

Chapter VII

Separation Axioms

1. Introduction

“Separation” refers here to whether or not objects like points or disjoint closed sets can be enclosed in disjoint open sets; “separation properties” have nothing to do with the idea of “separated sets” that appeared in our discussion of connectedness in Chapter 5 in spite of the similarity of terminology.. We have already met some simple separation properties of spaces: the T_0 , T_1 and T_2 (Hausdorff) properties. In this chapter, we look at these and others in more depth. As “more separation” is added to spaces, they generally become nicer and nicer – especially when “separation” is combined with other properties. For example, we will see that “enough separation” and “a nice base” guarantees that a space is metrizable.

“Separation axioms” translates the German term *Trennungsaxiome* used in the older literature. Therefore the standard separation axioms were historically named T_0 , T_1 , T_2 , T_3 , and T_4 , each stronger than its predecessors in the list. Once these were common terminology, another separation axiom was discovered to be useful and “interpolated” into the list: $T_{3\frac{1}{2}}$. It turns out that the $T_{3\frac{1}{2}}$ spaces (also called Tychonoff spaces) are an extremely well-behaved class of spaces with some very nice properties.

2. The Basics

Definition 2.1 A topological space X is called a

- 1) T_0 space if, whenever $x \neq y \in X$, there either exists an open set U with $x \in U, y \notin U$ or there exists an open set V with $y \in V, x \notin V$
- 2) T_1 space if, whenever $x \neq y \in X$, there exists an open set U with $x \in U, y \notin U$ and there exists an open set V with $y \in V, x \notin V$
- 3) T_2 space (or, Hausdorff space) if, whenever $x \neq y \in X$, there exist disjoint open sets U and V in X such that $x \in U$ and $y \in V$.

It is immediately clear from the definitions that $T_2 \Rightarrow T_1 \Rightarrow T_0$.

Example 2.2

- 1) X is a T_0 space iff whenever $x \neq y$, then $\mathcal{N}_x \neq \mathcal{N}_y$ – that is, different points in X have different neighborhood systems.
- 2) If $|X| > 1$ and X has the trivial topology, then X is not a T_0 space.
- 3) A pseudometric space (X, d) is metric iff (X, d) is a T_0 space.

Clearly, a metric space is T_0 . On the other hand, suppose (X, d) is T_0 and that $x \neq y$. Then for some $\epsilon > 0$ either $x \notin B_\epsilon(y)$ or $y \notin B_\epsilon(x)$. Either way, $d(x, y) \geq \epsilon$, so d is a metric.

4) In any topological space X we can define an equivalence relation $x \sim y$ iff $\mathcal{N}_x = \mathcal{N}_y$. and let $g : X \rightarrow Y = X / \sim$ by $g(x) = [x]$. Give Y the quotient topology. Then g is continuous, onto, open (not automatic for a quotient map!) and the quotient is a T_0 space:

If O is open in X , we want to show that $g[O]$ is open in Y , and because Y has the quotient topology this is true iff $g^{-1}[g[O]]$ is open in X . But $g^{-1}[g[O]]$
 $= \{x \in X : g(x) \in g[O]\} = \{x \in X : \text{for some } y \in O, g(x) = g(y)\}$
 $= \{x \in X : x \text{ is equivalent to some point } y \text{ in } O\} = O$.

If $[x] \neq [y] \in Y$, then x is not equivalent to y , so there is an open set $O \subseteq X$ with (say) $x \in O$ and $y \notin O$. Since g is open, $g[O]$ is open in Y and $[x] \in g[O]$. Moreover, $[y] \notin g[O]$ or else y would be equivalent to some point of O – implying $y \in O$.

Y is called the T_0 -identification of X . This identification turns any space into a T_0 space by identifying points that have identical neighborhoods. If X is a T_0 space to begin with, then g is 1–1 and g is a homeomorphism. Applied to a T_0 space, the T_0 -identification accomplishes nothing. If (X, d) is a pseudometric space, the T_0 -identification is the same as the metric identification discussed in Example VI.5.6 : in that case $\mathcal{N}_x = \mathcal{N}_y$ iff $d(x, y) = 0$.

5) For $i = 0, 1, 2$: if (X, \mathcal{T}) is a T_i space and $\mathcal{T}' \supseteq \mathcal{T}$ is a new topology on X , then (X, \mathcal{T}') is also a T_i space.

Example 2.3

1) (Exercise) It is easy to check that a space X is a T_1 space

iff for each $x \in X$, $\{x\}$ is closed
iff for each $x \in X$, $\{x\} = \bigcap \{O : O \text{ is open and } x \in O\}$

2) A finite T_1 space is discrete.

3) Sierpinski space $X = \{0, 1\}$ with topology $\mathcal{T} = \{\emptyset, \{1\}, \{0, 1\}\}$ is T_0 but not T_1 : $\{1\}$ is an open set that contains 1 and not 0; but there is no open set containing 0 and not 1.

4) \mathbb{R} , with the right-ray topology, is T_0 but not T_1 : if $x < y \in \mathbb{R}$, then $O = (x, \infty)$ is an open set that contains y and not x ; but there is no open set that contains x and not y .

5) With the cofinite topology, \mathbb{N} is T_1 but not T_2 : in an infinite cofinite space, any two nonempty open sets have nonempty intersection.

These separation properties are very well-behaved with respect to subspaces and products.

Theorem 2.4 For $i = 0, 1, 2$:

- a) A subspace of a T_i space is a T_i space
- b) If $X = \prod_{\alpha \in A} X_\alpha \neq \emptyset$, then X is a T_i space iff each X_α is a T_i space.

Proof All of the proofs are easy. We consider only the case $i = 1$, leaving the other cases as an exercise.

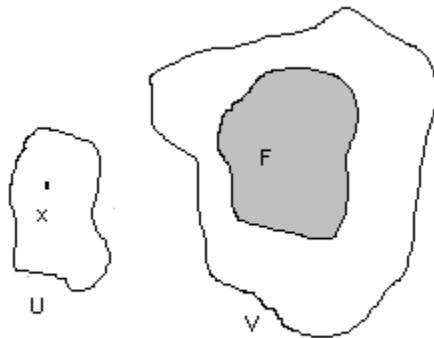
Suppose $a \neq b \in A \subseteq X$, where X is a T_1 space. If U' is an open set in X containing a but not b , then $U = U' \cap A$ is an open set in A containing a but not b . Similarly we can find an open set V in A containing b but not a . Therefore A is a T_1 space.

Suppose $X = \prod_{\alpha \in A} X_\alpha$ is a nonempty T_1 space. Each X_α is homeomorphic to a subspace of X , so each x_α is T_1 (by part a)). Conversely, suppose each X_α is T_1 and that $x \neq y \in X$. Then $x_\alpha \neq y_\alpha$ for some α . Pick an open set U_α in X_α containing x_α but not y_α . Then $U = \langle U_\alpha \rangle$ is an open set in X containing x but not y . Similarly, we find an open set V in X containing y but not x . Therefore X is a T_1 space. •

Exercise 2.5 Is a continuous image of a T_i space necessarily a T_i space? How about a quotient? A continuous open image?

We now consider a slightly different kind of separation axiom for a space X : formally, the definition is “just like” the definition of T_2 , but with a closed set replacing one of the points.

Definition 2.6 A topological space X is called regular if whenever F is a closed set and $x \notin F$, containing x , there exist disjoint open sets U and V such that $x \in U$ and $F \subseteq V$.

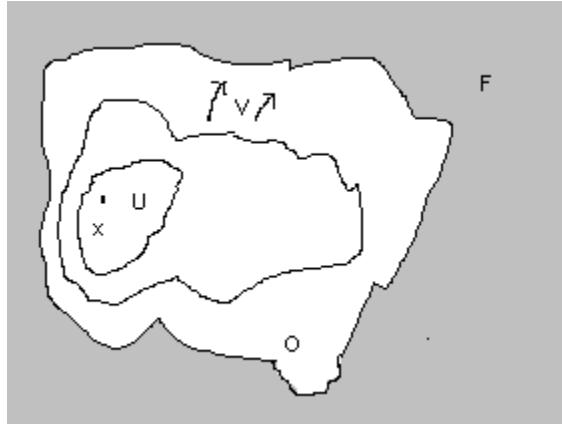


There are some easy variations on the definition of “regular” that are useful to recognize.

Theorem 2.7 The following are equivalent for any space X :

- i) X is regular
- ii) if O is an open set containing x , then there exists an open set $U \subseteq X$ such that $x \in U \subseteq \text{cl } U \subseteq O$
- iii) at each point $x \in X$ there exists a neighborhood base consisting of closed neighborhoods .

Proof i) \Rightarrow ii) Suppose X is regular and O is an open set with $x \in O$. Letting $F = X - O$, we use regularity to get disjoint open sets U, V with $x \in U$ and $F \subseteq V$ as illustrated below:



Then $x \in U \subseteq \text{cl } U \subseteq O$ (since $\text{cl } U \subseteq X - V$).

ii) \Rightarrow iii) If $N \in \mathcal{N}_x$, then $x \in O = \text{int } N$. By ii), we can find an open set U so that $x \in U \subseteq \text{cl } U \subseteq O$. Since $\text{cl } U$ is a neighborhood of x , the closed neighborhoods of x form a neighborhood base at x .

iii) \Rightarrow i) Suppose F is closed and $x \notin F$. By ii), there is a closed neighborhood K of x such that $x \in K \subseteq X - F$. We can choose $U = \text{int } K$ and $V = X - K$ to complete the proof that X is regular. •

Example 2.8 Every pseudometric space (X, d) is regular. Suppose $a \notin F$ and F is closed. We have a continuous function $f(x) = d(x, F)$ for which $f(a) = c > 0$ and $f|F = 0$. This gives us disjoint open sets with $a \in U = f^{-1}[(\frac{c}{2}, \infty)]$ and $F \subseteq V = f^{-1}[(- \infty, \frac{c}{2})]$. Therefore X is regular.

At first glance, one might think that regularity is a stronger condition than T_2 . But this is false: if (X, d) is a pseudometric space but not a metric space, then X is regular but not even T_0 .

To bring things into line, we make the following definition.

Definition 2.9 A topological space X is called a T_3 space if X is regular and T_1 .

It is easy to show that $T_3 \Rightarrow T_2$ ($\Rightarrow T_1 \Rightarrow T_0$): suppose X is T_3 and $x \neq y \in X$. Then $F = \{y\}$ is closed so, by regularity, there are disjoint open sets U, V with $x \in U$ and $y \in \{y\} \subseteq V$.

Caution The terminology varies from book to book. For some authors, the definition of “regular” includes T_1 : for them, “regular” means what we have called “ T_3 .” Check the definitions when reading other books.

Exercise 2.10 Show that a regular T_0 space must be T_3 (so it would have been equivalent to use “ T_0 ” instead of “ T_1 ” in the definition of “ T_3 ”).

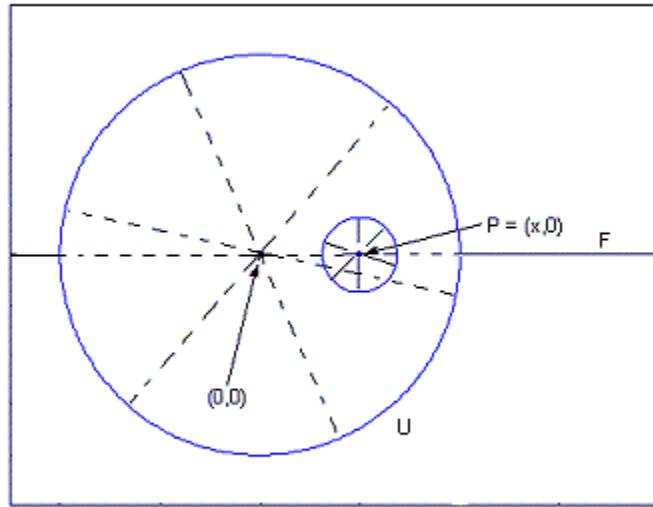
Example 2.11 $T_2 \neq T_3$. We will put a new topology on the set $X = \mathbb{R}^2$. At each point $x \in X$, let a neighborhood base \mathcal{B}_x consist of all sets N of the form

$$N = B_\epsilon(x) - (\text{a finite number of straight lines through } x) \cup \{x\} \text{ for some } \epsilon > 0.$$

(Check that the conditions in the Neighborhood Base Theorem III.5.2 are satisfied.) With the resulting topology, X is called the slotted plane. Note that $B_\epsilon(x) \in \mathcal{B}_x$ (because “0” is a finite number), so each $B_\epsilon(x)$ is among the basic neighborhoods in \mathcal{B}_x – so the slotted plane topology on \mathbb{R}^2 contains the usual Euclidean topology. It follows that X is T_2 .

