_n \cap F_m$). Therefore, to complete proof, we need only show that for some $i \neq m, n$, $C_n \cap F_i \neq \emptyset$.

Since $p \in \mathrm{cl}\, V \subseteq X - U$, we have that $\mathrm{cl}\, V \neq \emptyset$; and $\mathrm{cl}\, V \neq X$ because

$\emptyset \neq F_n \subseteq U$. Therefore, by Lemma 5.8, there is a point $q \in C_n \cap \mathrm{Fr}\,(\mathrm{cl}\, V)$. Since $q \in \mathrm{Fr}\,(\mathrm{cl}\, V)$ and $F_m \subseteq V \subseteq \mathrm{int}(\mathrm{cl}\, V)$, we have $q \notin F_m$. And since $q \in \mathrm{Fr}\,(\mathrm{cl}\, V) \subseteq \mathrm{cl}\, V \subseteq X - U$, we have that $q \notin F_n$. But X is covered by the F_i's, so $q \in F_i$ for some $i \neq m, n$. Therefore $C_n \cap F_i \neq \emptyset$.
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Proof of Theorem 5.10 We want to show that if $X = \bigcup_{n=1}^{\infty} F_n$ where the F_n's are disjoint closed sets, then <u>at most one</u> $F_n \neq \emptyset$. Looking for a contradiction, we suppose at least two F_n's are nonempty.

By Lemma 5.9, there is a continuum C_1 in X with $C_1 \cap F_1 = \emptyset$ and such that C_1 has nonempty intersection with a least two F_n's. We can write $C_1 = C_1 \cap X = C_1 \cap \bigcup_{n=1}^{\infty} F_n$ $= \bigcup_{n=1}^{\infty}(C_1 \cap F_n) = \bigcup_{n=2}^{\infty}(C_1 \cap F_n)$, where at least two of the sets $C_1 \cap F_n$ are nonempty.

Applying Lemma 5.9 again (to the continuum C_1) we find a continuum $C_2 \subseteq C_1$ such that $C_2 \cap (C_1 \cap F_2) = C_2 \cap F_2 = \emptyset$ and C_2 intersects at least two of the sets $C_1 \cap F_n$.
Then $C_2 = C_2 \cap C_1 = \bigcup_{n=2}^{\infty}(C_2 \cap (C_1 \cap F_n)) = \bigcup_{n=2}^{\infty}(C_2 \cap F_n) = \bigcup_{n=3}^{\infty}(C_2 \cap F_n)$, where at least two of the sets $C_2 \cap F_n$ are nonempty.

We continue this process inductively, repeatedly applying Lemma 5.9, to and generate a decreasing sequence of nonempty continua $C_1 \supseteq C_2 \supseteq \ldots \supseteq C_n \supseteq \ldots$ such that for each n, $C_n \cap F_n = \emptyset$. This gives $\emptyset = \bigcap_{n=1}^{\infty} C_n \cap \bigcup_{n=1}^{\infty} F_n = \bigcap_{n=1}^{\infty} C_n \cap X = \bigcap_{n=1}^{\infty} C_n$. But this is impossible: the C_n's have the finite intersection property and X is compact, so $\bigcap_{n=1}^{\infty} C_n \neq \emptyset$. •

Example 5.11 By Theorem 5.10, we know that $[0,1]$ cannot be written as the union of m pairwise disjoint nonempty closed sets if $1 < m \leq \aleph_0$. And, of course, $[0,1]$ can easily be written as the union of $m = c$ such sets: for example, $[0,1] = \bigcup_{x \in [0,1]} \{x\}$. What if $\aleph_0 < m < c$?

There are other related questions you could ask yourself. For example, can $[0,1]$ be written as the union of c disjoint closed sets each of which is uncountable? The answer is "yes." For example, take a continuous onto function $f : [0,1] \to [0,1]^2$ (*a space-filling curve, whose existence you should have seen in an advanced calculus course*). For each $x \in [0,1]$, let $L_x = \{x\} \times [0,1] =$ "the vertical line segment at x in $[0,1]^2$. Then the sets $f^{-1}[L_x]$ do the job.