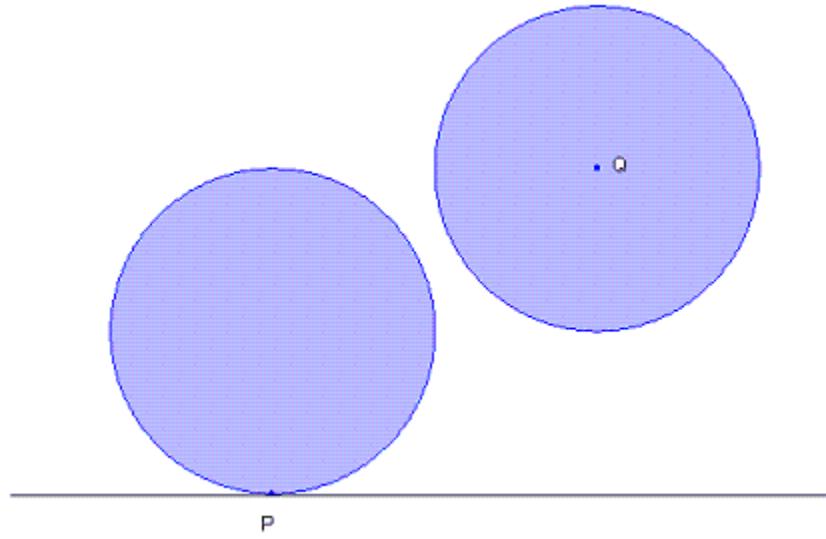
The set $F = \{(x, 0) : x \neq 0\} = \text{“the } x\text{-axis with the origin deleted”}$ is a closed set in X (why?).

If U is any open set containing the origin $(0, 0)$, then there is a basic neighborhood N with $(0, 0) \in N \subseteq U$. Using the ϵ in the definition of N , we can choose a point $P = (x, 0) \in F$ with $0 < x < \epsilon$. Any basic neighborhood set of P must intersect N (why?) and therefore must intersect U . It follows that $(0, 0)$ and F cannot be separated by disjoint open sets, so the slotted plane is not regular (and therefore not T_3).



Note: The usual topology in \mathbb{R}^2 is regular. This example shows that an “enlargement” of a regular (or T_3) topology may not be regular (or T_3). Although the enlarged topology has more open sets to work with, there are also more “point/closed set pairs x, F ” that need to be separated. Of course it is easy to see that an “enlargement” of a T_i topology ($i = 0, 1, 2$) is also T_i .

Example 2.12 The Moore plane Γ (Example III.5.6) is clearly T_2 . In fact, at each point, there is a neighborhood base of closed neighborhoods. The figure illustrates this for a point P on the x -axis and a point Q above the x -axis. Therefore Γ is T_3 .



Theorem 2.13 a) A subspace of a regular (T_3) space is regular (T_3) .
b) Suppose $X = \prod_{\alpha \in A} X_\alpha \neq \emptyset$. X is regular (T_3) iff each X_α is regular (T_3) .

Proof a) Let $A \subseteq X$ where X is regular. Suppose $a \in A$ and that F is a closed set in A that does not contain a . There exists a closed set F' in X such that $F' \cap A = F$. Choose disjoint open sets U' and V' in X with $a \in U'$ and $V' \supseteq F'$. Then $U = U' \cap A$ and $V = V' \cap A$ are open in A , disjoint, $a \in U$, and $F \subseteq V$. Therefore A is regular.

b) If $X = \prod_{\alpha \in A} X_\alpha \neq \emptyset$ is regular, then part a) implies that each X_α is regular – since each X_α is homeomorphic to a subspace of X . Conversely, suppose each X_α is regular and that $U = \langle U_{\alpha_1}, \dots, U_{\alpha_n} \rangle$ is a basic open set containing x . For each α_i , we can pick an open set V_{α_i} in X_{α_i} such that $x_{\alpha_i} \in V_{\alpha_i} \subseteq \text{cl } V_{\alpha_i} \subseteq U_{\alpha_i}$. Then $x \in V = \langle V_{\alpha_1}, \dots, V_{\alpha_n} \rangle \subseteq \text{cl } V \subseteq \langle \text{cl } V_{\alpha_1}, \dots, \text{cl } V_{\alpha_n} \rangle \subseteq U$. (Why is the last inclusion true?) Therefore X is regular.

Since “ T_1 ” is hereditary and productive, a) and b) also hold for “ T_3 ” •

The obvious “next step up” in separation is the following:

Definition 2.14 A topological space X is called normal if, whenever A, B are disjoint closed sets in X , there exist disjoint open sets U, V in X with $A \subseteq U$ and $B \subseteq V$. X is called a T_4 space if X is normal and T_1 .

Example 2.15 a) Every pseudometric space (X, d) is normal (and therefore every metric space is T_4). In fact, if A and B are disjoint closed sets, we can define $f(x) = \frac{d(x, B)}{d(x, A) + d(x, B)}$. Since the

denominator cannot be 0, f is continuous and $f|A = 0$, $f|B = 1$. The open sets $U = \{x : f(x) < \frac{1}{2}\}$ and $V = \{x : f(x) > \frac{1}{2}\}$ are disjoint and contain A and B respectively. Therefore X is normal.

Note: the argument given is slick and clean. Can you show (X, d) is normal by constructing directly a pair of open sets separating A and B ?

- b) Let \mathbb{R} have the right ray topology $\mathcal{T} = \{(x, \infty); x \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$.
 - i) $(\mathbb{R}, \mathcal{T})$ is normal : the only possible pair of disjoint closed sets is \emptyset and X and we can separate these using the disjoint open sets $U = \emptyset$ and $V = X$.
 - ii) $(\mathbb{R}, \mathcal{T})$ is not regular: for example 1 is not in the closed set $F = (-\infty, 0]$, but every open set that contains F also contains 1. So “normal $\not\Rightarrow$ regular.”
 - iii) $(\mathbb{R}, \mathcal{T})$ is not a T_1 space : so T_4 is a stronger condition than normality.

But when we combine “normal + T_1 ” into T_4 , we have a property that fits perfectly into the separation hierarchy.

Theorem 2.16 $T_4 \Rightarrow T_3 (\Rightarrow T_2 \Rightarrow T_1 \Rightarrow T_0)$

Proof Suppose X is T_4 . If F is a closed set not containing x , then $\{x\}$ and F are disjoint closed sets. By normality, we can find disjoint open sets separating $\{x\}$ and F . It follows that X is regular and therefore T_3 . •

Exercises

E1. A topological space X is called a door space if every subset is either open or closed. Prove that a T_2 space is a door space if and only if it has at most one non-isolated point.

E2. A base for the closed sets in a space X is a collection of \mathcal{F} of closed subsets such that every closed set F is an intersection of sets from \mathcal{F} . Clearly, \mathcal{F} is a base for the closed sets in X iff $\mathcal{B} = \{O : O = X - F, F \in \mathcal{F}\}$ is a base for the open sets in X .

For a polynomial P in n real variables, define the zero set of P as

$$Z(P) = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : P(x_1, x_2, \dots, x_n) = 0\}$$

- a) Prove that $\{Z(P) : P \text{ a polynomial in } n \text{ real variables}\}$ is the base for the closed sets of a topology (called the Zariski topology) on \mathbb{R}^n .
- b) Prove that the Zariski topology on \mathbb{R}^n is T_1 but not T_2 .
- c) Prove that the Zariski topology on \mathbb{R} is the cofinite topology, but that if $n > 1$, the Zariski topology on \mathbb{R}^n is not the cofinite topology.

Note: The Zariski topology arises in studying algebraic geometry. After all, the sets $Z(P)$ are rather special geometric objects—those “surfaces” in \mathbb{R}^n which can be described by polynomial equations $P(x_1, x_2, \dots, x_n) = 0$.

E3. A space X is a $T_{5/2}$ space if, whenever $x \neq y \in X$, there exist open sets U and V such that $x \in U$, $y \in V$ and $\text{cl } U \cap \text{cl } V = \emptyset$. (Clearly, $T_3 \Rightarrow T_{5/2} \Rightarrow T_2$.)

- a) Prove that a subspace of a $T_{5/2}$ space is a $T_{5/2}$ space.
- b) Suppose $X = \prod X_\alpha \neq \emptyset$. Prove that X is $T_{5/2}$ iff each X_α is $T_{5/2}$.
- c) Let $S = \{(x, y) \in \mathbb{R}^2 : y \geq 0\}$ and $L = \{(x, 0) \in S : y = 0\}$. Define a topology on S with the following neighborhood bases:

$$\begin{aligned} \text{if } p \in S - L : \quad \mathcal{B}_p &= \{B_\epsilon(p) \cap S : \epsilon > 0\} \\ \text{if } p \in L : \quad \mathcal{B}_p &= \{B_\epsilon(p) \cap (S - L) \cup \{p\} : \epsilon > 0\} \end{aligned}$$

(You may assume that these \mathcal{B}_p 's satisfy the axioms for a neighborhood base.)

Prove that S is $T_{T_{5/2}}$ but not T_3 .

E4. If $A \subseteq X$, we can define a topology on X by

$$\mathcal{T} = \{O \subseteq X : O \supseteq A\} \cup \{\emptyset\}$$

Decide whether or not (X, \mathcal{T}) is normal.

E5. A function $f : X \rightarrow Y$ is called perfect if f is continuous, closed, onto, and, for each $y \in Y$, $f^{-1}(y)$ is compact. Prove that if X is regular and f is perfect, then Y is regular; and that if X is T_3 , the Y is also T_3 .

E6. a) Suppose X is finite. Prove that (X, \mathcal{T}) is regular iff there is a partition \mathcal{B} of X that is a base for the topology.

For parts b)-e) assume that X is a T_3 -space (not necessarily finite).

b) Suppose A is closed in X . Prove that

i) Prove that $A = \bigcap\{O : O \text{ is open and } A \subseteq O\}$.

ii) Define $x \sim y$ iff $x = y$ or $x, y \in A$. Prove that the quotient space X/\sim is Hausdorff.

c) Suppose B is an infinite subset of X . Prove that there exists a sequence of open sets U_n such that each $U_n \cap B \neq \emptyset$ and then $\text{cl } U_n \cap \text{cl } U_m = \emptyset$ if $n \neq m$.

d) Suppose each point y in a space Y has a neighborhood V such that $\text{cl } V$ is regular. Prove that Y is regular.

e) Give an example to show that a compact subset K of a space X need not be closed, but show that if X is regular then $\text{ck } K$ is compact.

3. Completely Regular Spaces and Tychonoff Spaces

The T_3 property is well-behaved. For example, we saw in Theorem 2.13 that the T_3 property is hereditary and productive. However, it is not a strong enough property to give us really nice theorems.

For example, it's very useful in studying a space X if there are a lot of (nonconstant) continuous real-valued functions on X available. Remember how many times we have used the fact that continuous real-valued functions f can be defined on a metric space (X, d) using formulas like $f(x) = d(x, a)$ or $f(x) = d(x, F)$; when $|X| > 1$, we get many nonconstant real functions defined on (X, d) . But a T_3 space can be very deficient in continuous real-valued functions – in 1946, Hewett gave an example of a infinite T_3 space H on which the only continuous real-valued functions are the constant functions.

On the other hand, the T_4 property is strong enough to give some really nice theorems (for example, see Theorems 5.2 and 5.6 later in this chapter). But T_4 spaces turn out to also have some very bad behavior: the T_4 property is not hereditary (*explain why a proof analogous to the one given for Theorem 2.13b doesn't work*) and not even finitely productive. Examples of this bad behavior are a little hard to get right now, but they will appear rather naturally later.

These observations lead us (out of historical order) to look at a class of spaces with separation somewhere “between T_3 and T_4 .” We want a group of spaces that is well-behaved, but also with enough separation to give us some very nice theorems. We begin with some notation and a lemma.

Recall that $C(X) = \{f \in X^{\mathbb{R}} : f \text{ is continuous}\} = \text{the collection of continuous real-valued functions on } X$
 $C^*(X) = \{f \in C(X) : f \text{ is continuous and bounded}\}.$

Lemma 3.1 Suppose $f, g \in C(X)$. Define real-valued functions $f \vee g$ and $f \wedge g$ by

$$\begin{aligned}(f \vee g)(x) &= \max \{f(x), g(x)\} \\ (f \wedge g)(x) &= \min \{f(x), g(x)\}\end{aligned}$$

Then $f \vee g$ and $f \wedge g$ are in $C(X)$.

Proof We want to prove that the max or min of two continuous real-valued functions is continuous. But this follows immediately from the formulas

$$\begin{aligned}(f \vee g)(x) &= \frac{f(x) + g(x)}{2} + \frac{|f(x) - g(x)|}{2} \\ (f \wedge g)(x) &= \frac{f(x) + g(x)}{2} - \frac{|f(x) - g(x)|}{2}\end{aligned} \bullet$$

Definition 3.2 A space X is called completely regular if whenever F is a closed set and $x \notin F$, there exists a function $f \in C(X)$ such that $f(x) = 0$ and $f|F = 1$.

Note i) The definition requires only that $f|F = 1$, in other words, that $f^{-1}[\{1\}] \supseteq F$. However, the two sets might not be equal.

ii) If there is such a function f , there is also a continuous $g : X \rightarrow [0, 1]$ such that $g(x) = 0$ and $g|F = 1$. For example, we could use $g = (f \vee 0) \wedge 1$ which, by Lemma 3.1, is continuous.

iii) If $g(x) = 0$, $g|F = 1$ and $g : X \rightarrow [0, 1]$, we could compose g with a linear homeomorphism $\phi : [0, 1] \rightarrow [a, b]$ to get a continuous function $h = \phi \circ g : X \rightarrow [a, b]$ where $h(x) = a$ and $h|F = b$ (or vice versa). (*It must be true that either $\phi(0) = a$ and $\phi(1) = b$, or vice versa: why?*)

Putting these observations together, we see Definition 3.2 is equivalent to:

Definition 3.2' A space X is called completely regular if whenever F is a closed set, $x \notin F$, and $a \neq b$ are real numbers, then there exists a continuous function $f : X \rightarrow \mathbb{R}$ for which $f(x) = a$ and $f|F = b$, and for all x , $a \leq f(x) \leq b$.

Informally, “completely regular” means that “ x and F can be separated by a bounded continuous real-valued function.”

The definition of “Tychonoff space” is quite different from the definitions of the other separation properties since the definition isn’t “internal” – it makes use of an “outside” topological space, \mathbb{R} , as part of its definition. Although it is possible to contrive a purely internal definition of “Tychonoff space,” the definition is complicated and seems completely unnatural: it simply imposes some fairly unintuitive conditions to force the existence of enough functions in $C(X)$.

Example 3.3 Suppose $a \notin F \subseteq (X, d)$ where (X, d) is a pseudometric space. Then $f(x) = \frac{d(x, F)}{d(a, F)}$ is continuous, $f(a) = 1$ and $f|F = 0$. So (X, d) is completely regular (*and so a completely regular space might not even be T_0*).

Definition 3.4 A completely regular T_1 space X is called a Tychonoff space (or $T_{3\frac{1}{2}}$ space).

Theorem 3.5 $T_{3\frac{1}{2}} \Rightarrow T_3$ ($\Rightarrow T_2 \Rightarrow T_1 \Rightarrow T_0$)

Proof Suppose F is a closed set in X not containing x . If X is $T_{3\frac{1}{2}}$, we can choose $f \in C(X)$ with $f(x) = 0$ and $f|F = 1$. Then $U = f^{-1}((-\infty, \frac{1}{2}))$ and $V = f^{-1}([\frac{1}{2}, \infty))$ are disjoint open sets with $x \in U$, $F \subseteq V$. Therefore X is regular. Since X is T_1 , X is T_3 . •

Hewitt's example of a T_3 space on which every continuous real-valued function is constant is more than enough to show that a T_3 space may not be $T_{3\frac{1}{2}}$ (the example, in *Ann. Math.*, 47(1946) 503-509, is rather complicated.). For that purpose, it is a little easier – but still nontrivial – to find a T_3 space X containing two points p, q such that for all $f \in C(X)$, $f(p) = f(q)$. Then p and $\{q\}$ cannot be separated by a function from $C(X)$ so X is not $T_{3\frac{1}{2}}$. (DJ. Thomas, *A regular space, not completely regular*, American Mathematical Monthly, 76(1969), 181-182). The space X can then be used to construct an infinite T_3 space H (simpler than Hewitt's example) on which every continuous real-valued function is constant (see Gantner, *A regular space on which every continuous real-valued function is constant*, American Mathematical Monthly, 78(1971), 52.) Although we will not present these constructions here, we will occasionally refer to H in comments later in this section.

Note: We have not shown that $T_4 \Rightarrow T_{3\frac{1}{2}}$. This statement is true (as the notation suggests), but it is not at all easy to prove: try it! This result will follow as a Corollary in Section 4.

Tychonoff spaces continue the pattern of “good behavior” that we saw earlier and will also turn out to be a rich class of spaces to study.

Theorem 3.7 a) A subspace of a completely regular ($T_{3\frac{1}{2}}$) space is completely regular ($T_{3\frac{1}{2}}$).

b) Suppose $X = \prod_{\alpha \in A} X_\alpha \neq \emptyset$. X is completely regular ($T_{3\frac{1}{2}}$) iff each X_α is completely regular ($T_{3\frac{1}{2}}$).

Proof Suppose $a \notin F \subseteq A \subseteq X$, where X is completely regular and F is a closed set in A . Pick a closed set K in X such that $K \cap A = F$ and an $f \in C(X)$ such that $f(a) = 0$ and $f|K = 1$. Then $g = f|A \in C(A)$, $g(a) = 0$ and $g|F = 1$. Therefore A is completely regular.

If $\emptyset \neq X = \prod_{\alpha \in A} X_\alpha$ is completely regular, then each X_α is homeomorphic to a subspace of X so each X_α is completely regular. Conversely, suppose each X_α is completely regular and that F is a closed set in X not containing a . There is a basic open set U such that

$$a \in U = \langle U_{\alpha_1}, \dots, U_{\alpha_n} \rangle \subseteq X - F$$

For each $i = 1, \dots, n$ we can pick a continuous function $f_{\alpha_i} : X_{\alpha_i} \rightarrow [0, 1]$ with $f_{\alpha_i}(a_{\alpha_i}) = 0$ and $f_{\alpha_i}|(X_{\alpha_i} - U_{\alpha_i}) = 1$. Define $f : X \rightarrow [0, 1]$ by

$$f(x) = \max \{(f_{\alpha_i} \circ \pi_{\alpha_i})(x) : i = 1, \dots, n\} = \max \{f_{\alpha_i}(x_{\alpha_i}) : i = 1, \dots, n\}$$

Then f is continuous and $f(a) = \max \{f_{\alpha_i}(a_{\alpha_i}) : i = 1, \dots, n\} = 0$. If $x \in F$, then for some i , $x_{\alpha_i} \notin U_{\alpha_i}$ and $f_{\alpha_i}(x_{\alpha_i}) = 1$, so $f(x) = 1$. Therefore $f|F = 1$ and X is completely regular.

Since the T_1 property is both hereditary and productive, the statements in a) and b) also hold for $T_{3\frac{1}{2}}$. •

Corollary 3.8 $[0, 1]^m$ and all its subspaces are $T_{3\frac{1}{2}}$.

Since a Tychonoff space X is defined using $C(X)$, we expect that continuous real-valued functions have a close relationship to the topology on X . We want to explore that connection.

Definition 3.9 Suppose $f \in C(X)$. Then $Z(f) = f^{-1}[\{0\}] = \{x \in X : f(x) = 0\}$ is called the zero set of f . If $A = Z(f)$ for some $f \in C(X)$, we call A a zero set in X . The complement of a zero set in X is called a cozero set: $\text{coz}(f) = X - Z(f) = \{x \in X : f(x) \neq 0\}$.

A zero set $Z(f)$ in X is closed because f is continuous. In addition, $Z(f) = \bigcap_{n=1}^{\infty} O_n$, where $O_n = \{x \in X : |f(x)| < \frac{1}{n}\}$. Each O_n is open. Therefore a zero set is always a closed G_δ -set. Taking complements shows that $\text{coz}(f)$ is always an open F_σ -set in X .

For $f \in C(X)$, let $g = (-1 \vee f) \wedge 1 \in C^*(X)$. Then $Z(f) = Z(g)$. Therefore $C(X)$ and $C^*(X)$ produce the same zero in X (and therefore also the same cozero sets).

Example 3.10

- 1) A closed set F in a pseudometric space (X, d) is a zero set: $F = Z(f)$, where $f(x) = d(x, F)$.
- 2) In general, a closed set might not be a zero set – if fact, a closed set in X might not even be a G_δ set.

Suppose X is uncountable and $p \in X$. Define a topology on X by letting $\mathcal{B}_x = \{\{x\}\}$ be a neighborhood at each point $x \neq p$ and letting $\mathcal{B}_p = \{B : p \in B \text{ and } X - B \text{ is countable}\}$ be the neighborhood base at p . (*Check that the conditions of the Neighborhood Base Theorem III.5.2 are satisfied.*)

All points in $X - \{p\}$ are isolated and X is clearly T_1 . In fact, X is T_4 .

If A and B are disjoint closed sets in X , then one of them (say A) satisfies $A \subseteq X - \{p\}$ and therefore A is clopen. We then have open sets $U = A$ and $V = X - A \supseteq B$ that work in the definition of normality.

We do not know, at this point that $T_4 \Rightarrow T_{3\frac{1}{2}}$, but we can argue directly that this X is also $T_{3\frac{1}{2}}$.

If F is a closed set not containing x , then either $F \subseteq X - \{p\}$ or $\{x\} \subseteq X - \{p\}$. So one of the sets F or $\{p\}$ is clopen. The characteristic function of this clopen set is continuous and works to show that X is completely regular.

The set $\{p\}$ is a closed set but $\{p\}$ is not a G_δ set in X (so $\{p\}$ is not a zero set in X).

Suppose $\{p\} = \bigcap_{n=1}^{\infty} O_n$ where O_n is open. For each n , $p \in B_n \subseteq O_n$ for some $B_n \in \mathcal{B}_p$. Therefore $X - \{p\} = X - \bigcap_{n=1}^{\infty} O_n = \bigcup_{n=1}^{\infty} (X - O_n) \subseteq \bigcup_{n=1}^{\infty} (X - B_n)$ is a countable set – which is impossible since X is uncountable.

Even when F is both a closed set and a G_δ set, F might not be a zero set. We will see examples later.

For technical purposes, it is convenient to notice that zero sets and cozero sets can be described in a many different forms. For example, if $f \in C(X)$, then each set in the left column is a zero set obtained from some other function $g \in C(X)$:

$$\begin{array}{lll}
 Z = \{x : f(x) = r\} & = Z(g), & \text{where } g(x) = f(x) - r \\
 Z = \{x : f(x) \geq 0\} & = Z(g), & \text{where } g(x) = f(x) - |f(x)| \\
 Z = \{x : f(x) \leq 0\} & = Z(g), & \text{where } g(x) = f(x) + |f(x)| \\
 Z = \{x : f(x) \geq r\} & = Z(g), & \text{where } g(x) = (f(x) - r) - |f(x) - r| \\
 Z = \{x : f(x) \leq r\} & = Z(g), & \text{where } g(x) = - (f(x) - r) - |f(x) - r|
 \end{array}$$

On the other hand, if $g \in C(X)$, we can write $Z(g)$ in any of the forms listed above by choosing an appropriate function $f \in C(X)$:

$$\begin{aligned}
 Z(g) &= \{x : f(x) = r\} & \text{where } f(x) = r + g(x) + r \\
 Z(g) &= \{x : f(x) \geq 0\} & \text{where } f(x) = -|g(x)| \\
 Z(g) &= \{x : f(x) \leq 0\} & \text{where } f(x) = |g(x)| \\
 Z(g) &= \{x : f(x) \geq r\} & \text{where } f(x) = r - |g(x)| \\
 Z(g) &= \{x : f(x) \leq r\} & \text{where } f(x) = r + |g(x)|
 \end{aligned}$$

Taking complements, we see that if $f \in C(X)$ then a set with any one of the forms $\{x : f(x) \neq r\}$, $\{x : f(x) < 0\}$, $\{x : f(x) > 0\}$, $\{x : f(x) < r\}$, $\{x : f(x) > r\}$ is a cozero set, and that any given cozero set can be written in any one of these forms.