We could also ask: is it possible to write $[0,1]$ as the union of uncountably many pairwise disjoint closed sets each of which is countably infinite? (See Exercise VIII.E27).

Exercises

E1. Suppose $X = A \cup B$ where $A - B$ and $B - A$ are separated.

a) Prove that for any $C \subseteq X$, $\text{cl}_X C = \text{cl}_A(A \cap C) \cup \text{cl}_B(B \cap C)$.
b) Conclude that C is closed if $C \cap A$ is closed in A and $C \cap B$ is closed in B.
c) Conclude that C is open if $C \cap A$ is open in A and $C \cap B$ is open in B.
d) Suppose $X = A \cup B$ where $A - B$ and $B - A$ are separated. Prove that if $f : X \to Y$ and both $f|A$ and $f|B$ are continuous, then f is continuous.

E2. Prove that $[0, 1]$ is connected directly from the definition of connected.
(*Use the least upper bound property of* $\mathbb{R}$.)

E3. Suppose both A, B are closed subsets of $(X, \mathcal{T})$. Prove that $A - B$ is separated from $B - A$. Do the same assuming instead that A and B are both open.

E4. Suppose S is a connected subset of $(X, \mathcal{T})$. Prove that if $S \cap E \neq \emptyset$ and $S \cap (X - E) \neq \emptyset$, then $S \cap \text{Fr}\, E \neq \emptyset$.

E5. Let (X, d) and (Y, d') be two connected metric spaces. Suppose $k > 0$ and that $(a, b) \in X \times Y$. Let $K = \{(x, y) \in X \times Y : d(x, a) \leq k$ and $d'(y, b) \leq k\}$.

a) Give an example to show that the complement of K in $X \times Y$ might not be connected.
b) Prove that the complement of K in $X \times Y$ is connected if (X, d) and $Y, d')$ are unbounded spaces.

E6. Prove that there does not exist a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $f[\mathbb{P}] \subseteq \mathbb{Q}$ and $f[\mathbb{Q}] \subseteq \mathbb{P}$.
Hint: One method: What do you know about ran f? What else do you know?)
Another method: if such an f exists, let $g = \frac{1}{1+|f|}$ and let $h = g|[0,1]$. What do you know about h?

E7. a) Find the cardinality of the collection of all compact connected subsets of $\mathbb{R}^2$.
b) Find the cardinality of the collection of all connected subsets of $\mathbb{R}^2$.

E8. Suppose (X, d) is a connected metric space with $|X| > 1$. Prove that $|X| \geq c$.

E9. Suppose each point in a metric space (X, d) has a neighborhood base consisting of clopen sets (such a metric space is sometimes called *zero-dimensional*). Prove that (X, d) is totally disconnected.

E10. A metric space (X, d) satisfies the ϵ-chain condition if for all $\epsilon > 0$ and all x, $y \in X$, there exists a finite set of points x_1, x_2, ... , x_{n-1}, x_n where $x_1 = x$, $x_n = y$, and $d(x_i, x_{i+1}) < \epsilon$ for all $i = 1$, ..., $n - 1$.

a) Give an example of a metric space which satisfies the ϵ-chain condition but which is not connected.

b) Prove that if (X, d) is connected, then (X, d) satisfies the ϵ-chain condition.

c) Prove that if (X, d) is compact and satisfies the ϵ-chain condition, then X is connected.

d) Prove that $(\{0\} \times [-1, 1]) \cup \{(x, \sin\frac{1}{x}) : 0 < x \leq 1\} \subseteq \mathbb{R}^2$ is connected.
(*Use c) to give a different proof than the one given in Example 3.4.*)

e) In any space $(X, \mathcal{T})$, a simple chain from a to b is a finite collection of sets $\{A_1, ..., A_n\}$ such that:

$a \in A_1$ and $a \notin A_i$ if $i \neq 1$
$b \in A_n$ and $b \notin A_i$ if $i \neq n$
for all $i = 1, ..., n - 1,\ A_i \cap A_{i+1} \neq \emptyset$
$A_i \cap A_j = \emptyset$ if $|j - i| \neq 1$

Prove X is connected iff for every open cover $\mathcal{U}$ and every pair of points $a, b \in X,$ there is a simple chain from a to b consisting of sets taken from $\mathcal{U}$.