Using the terminology of cozero sets, we can see a nice comparison/contrast between regularity and complete regularity. Suppose $x \notin F$, where F is closed in X . If X is regular, we can find disjoint open sets U and V with $x \in U$ and $F \subseteq V$. If X is completely regular and we choose $f \in C(X)$ with $f(x) = 0$ and $f|F = 1$, then

$$x \in U = \{x : f(x) < \frac{1}{2}\} \text{ and } F \subseteq V = \{x : f(x) > \frac{1}{2}\}$$

Thus, in a completely regular space we can “separate” x and F with special disjoint open sets: cozero sets. In fact this observation (according to Theorem 3.12, below) characterizes completely regular spaces – that is, if a regular space fails to be completely regular, it is because there is a “shortage” of cozero sets: because $C(X)$ contains “too few” functions. In the extreme case of a T_3 space H on which the only continuous real valued functions are constant (*see the remarks at the beginning of this section*), the only cozero sets are \emptyset and H !

The next theorem gives the connections between the cozero sets, $C(X)$ and the weak topology on X .

Theorem 3.11 For any space (X, \mathcal{T}) , $C(X)$ and $C^*(X)$ induce the same weak topology on X . A base for this weak topology is the collection of all cozero sets in X .

Proof A subbase for the weak topology generated by $C(X)$ consists of all sets of the form $f^{-1}[U]$, where U is open in \mathbb{R} and $f \in C(X)$. Without loss of generality, we can assume the sets U are subbasic open sets of the form (a, ∞) and $(-\infty, b)$, so the sets $f^{-1}[U]$ have the form $\{x \in X : f(x) > a\}$ or $\{x \in X : f(x) < b\}$. But these are cozero sets of X , and every cozero set in X has this form. So the cozero sets are a subbase for the weak topology generated by $C(X)$. In fact, the cozero sets are actually a base because $\text{coz}(f) \cap \text{coz}(g) = \text{coz}(fg)$: the intersection of two cozero sets is a cozero set.

The same argument, with $C^*(X)$ replacing $C(X)$, shows that the cozero sets of $C^*(X)$ are a base for the weak topology on X generated by $C^*(X)$. But $C(X)$ and $C^*(X)$ have the same cozero sets in X , and therefore generate the same weak topology on X . •

Now we can now see the close connection between X and $C(X)$ in completely regular spaces. For any space (X, \mathcal{T}) , the functions in $C(X)$ are continuous with respect to \mathcal{T} (by definition of $C(X)$), but is \mathcal{T} the smallest topology making this collection of functions continuous? In other words, is \mathcal{T}

the weak topology on X generated by $C(X)$? The next theorem says that is true precisely when X is completely regular.

Theorem 3.12 For any space (X, \mathcal{T}) , the following are equivalent:

- a) X is completely regular
- b) The cozero sets of X are a base for the topology on X (equivalently, the zero sets of X are a base for the closed sets—meaning that every closed set is an intersection of zero sets)
- c) X has the weak topology from $C(X)$ (equivalently, from $C^*(X)$)
- d) $C(X)$ (equivalently, $C^*(X)$) separates points from closed sets.

Proof The preceding theorem shows that b) and c) are equivalent.

a) \Rightarrow b) Suppose $x \in O$ where O is open. Let $F = X - O$. Then we can choose $f \in C(X)$ with $f(x) = 0$ and $f|F = 1$. Then $U = \{x : f(x) < \frac{1}{2}\}$ is a cozero set for which $x \in U \subseteq O$. Therefore the cozero sets are a base for X .

b) \Rightarrow d) Suppose F is a closed set not containing x . By b), we can choose $f \in C(X)$ so that $x \in \text{coz}(f) \subseteq X - F$. Then $f(x) = r \neq 0$, so $f(x) \notin \text{cl } f[F] = \{0\}$. Therefore $C(X)$ separates points and closed sets.

d) \Rightarrow a) Suppose F is a closed set not containing x . For some $f \in C(X)$ we have $f(x) \notin \text{cl } f[F]$. Without loss of generality (why?), we can assume $f(x) = 0$. Then, for some $\epsilon > 0$, $(-\epsilon, \epsilon) \cap f[F] = \emptyset$, so that for $x \in F$, $|f(x)| \geq \epsilon$. Define $g \in C^*(X)$ by $g(x) = \min\{|f(x)|, \epsilon\}$. Then $g(x) = 0$ and $g|F = \epsilon$, so X is completely regular.

At each step of the proof, $C(X)$ can be replaced by $C^*(X)$ (check!) •

The following corollary is curious and the proof is a good test of whether one understands the idea of “weak topology.”

Corollary 3.13 Suppose X is a set and let \mathcal{T}_F be the weak topology on X generated by any family of functions $\mathcal{F} \subseteq \mathbb{R}^X$. Then (X, \mathcal{T}_F) is completely regular.

Proof Give X the topology the weak topology \mathcal{T}_F generated by \mathcal{F} . Now that X has a topology, the collection $C(X)$ makes sense. Let \mathcal{T}_C be the weak topology on X generated by $C(X)$.

The topology \mathcal{T}_F does make all the functions in $C(X)$ continuous, so $\mathcal{T}_C \subseteq \mathcal{T}_F$.

On the other hand: $\mathcal{F} \subseteq C(X)$ by definition of \mathcal{T}_F , and the larger collection of functions $C(X)$ generates a (potentially) larger weak topology. Therefore $\mathcal{T}_F \subseteq \mathcal{T}_C$.

Therefore $\mathcal{T}_F = \mathcal{T}_C$. By Theorem 3.12, (X, \mathcal{T}_F) is completely regular. •

Example 3.14

1) If $\mathcal{F} = \{f \in \mathbb{R}^{\mathbb{R}} : f \text{ is nowhere differentiable}\}$, then the weak topology $\mathcal{T}_{\mathcal{F}}$ on \mathbb{R} generated by \mathcal{F} is completely regular.

2) If H is an infinite T_3 space on which every continuous real-valued function is constant (see the comments at the beginning of this Section 3), then the weak topology generated by $C(X)$ has for a base the collection of cozero sets $\{\emptyset, H\}$. So the weak topology generated by $C(X)$ is the trivial topology, not the original topology on X .

Theorem 3.12 leads to a lovely characterization of Tychonoff spaces.

Corollary 3.15 Suppose X is a Tychonoff space. For each $f \in C^*(X)$, we have $\text{ran}(f) \subseteq [a_f, b_f] = I_f$ for some $a_f < b_f \in \mathbb{R}$. The evaluation map $e : X \rightarrow \prod \{I_f : f \in C^*(X)\}$ is an embedding.

Proof X is T_1 , the f 's are continuous and the collection of f 's ($= C^*(X)$) separates points and closed sets. By Corollary VI.4.11, e is an embedding. •

Since each I_f is homeomorphic to $[0, 1]$, $\prod \{I_f : f \in C^*(X)\}$ is homeomorphic to $[0, 1]^m$, where $m = |C^*(X)|$. Therefore any Tychonoff space can be embedded in a “cube.” On the other hand (Corollary 3.8) $[0, 1]^m$ and all its subspaces are Tychonoff. So we have:

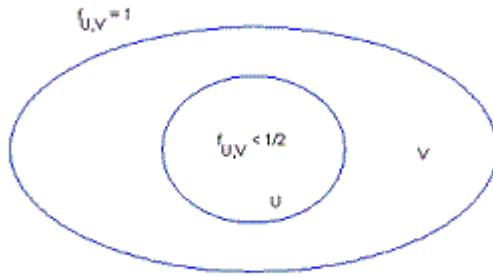
Corollary 3.16 A space X is Tychonoff iff X is homeomorphic to a subspace of the cube $[0, 1]^m$ for some cardinal number m .

The exponent $m = |C^*(X)|$ in the corollary may not be the smallest possible. In an extreme case, for example, we have $c = |C^*(\mathbb{R})|$, even though we can embed \mathbb{R} in $[0, 1] = [0, 1]^1$. However, the following theorem improves the value for m in certain cases. (We proved a similar result for metric spaces (X, d) : see Example VI.4.5.)

Theorem 3.17 Suppose X is Tychonoff with a base \mathcal{B} of cardinality m . Then X can be embedded in $[0, 1]^m$. In particular, X can be embedded in $[0, 1]^{w(X)}$.

Proof Suppose m is finite. Since X is T_1 , $\{x\} = \bigcap \{B : B \text{ is a basic open set containing } x\}$. Only finitely many such intersections are possible, so X is finite and therefore discrete. Hence $X \stackrel{\text{top}}{\subseteq} [0, 1] \stackrel{\text{top}}{\subseteq} [0, 1]^m$.

Suppose \mathcal{B} is a base of cardinal m where m is infinite. Call a pair $(U, V) \in \mathcal{B} \times \mathcal{B}$ distinguished if there exists a continuous $f_{U,V} : X \rightarrow [0, 1]$ with $f_{U,V}(x) < \frac{1}{2}$ for all $x \in U$ and $f_{U,V}(x) = 1$ for all $x \in X - V$. Clearly, $U \subseteq V$ for a distinguished pair (U, V) . For each distinguished pair, pick such a function $f_{U,V}$ and let $\mathcal{F} = \{f_{U,V} : (U, V) \in \mathcal{B} \times \mathcal{B} \text{ is distinguished}\}$.



We note that if $x \in V \in \mathcal{B}$, then there must exist $U \in \mathcal{B}$ such that $x \in U$ and (U, V) is distinguished. To see this, pick an $f : X \rightarrow [0, 1]$ so that $f(x) = 0$ and $f|X - V = 1$. Then choose $U \in \mathcal{B}$ so that $x \in U \subseteq f^{-1}[[0, \frac{1}{2}]] \subseteq V$.

We claim that \mathcal{F} separates points and closed sets:

Suppose F is a closed set not containing x . Choose a basic set $V \in \mathcal{B}$ with $x \in V \subseteq X - F$. There is a distinguished pair (U, V) with $x \in U \subseteq V \subseteq X - F$. Then $f_{U,V}(x) = r < \frac{1}{2}$ and $f_{U,V}|F = 1$, so $f_{U,V}(x) \notin \text{cl } f_{U,V}[F] = \{1\}$.

By Corollary VI.4.11, $e : X \rightarrow [0, 1]^{|\mathcal{F}|}$ is an embedding. Since m is infinite, $|\mathcal{F}| \leq |\mathcal{B} \times \mathcal{B}| = m^2 = m$. •

A “metrization theorem” is one that states that certain topological properties of a space X imply that X is metrizable. Typically the hypotheses of a metrization theorem involve asking that 1) X has “enough separation” and 2) X has a “sufficiently nice base.” The following theorem is a simple example.

Corollary 3.18 (“Baby Metrization Theorem”) A second countable Tychonoff space X is metrizable.

Proof By the theorem, $X \stackrel{\text{top}}{\subseteq} [0, 1]^{\aleph_0}$. Since $[0, 1]^{\aleph_0}$ is metrizable, so is X . •

In Corollary 3.18, X turns out to be metrizable and separable (since X is second countable). On the other hand $[0, 1]^{\aleph_0}$ and all its subspaces are separable metrizable spaces. Thus, the corollary tells us that, topologically, the separable metrizable spaces are precisely the second countable Tychonoff spaces.

Exercises

E7. Prove that if X is a countable Tychonoff space, then there is a neighborhood base of clopen sets at each point. (Such a space X is sometimes called zero-dimensional.)

E8. Prove that in any space X , a countable union of cozero sets is a cozero set – equivalently, that a countable intersection of zero sets is a zero set.

E9. Prove that the following are equivalent in any Tychonoff space X :

- a) every zero set is open
- b) every G_δ set is open
- c) for each $f \in C(X)$: if $f(p) = 0$ then there is a neighborhood N of p such that $f|N \equiv 0$

E10. Let $i : \mathbb{R} \rightarrow \mathbb{R}$ be the identity map and let

$$(i) = \{f \in C(\mathbb{R}) : f = gi \text{ for some } g \in C(\mathbb{R})\}.$$

For those who know a bit of algebra: $C(\mathbb{R})$ (or, more generally, $C(X)$) with addition and multiplication defined pointwise, is a commutative ring with unit. (i) is called the ideal generated by the element i .

- a) Prove that $(i) = \{f \in C(\mathbb{R}) : f(0) = 0 \text{ and the derivative } f'(0) \text{ exists}\}$.
- b) Exhibit two functions f, g in $C(\mathbb{R})$ for which $fg \in (i)$ yet $f \notin (i)$ and $g \notin (i)$.
- c) Let X be a Tychonoff space with more than one point. Prove that there are two functions $f, g \in C(X)$ such that $fg \equiv 0$ on X yet neither f nor g is identically 0 on X .
(So the ring $C(X)$ has zero divisors.)
- d) Prove that there are exactly two functions $f \in C(\mathbb{R})$ for which $f^2 = f$. (Here, $f^2(x)$ means $f(x) \cdot f(x)$, not $f(f(x))$.)
- e) Prove that there are exactly c functions f in $C(\mathbb{Q})$ for which $f^2 = f$.

An element which equals its own square is called an idempotent in $C(X)$). Part d) shows that $C(\mathbb{R})$ and $C(\mathbb{Q})$ are not isomorphic rings since they have different numbers of idempotents. Is either isomorphic to $C(\mathbb{N})$?

A classic part of general topology is the exploration of the relationship between the space X and the rings $C(X)$ and $C^*(X)$. For example, if X and Y are homeomorphic, then $C(X)$ is isomorphic to $C(Y)$. This necessarily implies that $C^*(X)$ is isomorphic to $C^*(Y)$ also (why?). The question “when does isomorphism imply homeomorphism” is more difficult. Another important area of study is how

the maximal ideals of the ring $C(X)$ are related to the topology of X . The best introduction to this material is the classic Rings of Continuous Functions (Gillman-Jerison).)

f) Let $D(\mathbb{R})$ be the set of differentiable functions $f : \mathbb{R} \rightarrow \mathbb{R}$. Are the rings $C(\mathbb{R})$ and $D(\mathbb{R})$ isomorphic? Hint: An isomorphism between $C(\mathbb{R})$ and $D(\mathbb{R})$ preserves cube roots.

E11. Suppose X is a connected Tychonoff space with more than one point. Prove $|X| \geq c..$

E12. Let X be a topological space. Suppose $f, g \in C(X)$ and that $Z(f)$ is a neighborhood of $Z(g)$ (that is, $Z(f) \subseteq \text{int } Z(g)$.)

a) Prove that there is a function $h \in C(X)$ such that $f(x) = g(x)h(x)$ for all $x \in X$, i.e., that f is a multiple of g in $C(X)$. Also,

b) Give an example where $Z(f) \supseteq Z(g)$ but f is not a multiple of g in $C(X)$.

E13. Let X be a Tychonoff space. Suppose $F, A \subseteq X$, where F is closed and A is countable. Prove that if $F \cap A = \emptyset$, then A is disjoint from some zero set containing F .

E14. A space X is called pseudocompact (*Definition IV.8.7*) if every continuous $f : X \rightarrow \mathbb{R}$ is bounded, that is, if $C(X) = C^*(X)$. Consider the following condition on a topological space X :

(*) If $V_1 \supseteq V_2 \supseteq \dots \supseteq V_n \supseteq \dots$ is a decreasing sequence of nonempty open sets, then $\bigcap_{n=1}^{\infty} \text{cl } V_n \neq \emptyset$.

a) Prove that if X satisfies (*), then X is pseudocompact.

b) Prove that if X is Tychonoff and pseudocompact, then X satisfies (*).

Note: For Tychonoff spaces, part b) gives an “internal” characterization of pseudocompactness – that is, one that makes no explicit mention of \mathbb{R} .

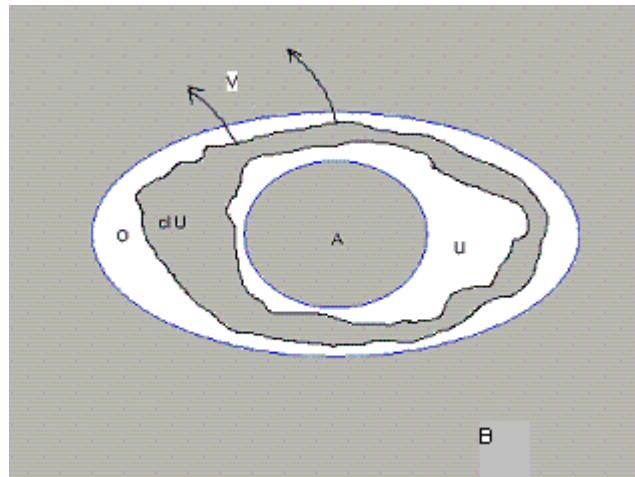
4. Normal and T_4 Spaces

We now return to a topic in progress: normal spaces (and T_4 spaces). Even though normal spaces are badly behaved in some ways, there are still some very nice classical (and nontrivial) theorems we can prove. One of these will have " $T_4 \Rightarrow T_{3\frac{1}{2}}$ " as a corollary.

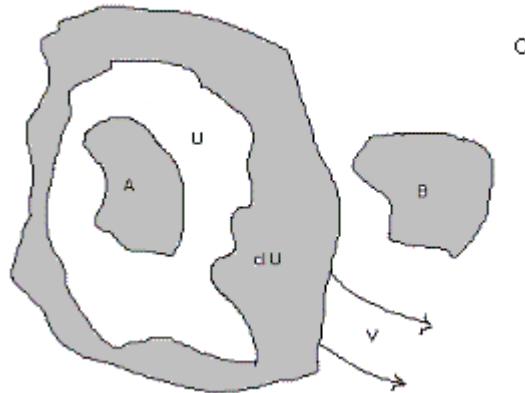
To begin, the following technical variation on the definition of normality is very useful.

Lemma 4.1 A space X is normal iff whenever O is open, A closed, and $A \subseteq O$, then there exists an open set U with $A \subseteq U \subseteq \text{cl } U \subseteq O$.

Proof Suppose X is normal and that O is an open set containing the closed set A . Then A and $B = X - O$ are disjoint closed sets. By normality, there are disjoint open sets U and V with $A \subseteq U$ and $B \subseteq V$. Then $A \subseteq U \subseteq \text{cl } U \subseteq X - V \subseteq O$.



Conversely, suppose X satisfies the stated condition and that A, B are disjoint closed sets.



Then $A \subseteq O = X - B$, so there is an open set U with $A \subseteq U \subseteq \text{cl } U \subseteq X - B$. Let $V = X - \text{cl } U$. U and V are disjoint closed sets containing A and B respectively, so X is normal. •

Theorem 4.2 a) A closed subspace of a normal (T_4) space is normal (T_4).
b) A continuous closed image of a normal (T_4) space is normal (T_4).

Proof a) Suppose F is a closed subspace of a normal space X and let A and B be disjoint closed sets in F . Then A, B are also closed in X so we can find disjoint open sets U' and V' in X containing A and B respectively. Then $U = U' \cap F$ and $V = V' \cap F$ are disjoint open sets in F that contain A and B , so F is normal.

b) Suppose X is normal and that $f : X \rightarrow Y$ is continuous, closed and onto. If A and B are disjoint closed sets in Y , then $f^{-1}[A]$ and $f^{-1}[B]$ are disjoint closed sets in X . Pick U' and V' disjoint open sets in X with $f^{-1}[A] \subseteq U'$ and $f^{-1}[B] \subseteq V'$. Then $U = Y - f[X - U']$ and $V = Y - f[X - V']$ are open sets in Y .

If $y \in U$, then $y \notin f[X - U']$. Since f is onto, $y = f(x)$ for some $x \in U' \subseteq X - V'$. Therefore $y \in f[X - V']$ so $y \notin V$. Hence $U \cap V = \emptyset$.

If $y \in A$, then $f^{-1}[\{y\}] \subseteq f^{-1}[A] \subseteq U'$, so $f^{-1}[\{y\}] \cap (X - U') = \emptyset$. Therefore $y \notin f[X - U']$ so $y \in Y - f[X - U'] = U$. Therefore $A \subseteq U$ and, similarly, $B \subseteq V$ so Y is normal.

Since the T_1 property is hereditary and is preserved by closed onto maps, the statements in a) and b) hold for T_4 as well as normality. •

The next theorem gives us more examples of normal (and T_4) spaces.

Theorem 4.3 Every regular Lindelöf space X is normal (and therefore every Lindelöf T_3 space is T_4).

Proof Suppose A and B are disjoint closed sets in X . For each $x \in A$, use regularity to pick an open set U_x such that $x \in U_x \subseteq \text{cl } U_x \subseteq X - B$. Since the Lindelöf property is hereditary on closed subsets, a countable number of the U_x 's cover A : relabel these as $U_1, U_2, \dots, U_n, \dots$. For each n , we have $\text{cl } U_n \cap B = \emptyset$. Similarly, choose a sequence of open sets $V_1, V_2, \dots, V_n, \dots$ covering B such that $\text{cl } V_n \cap A = \emptyset$ for each n .

We have that $\bigcup_{n=1}^{\infty} U_n \supseteq A$ and $\bigcup_{n=1}^{\infty} V_n \supseteq B$, but these unions may not be disjoint. So we define

$$\begin{array}{ll} U_1^* = U_1 - \text{cl } V_1 & V_1^* = V_1 - \text{cl } U_1 \\ U_2^* = U_2 - (\text{cl } V_1 \cup \text{cl } V_2) & V_2^* = V_2 - (\text{cl } U_1 \cup \text{cl } U_2) \\ \vdots & \vdots \\ U_n^* = U_n - (\text{cl } V_1 \cup \text{cl } V_2 \cup \dots \cup \text{cl } V_n) & V_n^* = V_n - (\text{cl } U_1 \cup \text{cl } U_2 \cup \dots \cup \text{cl } U_n) \\ \vdots & \vdots \end{array}$$

Let $U = \bigcup_{n=1}^{\infty} U_n^*$ and $V = \bigcup_{n=1}^{\infty} V_n^*$.

If $x \in A$, then $x \notin \text{cl } V_n$ for all n . But $x \in U_k$ for some k , so $x \in U_k^* \subseteq U$. Therefore $A \subseteq U$ and, similarly, $B \subseteq V$.

To complete the proof, we show that $U \cap V = \emptyset$. Suppose $x \in U$.

Then $x \in U_k^*$ for some k ,
so $x \notin \text{cl } V_1 \cup \text{cl } V_2 \cup \dots \cup \text{cl } V_k$,
so $x \notin V_1 \cup V_2 \cup \dots \cup V_k$,
so $x \notin V_1^* \cup V_2^* \cup \dots \cup V_k^*$,
so $x \notin V_n^*$ for any $n \leq k$.

Since $x \in U_k^*$, then $x \in U_k$. So, if $n > k$, then $x \notin V_n^* = V_n - (\text{cl } U_1 \cup \dots \cup \text{cl } U_k \cup \dots \cup \text{cl } U_n)$

So $x \notin V_n^*$ for all n , so $x \notin V$ and therefore $U \cap V = \emptyset$. •

Example 4.4 The Sorgenfrey line S is regular because the sets $[a, b)$ form a base of closed neighborhoods at each point a . We proved in Example VI.3.2 that S is Lindelöf, so S is normal. Since S is T_1 , we have that S is T_4 .

5. Urysohn's Lemma and Tietze's Extension Theorem

We now turn our attention to the issue of “ $T_4 \Rightarrow T_{3\frac{1}{2}}$ ”. Proving this is hard because to show that a space X is $T_{3\frac{1}{2}}$, we need to prove that certain continuous functions exist; but the hypothesis “ T_4 ” gives us no continuous functions to work with. As far as we know at this point, there could even be T_4 spaces on which every continuous real-valued function is constant! If T_4 spaces are going to have a rich supply of continuous real-valued functions, we will have to show that these functions can be “built from scratch” in a T_4 space. This will lead us to two of the most well-known classical theorems of general topology.