E11. a) Prove that X is locally connected iff the components of every open set O are also open in X.

b) The path components of a space are its maximal path connected subsets. Show that X is locally path connected iff the path components of every open set O are also open in X.

E12. Let $\mathcal{T}$ be the cofinite topology on $\mathbb{N}$. We know that $(\mathbb{N}, \mathcal{T})$ is not path connected (because of Sierpinski's Theorem applied to the closed interval $[0, 1]$). Prove that the statement "$(\mathbb{N}, \mathcal{T})$ is not path connected" is equivalent to "Sierpinski's Theorem for the case $X = [0, 1]$."

E13. Let $n > 1$ and suppose $f : [0, 1] \to \mathbb{R}^n$ is a homeomorphism (into); then ran(f) is called an arc in $\mathbb{R}^n$. Use a connectedness argument to prove that an arc is nowhere dense in $\mathbb{R}^n$. Is the same true if $[0, 1]$ is replaced by the circle S^1?

E14. a) Prove that for any space X and $n \geq 2$,

if X has $\geq n$ components, then there are nonempty pairwise separated sets $H_1, ..., H_n$ for which $X = H_1 \cup ... \cup H_n$ (**)

Hints. For a given n, do not start with the components and try to group them to form the H_n's. Start with the fact that X is not connected. Use induction. When X has infinitely many components, then X has $\geq n$ components for every n.

b) Recall that a <u>disconnection</u> of X means a pair of nonempty separated sets A, B for which $X = A \cup B$. Remember also that if C is a component of X, C is not necessarily "one piece in a disconnection of X" *(see Example 4.4)*.

Prove that X has only finitely many components $n\,(n \geq 2)$ iff X has only finitely many disconnections.

E15. A metric space (X, d) is called locally separable if, for each $x \in X$, there is an open set U containing x such that (U, d) is separable. Prove that a connected, locally separable metric space is separable.

E16. In $(X, \mathcal{T})$, define $x \sim y$ if there does <u>not</u> exist a disconnection $X = A \cup B$ with $x \in A$ and $y \in B$, i.e., if "X can't be split between x and y." Prove that $\sim$ is an equivalence relation and that the equivalence class of a point p is the quasicomponent Q_p. (*It follows that X is the disjoint union of its quasicomponents.*)

E17. For the following alphabet (capital Arial font), decide which letters are homeomorphic to each other:

A B C D E F G H I J K L M N O P Q R S T U V W X Y Z

E18. Suppose $f : \mathbb{R} \to X = \{(x, y) \in \mathbb{R}^2 : x = 0 \text{ or } y = 0\}$ is continuous and onto. Prove that $f^{-1}[\{(0,0)\}]$ contains at least 3 points.

E19. Show how to write $\mathbb{R}^2 = A \cup B$ where A and B are nonempty, disjoint, connected, dense and congruent by translation (i.e., $\exists\, (u, v) \in \mathbb{R}^2$ such that $B = \{(x+u, y+v) : (x, y) \in A\}$).

E20. Suppose X is connected and $|X| \geq 2$. Show that X can be written as $A \cup B$ where A and B are connected proper subsets of X.

E21. Prove or disprove: a nonempty product $X \times Y$ is totally disconnected iff both X and Y are totally disconnected.

Chapter V Review

Explain why each statement is true, or provide a counterexample.

1. There exists a continuous function $f : \mathbb{R} \to \mathbb{R}$ which is onto and for which $f[\mathbb{Q}] \subseteq \mathbb{N}$.