We begin with the following technical lemma. It gives a way to use a certain collection of open sets $\{U_r : r \in Q\}$ to construct a function $f \in C(X)$. The idea in the proof is quite straightforward, but I attribute its elegant presentation (and that of Urysohn's Lemma which follows) primarily to Leonard Gillman and Meyer Jerison.

Lemma 5.1 Suppose X is any topological space and let Q be any dense subset of \mathbb{R} . Suppose open sets $U_r \subseteq X$ have been defined, for each $r \in Q$, in such a way that:

- i) $X = \bigcup_{r \in Q} U_r$ and $\bigcap_{r \in Q} U_r = \emptyset$
- ii) if $r, s \in Q$ and $r < s$, then $\text{cl } U_r \subseteq U_s$.

For $x \in X$, define $f(x) = \inf \{r \in Q : x \in U_r\}$. Then $f : X \rightarrow \mathbb{R}$ is continuous.

We will only use the Lemma once, with $Q = \mathbb{Q}$. So if you like, there is no harm in assuming that $Q = \mathbb{Q}$ in the proof.

Proof First note: suppose $x \in X$. By i) we know that $x \in U_r$ for some r , so $\{r \in Q : x \in U_r\} \neq \emptyset$. And by ii), we know that $x \notin U_s$ for some s . For that s : if $x \in U_r$, then (by ii) $s \leq r$, so s is a lower bound for $\{r \in Q : x \in U_r\}$. Therefore $\{r \in Q : x \in U_r\}$ has a greatest lower bound, so the definition of $f(x) = \inf \{r \in Q : x \in U_r\}$ makes sense.

From the definition of f , we get that for $r, s \in Q$,

- a) if $x \in \text{cl } U_r$, then $x \in U_s$ for all $s > r$ so $f(x) \leq r$

b) if $f(x) < s$, then $x \in U_s$.

We want to prove f is continuous at each point $a \in X$. Since Q is dense in \mathbb{R} ,

$$\{[r, s] : r, s \in Q \text{ and } r < f(a) < s\}$$

is a neighborhood base at $f(a)$ in \mathbb{R} . Therefore it is sufficient to show that whenever $r < f(a) < s$, then there is a neighborhood U of a such that $f[U] \subseteq [r, s]$.

Since $f(a) < s$, we have $a \in U_s$, and $f(a) > r$ gives us that $a \notin \text{cl } U_r$. Therefore $U = U_s - \text{cl } U_r$ is an open neighborhood of a . If $z \in U$, then $z \in U_s \subseteq \text{cl } U_s$, so $f(z) \leq s$; and $z \notin \text{cl } U_r$, so $z \notin U_r$ and $f(z) \geq r$. Therefore $f[U] \subseteq [r, s]$. •

Our first major theorem about normal spaces is traditionally referred to as a “lemma” because it was a lemma in the paper where it originally appeared. (*Its author, Paul Urysohn, died at age 26, on the morning of 17 August 1924, while swimming off the coast of Brittany.*)

Theorem 5.2 (Urysohn's Lemma) A space X is normal iff whenever A, B are disjoint closed sets in X , there exists a function $f \in C(X)$ with $f|A = 0$ and $f|B = 1$. (When such an f exists, we say that A and B are completely separated.)

Notice: the theorem says that $A \subseteq f^{-1}(0)$ and $B \subseteq f^{-1}(1)$, but equality might not be true. If fact, if we had $A = f^{-1}(0)$ and $B = f^{-1}(1)$, then A and B would have been zero sets in the first place. And in that case, normality would not even be necessary because, in any space X :

If $A = Z(g)$ and $B = Z(h)$ are disjoint zero sets, then the function $f(x) = \frac{g^2(x)}{g^2(x) + h^2(x)}$ completely separates A and B .

This shows again that zero sets are very special closed sets: disjoint zero sets are always completely separated. So, given Urysohn's Lemma, we can conclude that every nonnormal space must contain a closed set that is not a zero set.

Proof The proof of Urysohn's Lemma in one direction is almost trivial. If such a function f exists, then $U = \{x : f(x) < \frac{1}{2}\}$ and $V = \{x : f(x) > \frac{1}{2}\}$ are disjoint open sets (in fact, cozro sets) containing A and B respectively. It is the other half of Urysohn's Lemma for which Urysohn deserves credit.

Let A and B be disjoint closed sets in a normal space X . We will define sets open sets U_r ($r \in \mathbb{Q}$) in such a way that the Lemma 5.1 applies. To start, let $U_r = \emptyset$ for $r < 0$ and $U_r = X$ for $r > 1$.

Enumerate the remaining rationals in $\mathbb{Q} \cap [0, 1]$ as $r_1, r_2, \dots, r_n, \dots$, beginning the list with $r_1 = 1$ and $r_2 = 0$. We begin by defining $U_{r_1} = U_1 = X - B$. Then use normality to define $U_{r_2} (= U_0)$: since $A \subseteq U_{r_1} = X - B$, we can pick U_{r_2} so that

$$A \subseteq U_{r_2} \subseteq \text{cl } U_{r_2} \subseteq U_{r_1} = X - B$$

Then $0 = r_2 < r_3 < r_1 = 1$, and we use normality to pick an open set U_{r_3} so that

$$A \subseteq U_{r_2} \subseteq \text{cl } U_{r_2} \subseteq U_{r_3} \subseteq \text{cl } U_{r_3} \subseteq U_{r_1} = X - B$$

We continue by induction. Suppose $n \geq 3$ and that we have already defined open sets $U_{r_1}, U_{r_2}, \dots, U_{r_n}$ in such a way that whenever $r_i < r_j < r_k$ ($i, j, k \leq n$), then

$$\text{cl } U_{r_i} \subseteq U_{r_j} \subseteq \text{cl } U_{r_j} \subseteq U_{r_k} \quad (*)$$

We need to define $U_{r_{n+1}}$ so that $(*)$ holds for $i, j, k \leq n + 1$.

Since $r_1 = 1$ and $r_2 = 0$, and $r_{n+1} \in (0, 1)$, it makes sense to define

$$\begin{aligned} r_k &= \text{the largest among } r_1, r_2, \dots, r_n \text{ that is smaller than } r_{n+1}, \text{ and} \\ r_l &= \text{the smallest among } r_1, r_2, \dots, r_n \text{ that is larger than } r_{n+1}. \end{aligned}$$

By the induction hypothesis, we already have $\text{cl } U_{r_k} \subseteq U_{r_l}$. Then use normality to pick an open set $U_{r_{n+1}}$ so that

$$\text{cl } U_{r_k} \subseteq U_{r_{n+1}} \subseteq \text{cl } U_{r_{n+1}} \subseteq U_{r_l}.$$

The U_r 's defined in this way satisfy the conditions of Lemma 5.1, so the function $f : X \rightarrow \mathbb{R}$ defined by $f(x) = \inf \{r \in \mathbb{Q} : x \in U_r\}$ is continuous. If $x \in A$, then $x \in U_{r_2} = U_0$ and $x \notin U_r$ if $r < 0$, so $f(x) = 0$. If $x \in B$ then $x \notin U_1$, but $x \in U_r = X$ for $r > 1$, so $f(x) = 1$. •

Once we have the function f we can replace it, if we like, by $g = (0 \vee f) \wedge 1$ so that A and B are completely separated by a function $g \in C^*(X)$. It is also clear that we can modify g further to get an $h \in C^*(X)$ for which $h|A = a$ and $h|B = b$ where a and b are any two real numbers.

With Urysohn's Lemma, the proof of the following corollary is obvious.

Corollary 5.3 $T_4 \Rightarrow T_{3\frac{1}{2}}$.

There is another famous characterization of normal spaces in terms of $C(X)$. It is a result about “extending” continuous real-valued functions defined on closed subspaces.

We begin with the following two lemmas. Lemma 5.4, called the “Weierstrass M -Test” is a slight generalization of a theorem with the same name in advanced calculus. It can be useful in “piecing together” infinitely many real-valued continuous functions to get a new one. Lemma 5.5 will be used in the proof of Tietze's Extension Theorem (Theorem 5.6).

Lemma 5.4 (Weierstrass M -Test) Let X be a topological space. Suppose $f_n : X \rightarrow \mathbb{R}$ is continuous for each $n \in \mathbb{N}$ and that $|f_n(x)| \leq M_n$ for all $x \in X$. If $\sum_{n=1}^{\infty} M_n < \infty$, then $f(x) = \sum_{n=1}^{\infty} f_n(x)$ converges (absolutely) for all x and $f : X \rightarrow \mathbb{R}$ is continuous.

Proof For each x , $\sum_{n=1}^{\infty} |f_n(x)| \leq \sum_{n=1}^{\infty} M_n < \infty$, so $\sum_{n=1}^{\infty} f_n(x)$ converges (absolutely) by the Comparison Test.

Suppose $a \in X$ and $\epsilon > 0$. Choose N so that $\sum_{n=N+1}^{\infty} M_n < \frac{\epsilon}{4}$. Each f_n is continuous, so for $n = 1, \dots, N$ we can pick a neighborhood U_n of a such that for $x \in U_n$, $|f_n(x) - f_n(a)| < \frac{\epsilon}{2N}$. Then $U = \bigcap_{n=1}^N U_n$ is a neighborhood of a , and for $x \in U$ we get $|f(x) - f(a)|$

$$\begin{aligned} &= \left| \sum_{n=1}^N (f_n(x) - f_n(a)) + \sum_{n=N+1}^{\infty} (f_n(x) - f_n(a)) \right| \leq \sum_{n=1}^N |f_n(x) - f_n(a)| + \sum_{n=N+1}^{\infty} |f_n(x) - f_n(a)| \\ &\leq \sum_{n=1}^N |f_n(x) - f_n(a)| + \sum_{n=N+1}^{\infty} |f_n(x)| + |f_n(a)| < N \cdot \frac{\epsilon}{2N} + \sum_{n=N+1}^{\infty} 2M_n < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Therefore f is continuous at a . •

Lemma 5.5 Let A be a closed set in a normal space X and let r be a positive real number. Suppose $h : A \rightarrow [-r, r]$ is continuous. Then there exists a continuous $\phi : X \rightarrow [-\frac{r}{3}, \frac{r}{3}]$ such that $|h(x) - \phi(x)| \leq \frac{2r}{3}$ for each $x \in A$.

Proof Let $A_1 = \{x \in A : h(x) \leq -\frac{r}{3}\}$ and $B_1 = \{x \in A : h(x) \geq \frac{r}{3}\}$. A_1 and B_1 are disjoint closed sets in A , and since A is closed, A_1 and B_1 are closed in X . By Urysohn's Lemma, there exists a continuous function $\phi : X \rightarrow [-\frac{r}{3}, \frac{r}{3}]$ such that $\phi|A_1 = -\frac{r}{3}$ and $\phi|B_1 = \frac{r}{3}$.

If $x \in A_1$, then $-r \leq h(x) \leq -\frac{r}{3}$ and $\phi(x) = -\frac{r}{3}$, so $|h(x) - \phi(x)| \leq |-r - (-\frac{r}{3})| = \frac{2r}{3}$; and similarly if $x \in B_1$, $|h(x) - \phi(x)| \leq \frac{2r}{3}$. If $x \in A - (A_1 \cup B_1)$, then $h(x)$ and $\phi(x)$ are both in $[-\frac{r}{3}, \frac{r}{3}]$ so $|h(x) - \phi(x)| \leq \frac{2r}{3}$. •

Theorem 5.6 (Tietze's Extension Theorem) A space X is normal iff whenever A is a closed set in X and $f \in C(A)$, then there exists a function $g \in C(X)$ such that $g|A = f$.

Note: if A is a closed subset of \mathbb{R} , then it is quite easy to prove that each $f \in C(A)$ can be extended to a function g defined on all of \mathbb{R} . In that case, the open set $\mathbb{R} - A$ can be written as a countable union of disjoint open intervals I , where each $I = (a, b)$ or $(-\infty, b)$ or (a, ∞) (see Theorem II.3.4). Any endpoints of I are in A , where f is already defined. If $I = (a, b)$ then extend the definition of f over I by using a straight line segment to join $(a, f(a))$ and $(b, f(b))$ on the graph of f . If $I = (a, \infty)$, then extend the graph of f over I using a horizontal right ray at height $f(a)$; if $I = (-\infty, b)$, then extend the graph of f over I using a horizontal left ray at height $f(b)$.

As with Urysohn's Lemma, half of the proof is easy. The significant part of theorem is proving the existence of the extension g when X is normal.

Proof (\Leftarrow) Suppose A and B are disjoint closed sets in X . A and B are clopen in the subspace $A \cup B$ so the function $f : A \cup B \rightarrow [0, 1]$ defined by $f|A = 0$ and $f|B = 1$ is continuous. Since $A \cup B$ is closed in X , there is a function $g \in C(X)$ such that $g|(A \cup B) = f$. Then $U = \{x : g(x) < \frac{1}{2}\}$ and $V = \{x : g(x) > \frac{1}{2}\}$ are disjoint open sets (cozero sets, in fact) that contain A and B respectively. Therefore X is normal.

(\Rightarrow) The idea is to find a sequence of functions $g_i \in C(X)$ such that

$|f(x) - \sum_{i=1}^n g_i(x)| \rightarrow 0$ as $n \rightarrow \infty$ for each $x \in A$ (where f is defined). The sums $\sum_{i=1}^n g_i(x)$ are defined on all of X and as $n \rightarrow \infty$ we can think of them as giving better and better approximations to the extension g that we want. Then we can let $g(x) = \lim_{n \rightarrow \infty} \sum_{i=1}^n g_i(x) = \sum_{i=1}^{\infty} g_i(x)$. The details follow. We proceed in three steps, but the heart of the argument is in Step I.

Step I Suppose $f : A \rightarrow [-1, 1]$ is continuous. We claim there is a continuous function $g : X \rightarrow [-1, 1]$ with $g|A = f$.

Using Lemma 5.5 (with $h = f$, $r = 1$) we get a function $g_1 = \phi : X \rightarrow [-\frac{1}{3}, \frac{1}{3}]$ such that for $x \in A$, $|f(x) - g_1(x)| \leq \frac{2}{3}$. Therefore $f - g_1 : A \rightarrow [-\frac{2}{3}, \frac{2}{3}]$.

Using Lemma 5.5 again (with $h = f - g_1$, $r = \frac{2}{3}$), we get a function $g_2 : X \rightarrow [-\frac{2}{9}, \frac{2}{9}]$ such that for $x \in A$, $|f(x) - g_1(x) - g_2(x)| \leq \frac{4}{9} = (\frac{2}{3})^2$. So $f - (g_1 + g_2) : A \rightarrow [-\frac{4}{9}, \frac{4}{9}]$.

Using Lemma 5.5 again (with $h = f - g_1 - g_2$, $r = \frac{4}{9}$), we get a function $g_3 : X \rightarrow [-\frac{4}{27}, \frac{4}{27}]$ such that for $x \in A$, $|f(x) - g_1(x) - g_2(x) - g_3(x)| \leq \frac{8}{27} = (\frac{2}{3})^3$. So $f - (g_1 + g_2 + g_3) : A \rightarrow [-\frac{8}{27}, \frac{8}{27}]$.

We continue, using induction, to find for each i a continuous function

$g_i : X \rightarrow [-\frac{2^{i-1}}{3^i}, \frac{2^{i-1}}{3^i}]$ such that $|f(x) - \sum_{i=1}^n g_i(x)| \leq (2/3)^n$ for $x \in A$.

Since $\sum_{i=1}^{\infty} |g_i(x)| \leq \sum_{i=1}^{\infty} \frac{2^{i-1}}{3^i} < \infty$, the series $g(x) = \sum_{i=1}^{\infty} g_i(x)$ converges (absolutely) for every $x \in X$, and g is continuous by the Weierstrass M -Test. Since $|g(x)| = |\sum_{i=1}^{\infty} g_i(x)| \leq \sum_{i=1}^{\infty} |g_i(x)| \leq \sum_{i=1}^{\infty} \frac{2^{i-1}}{3^i} = 1$, we have $g : X \rightarrow [-1, 1]$.

Finally, for $x \in A$, $|f(x) - g(x)| = \lim_{n \rightarrow \infty} |f(x) - \sum_{i=1}^n g_i(x)| \leq \lim_{n \rightarrow \infty} (\frac{2}{3})^n = 0$, so $g|A = f$ and the proof for Step I is complete.

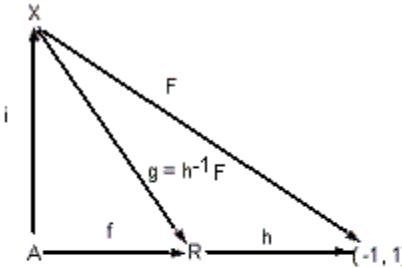
Step II Suppose $f : A \rightarrow (-1, 1)$ is continuous. We claim there is a continuous function $g : X \rightarrow (-1, 1)$ with $g|A = f$.

Since $f : A \rightarrow (-1, 1) \subseteq [-1, 1]$, we can apply Case I to find a continuous function $F : X \rightarrow [-1, 1]$ with $F|A = f$. To get g , we merely make a slight modification to F to get a g that still extends f but where g has all its values in $(-1, 1)$.

Let $B = \{x \in X : F(x) = \pm 1\}$. A and B are disjoint closed sets in X , so by Urysohn's Lemma there is a continuous $h : X \rightarrow [-1, 1]$ such that $h|B = 0$ and $h|A = 1$. If we let $g(x) = F(x)h(x)$, then $g : X \rightarrow (-1, 1)$ and $g|A = f$, completing the proof of Case II.

Step III (the full theorem) Suppose $f : A \rightarrow \mathbb{R}$ is continuous. We claim there is a continuous function $g : X \rightarrow \mathbb{R}$ with $g|A = f$.

Let $h : \mathbb{R} \rightarrow (-1, 1)$ be a homeomorphism. Then $h \circ f : A \rightarrow (-1, 1)$ and, by Step II, there is a continuous $F : X \rightarrow (-1, 1)$ with $F|A = h \circ f$.



Let $g = h^{-1} \circ F : X \rightarrow \mathbb{R}$. Then for $x \in A$ we have $g(x) = h^{-1}(F(x)) = h^{-1}((h \circ f))(x) = f(x)$. •

It is easy to see that $C^(X)$ can replace $C(X)$ in the statement of Tietze's Extension Theorem.*

Example 5.7 We now know enough about normality to see some of its bad behavior. The Sorgenfrey line S is normal (Example 4.4) but the plane $S \times S$ is not normal.

To see this, let $D = \mathbb{Q} \times \mathbb{Q}$, a countable dense set in $S \times S$. Every continuous real-valued function on $S \times S$ is completely determined by its values on D . (See Theorem II.5.12. The theorem is stated for the case of functions defined on a pseudometric space, but the proof is written in a way that applies just as well to functions with any space X as domain.) Therefore the mapping $C(S \times S) \rightarrow C(D)$ given by $f \mapsto f|D$ is one-to-one, so $|C(S \times S)| \leq |C(D)| \leq |\mathbb{R}^D| = c^{\aleph_0} = c$.

$A = \{(x, y) \in S \times S : x + y = 1\}$ is closed and discrete in the subspace topology, so every function defined on A is continuous, that is, $\mathbb{R}^A = C(A)$ and so $|C(A)| = c^c = 2^c$. If $S \times S$ were normal, then each $f \in C(A)$ could be extended (by Tietze's Theorem) to a continuous function in $C(S \times S)$. This would mean that $|C(S \times S)| \geq |\mathbb{R}^A| = c^c = 2^c > c$, which is false. Therefore normality is not even finitely productive.

The comments following the statement of Urysohn's Lemma imply that $S \times S$ must contain closed sets that are not zero sets.

A completely similar argument “counting continuous real-valued functions” shows that the Moore plane Γ (Example III.5.6) is not normal: use that Γ is separable and the x -axis in Γ is a closed discrete subspace.

Questions about the normality of products are difficult. For example, it was an open question for a long time whether the product of a normal space X with $[0, 1]$ (*a very nice, well-behaved space*) must be normal. In the 1950's, Dowker proved that $X \times [0, 1]$ is normal iff X is normal and "countably paracompact."

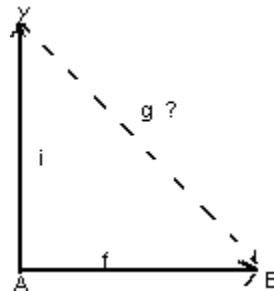
However, this result was unsatisfying — because no one knew whether a normal space was automatically "countably paracompact." Then, in the 1960's, Mary Ellen Rudin constructed a normal space X which was not countably paracompact. This example was still unsatisfying because the construction assumed the existence of a space called a "Souslin line" — and whether a Souslin line exists cannot be decided in the ZFC set theory! In other words, constructing her space X required adding a new axiom to ZFC.

Things were finally settled in 1971 when Mary Ellen Rudin constructed a "real" example of a normal space X whose product with $[0, 1]$ is not normal. By "real," we mean that X can be constructed in ZFC with no additional set theoretic assumptions. Among other things, this example makes use of the box topology on a product.

Example 5.8 The Sorgenfrey line S is T_4 , so S is $T_{3\frac{1}{2}}$ and therefore the Sorgenfrey plane $S \times S$ is also $T_{3\frac{1}{2}}$. So $S \times S$ is an example that shows $T_{3\frac{1}{2}}$ does not imply T_4 .

Extension theorems are an important idea in mathematics. In general, an "extension theorem" has the following form:

$A \subseteq X$ and $f : A \rightarrow B$, then there is a function $g : X \rightarrow B$ such that $g|A = f$.



If we let $i : A \rightarrow X$ be the injection $i(a) = a$, then the condition " $g|A = f$ " can be rewritten as $g \circ i = f$. In the language of algebra, we are asking whether there is a suitable function g which "makes the diagram commute."

Specific extension theorems impose conditions on A and X , and usually we want g to share some property of f such as continuity. Here are some illustrations, without the specifics.

- 1) Extension theorems that generalize of Tietze's Theorem: by putting stronger hypotheses on X , we can relax the hypotheses on B .

Suppose A is closed in X and $f : A \rightarrow B$ is continuous.

$$\text{If } \begin{cases} X \text{ is normal} & \text{and } B = \mathbb{R} \text{ (Tietze's Theorem)} \\ X \text{ is normal} & \text{and } B = \mathbb{R}^n \\ X \text{ is collectionwise normal}^{**} & \text{and } B \text{ is a separable Banach space*} \\ X \text{ is paracompact}^{**} & \text{and } B \text{ is a Banach space*} \end{cases}$$

then f has a continuous extension $g : X \rightarrow B$.

The statement that \mathbb{R}^n can replace \mathbb{R} in Tietze's Theorem is easy to prove:

If X is normal and $f : A \rightarrow \mathbb{R}^n$ is continuous, write $f(x) = (f_1(x), f_2(x), \dots, f_n(x))$ where each $f_i : A \rightarrow \mathbb{R}$. By Tietze's Theorem, there exists for each i a continuous extension $g_i : X \rightarrow \mathbb{R}$ with $g_i|A = f_i$. If we let $g(x) = (g_1(x), \dots, g_n(x))$, then $g : X \rightarrow \mathbb{R}^n$ and $g|A = f$. In other words, we separately extend the coordinate functions in order to extend f . And in this example, n could even be an infinite cardinal.

* A *normed linear space* is a vector space V with a norm $|v|$ (= “absolute value”) that defines the “length” of each vector. Of course, a norm must satisfy certain axioms – for example, $|v_1 + v_2| \leq |v_1| + |v_2|$. These properties guarantee that a norm can be used to define a metric: $d(v_1, v_2) = |v_1 - v_2|$. A Banach space is a normed linear space which is complete in this metric d . For example, \mathbb{R}^n , with its usual norm $|(x_1, x_2, \dots, x_n)| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$ is a separable Banach space.