2. Let $\mathcal{T}$ be the right ray topology on $\mathbb{R}$. Then $(\mathbb{R}, \mathcal{T})$ is path connected.

3. There exists a countably infinite compact connected metric space.

4. The letter T is homeomorphic to the letter F.

5. If A and B are connected and not separated, then $A \cup B$ is connected.

6. If A and B are nonempty and $A \cup B$ is connected, then $\text{cl}\, A \cap \text{cl}\, B \neq \emptyset$.

7. If the graph of a function $f : \mathbb{R} \to \mathbb{R}$ is connected, then f is continuous.

8. $\mathbb{N}$, with the cofinite topology, is connected.

9. In a space $(X, \mathcal{T})$, the component containing the point p is a subset of the intersection of all clopen sets containing a point p.

10. If $A, B \subseteq (X, \mathcal{T})$ and A is clopen in $A \cup B$, then X is not connected.

11. If $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ is continuous and onto and $(X, \mathcal{T})$ is path connected, then $(Y, \mathcal{T}')$ is path connected.

12. Suppose $A \subseteq \mathbb{R}$, $|\, A \,| > 1$. If A is nowhere dense, then A is not connected.

13. Let C be the Cantor set. There exists a nonempty space X for which $X \times C$ is connected.

14. Let C be the Cantor set. Then $\mathbb{R}^2 - C^2$ is connected.

15. If C is a component of the complete metric space (X, d), then (C, d) is complete.

16. Let S^1 denote the unit circle in $\mathbb{R}^2$. S^1 is homeomorphic to a subspace of the Cantor set.

17. If X is the union of an uncountable collection of disjoint nonempty, connected closed sets C_α, then the C_α's are components of X.

18. If X is the union of a finite collection of disjoint nonempty connected closed sets C_α, then the C_α's are components of X.

19. If X is the union of a countable collection of disjoint connected closed sets C_α, then the C_α's are components of X.

20. If (X, d) is connected, then its completion $(\widetilde{X}, \widetilde{d})$ is connected.

21. If A is connected and B is clopen and $A \cap B \neq \emptyset$, then $A \subseteq B$.

22. Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous, that $f((0, 0, ..., 0)) = (0, 0, ..., 0)$ and $f((1, 0, ..., 0)) = (2, 0, ..., 0)$. Let A denote the set of all fixed points of f. It is possible that A is open in $\mathbb{R}^n$ (*Note: we are not assuming f is a contraction, so f may have more than one fixed point.)*

23. Suppose (X, d) is a connected separable metric space with $|X| > 1$. Then $|X| = c$.

24. If a subset A of $\mathbb{R}$ contains an open interval around each of its points, then A must be connected.

25. There exists a connected metric space (X, d) with $|X| = \aleph_0$.

26. If $\mathbb{R}^2 \supseteq A_1 \supseteq A_2 \supseteq ... \supseteq A_n \supseteq ...$ is a nested sequence of connected sets in the plane, then $\bigcap_{n=1}^{\infty} A_n$ is connected.

27. $(\mathbb{P} \times \mathbb{R}) \cup (\mathbb{R} \times \mathbb{P})$ is a connected set in $\mathbb{R}^2$.

28. Let $A = \{(x, \sin \frac{1}{x}) : x > 0\} \subseteq \mathbb{R}^2$ and suppose $f : \text{cl}\,(A) \to \{0, 1\}$ is continuous. Then f must be constant.

29. Suppose $X \neq \emptyset$. X is connected if and only if there are exactly two functions $f \in C(X)$ such that $f^2 = f$.

30. If X is nonempty, countable and connected, then every $f \in C(X)$ is constant.

31. Every path connected set in $\mathbb{R}^2$ is locally path connected.

32. If B is a dense connected set in $\mathbb{R}$, then $B = \mathbb{R}$.

33. In a metric space (X, d) the sets A and B are separated iff $d(A, B) > 0$.

34. A nonempty clopen subset of a space X must be a component of X.

35. Suppose $\mathcal{T}$ and $\mathcal{T}'$ are topologies on X and that $\mathcal{T} \subseteq \mathcal{T}'$. If $(X, \mathcal{T})$ is connected, then $(X, \mathcal{T}')$ is connected.

36. The letter X is homeomorphic to the letter Y.