** Roughly, a “collectionwise normal” space is one in which certain infinite collections of disjoint closed sets can be enclosed in disjoint open sets. We will not give definitions for “collectionwise normal” (or the stronger condition, “paracompactness”) here, but is true that

$$\begin{cases} \text{metric, or} \\ \text{compact } T_2 \end{cases} \Rightarrow \text{paracompact} \Rightarrow \text{collectionwise normal} \Rightarrow \text{normal}$$

Therefore, in the theorems cited above, a continuous map f defined on a closed subset of a metric space (or, compact T_2 space) and valued in a Banach space B and be continuously extended a function $g : X \rightarrow B$.

2) The Hahn-Banach Theorem is another example, taken from functional analysis, of an extension. Roughly, it states:

Suppose f is a continuous linear functional defined on a subspace M of a normed linear space X . For such a map, a “norm” $\|f\|$ can be defined. Then f can be extended to a continuous linear functional $g : X \rightarrow \mathbb{R}$ for which $\|g\| = \|f\|$.

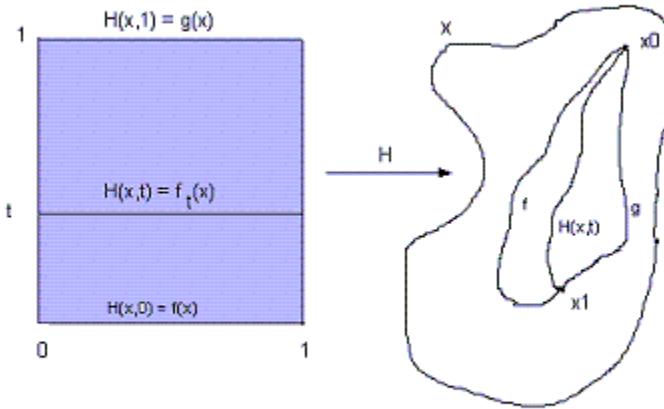
3) Homotopy is usually not formulated in terms of extension theorems, but extensions are really at the heart of the idea.

Let $f, g : [0, 1] \rightarrow X$ be continuous and suppose that $f(0) = g(0) = x_0$ and $f(1) = g(1) = x_1$. Then f and g are paths in X that start at x_0 and end at x_1 . Let B be the boundary of the square $[0, 1]^2 \subseteq \mathbb{R}^2$ and define $F : B \rightarrow X$ by

$$\begin{aligned} F(x, 0) &= f(x) & F(x, 1) &= g(x) \\ F(0, t) &= x_0 & F(1, t) &= x_1 \end{aligned}$$

Thus F agrees with f on the bottom edge of B and with g on the top edge. F is constant ($= x_0$) on the left edge of B and constant ($= x_1$) on the right edge of B . We ask whether F can be extended to a continuous map defined on the whole square, $H : [0, 1]^2 \rightarrow X$.

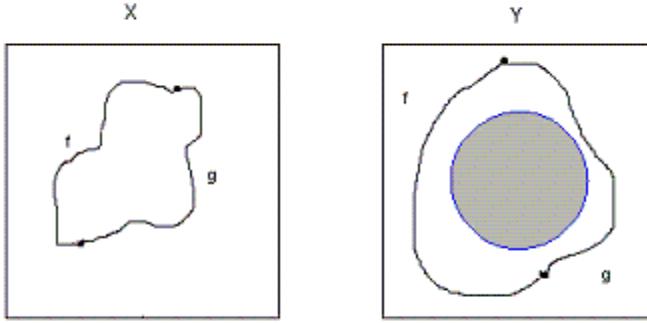
If H does exist, then we have



For each $t \in [0, 1]$, restrict H to the line segment at height t to define $f_t(x) = H(x, t)$. Then for each $t \in [0, 1]$, f_t is also a path in X from x_0 to x_1 . As t moves from 0 to 1, we can think of the f_t 's as a family of paths in X that continuously deform $f_0 = f$ into $f_1 = g$.

The continuous extension H (if it exists) is called a homotopy between f and g with fixed endpoints, and we say that the paths f and g are homotopic with fixed endpoints.

In the space X below, it seems intuitively clear that f can be continuously deformed (with endpoints held fixed) into g – in other words, that H exists.



However in the space Y , f and g together form a loop that surrounds a “hole” in Y , and it seems intuitively clear that the path f cannot be continuously deformed into the path g within the space Y — that is, the extension H does not exist.

In some sense, homotopy can be used to detect the presence of certain “holes” in a space, and is one important part of algebraic topology.

The next theorem shows us where compact Hausdorff spaces stand in the discussion of separation properties.

Theorem 5.9 A compact T_2 space X is T_4 .

Proof It is sufficient to show that X is regular because X is Lindelöf and a regular Lindelöf space is normal (Theorem 4.3). Suppose F is a closed set in X and $x \notin F$. For each $y \in F$ we can pick disjoint open sets U_y and V_y with $x \in U_y$ and $y \in V_y$. F is compact so a finite number of the V_y 's cover F — say $V_{y_1}, V_{y_2}, \dots, V_{y_n}$. Then $x \in \bigcap_{i=1}^n U_{y_i} = U$, $F \subseteq \bigcup_{i=1}^n V_{y_i} = V$, and U, V are disjoint open sets. •

Therefore, our results line up as:

$$(*) \text{ compact metric} \Rightarrow \begin{cases} \text{compact } T_2 \\ \text{or} \\ \text{metric} \end{cases} \Rightarrow T_4 \Rightarrow T_{3\frac{1}{2}} \Rightarrow T_3 \Rightarrow T_2 \Rightarrow T_1 \Rightarrow T_0$$

In particular, Urysohn's Lemma and Tietze's Extension Theorem hold in metric spaces and in compact T_2 spaces.

Notice that

- i) the space $(0, 1)$ is T_4 but not compact T_2
- ii) the Sorgenfrey line S is T_4 (see example 5.4) but not metrizable. (*If S were metrizable, then $S \times S$ would be metrizable and therefore T_4 – which is false : see Example 5.7.*)
- iii) $[0, 1]^c$ is compact T_2 (*assuming, for now, the Tychonoff Product Theorem VI.3.10*) but not metrizable (*why?*)
- iv) $(0, 1)$ is metrizable but not compact.

Combining these observations with earlier examples, we see that none of the implications in (*) is reversible.

Example 5.10 (*See Example 5.7*) The Sorgenfrey plane $S \times S$ is $T_{3\frac{1}{2}}$, so $S \times S$ can be embedded in a cube $[0, 1]^m$ and $[0, 1]^m$ is compact T_2 (*assuming the Tychonoff Product Theorem*). Since $S \times S$ is not normal, we see now that a normal space can have nonnormal subspaces. This example, admittedly, is not terribly satisfying since it is hard to visualize how $S \times S$ “sits” inside $[0, 1]^m$. In Chapter VIII (Example 8.10), we will look at an example of a T_4 space in which it's easy to “see” why a certain subspace isn't normal.

6. Some Metrization Results

We have enough information now to completely characterize separable metric spaces topologically.

Theorem 6.1 (Urysohn's Metrization Theorem) A second countable T_3 space is metrizable. (*Note: earlier we proved a similar metrization theorem” (Corollary 3.18), but the separation hypothesis then was $T_{3\frac{1}{2}}$ rather than T_3 .*)

Proof X is second countable so X is Lindelöf, and Theorem 4.3 tells us that a Lindelöf T_3 space is T_4 . Therefore X is $T_{3\frac{1}{2}}$. So by Corollary 3.18, X is metrizable. •

Because a separable metrizable space is second countable and T_3 , we have a complete characterization: X is a separable metrizable space iff X is a second countable T_3 space. So, with hindsight, we now see that the hypothesis “ $T_{3\frac{1}{2}}$ ” in Corollary 3.18 was unnecessarily strong. In fact, we see that T_3 and $T_{3\frac{1}{2}}$ are equivalent in a space that is second countable.

Further developments in metrization theory hinged on work of Arthur H. Stone in the late 1940's – in particular, his result that metric spaces have a property called “paracompactness.” This led quickly to a complete characterization of metrizable spaces came roughly a quarter century after Urysohn's work. We state this characterization here without a proof.

A family of sets \mathcal{B} in (X, T) is called locally finite if each point $x \in X$ has a neighborhood N that meets only finitely many sets in \mathcal{B} . The family \mathcal{B} is called σ -locally finite if we can write $\mathcal{B} = \bigcup_{n \in \mathbb{N}} \mathcal{B}_n$ where each subfamily \mathcal{B}_n is locally finite.

Theorem 6.2 (The Bing-Smirnov-Nagata Metrization Theorem) (X, \mathcal{T}) is metrizable iff X is T_3 and has a σ -locally finite base \mathcal{B} .

Note: If X is second countable, a countable base $\mathcal{B} = \{O_1, O_2, \dots, O_n, \dots\}$ is σ -locally finite – because we can write $\mathcal{B} = \bigcup \mathcal{B}_n$, where $\mathcal{B}_n = \{O_n\}$. Therefore this Metrization Theorem includes Urysohn's Metrization Theorem as a special case.

The Bing-Smirnov-Nagata Theorem has the typical form of most metrization theorems: X is metrizable iff “ X has enough separation” and “ X has a nice enough base.”

Exercises

E15. Let (X, d) be a metric space and $S \subseteq X$. Prove that if each continuous $f: S \rightarrow \mathbb{R}$ extends to a continuous $g: X \rightarrow \mathbb{R}$, then S is closed. (The converse, of course, follows from Tietze's Extension Theorem.)

E16. Let X be a Tychonoff space.

a) Suppose $F, K \subseteq X$ where F is closed, K is compact, X and $F \cap K = \emptyset$. Prove that there is an $f \in C(X)$ such that $f|K = 0$ and $f|F = 1$. (This is another example of the rule of thumb that “compact spaces act like finite spaces.” If necessary, try proving the result first for a finite set K .)

b) Suppose $p \in U$, where U is open in X . Prove $\{p\}$ is a G_δ set in X iff there exists a continuous function $f: X \rightarrow [0, 1]$ such that $f^{-1}(1) = \{p\}$ and $f|X - U = 0$.

E17. Suppose Y is a Hausdorff space. Define $x \sim y$ in Y iff there does not exist a continuous function $f: X \rightarrow [0, 1]$ such that $f(x) \neq f(y)$. Prove or disprove: Y/\sim is a Tychonoff space.

E18. Prove that a Hausdorff space X is normal iff for each finite open cover $\mathcal{U} = \{U_1, \dots, U_n\}$ of X , there exist continuous functions $f_i: X \rightarrow [0, 1]$ ($i = 1, \dots, n$) such that $\sum_{i=1}^n f_i(x) = 1$ for each $x \in X$ and such that, for each i , $f_i|X - U_i \equiv 0$. (Such a set of functions is called a partition of unity subordinate to the finite cover \mathcal{U} .)

Hint (\Rightarrow) First build a new open cover $\mathcal{V} = \{V_1, \dots, V_n\}$ that “shrinks” \mathcal{U} in the sense that, $V_i \subseteq \text{cl } V_i \subseteq U_i$ for each i . To begin the construction, let $F_1 = X - \bigcup_{i>1} U_i$. Pick an open V_1 so that $F_1 \subseteq V_1 \subseteq \text{cl } V_1 \subseteq U_1$. Then $\{V_1, U_2, \dots, U_n\}$ still covers X . Continue by looking at $F_2 = X - (V_1 \cup \bigcup_{i>2} U_i)$ and defining V_2 so that $\{V_1, V_2, U_3, \dots, U_n\}$ is still a cover and $V_2 \subseteq \text{cl } V_2 \subseteq U_2$. Continue in this way to replace the U_i 's one by one. Then use Urysohn's lemma to get functions g_i which can then be used to define the f_i 's. .

E19. Suppose X is a compact, countable Hausdorff space. Prove that X is completely metrizable.

Hint: 1) For each pair of points $x_n \neq x_m$ in X pick disjoint open sets $U_{m,n}$ and $V_{m,n}$ containing these points. Consider the collection of all finite intersections of such sets.

2) Or: Since X is, countable, every singleton $\{p\}$ is a G_δ set. Use regularity to find a descending sequence of open sets V_n containing p such that $\bigcap_{n=1}^{\infty} \text{cl } V_n = \{p\}$. Prove that the V_n 's are a neighborhood base at p .

E20. A space X is called completely normal if every subspace of X is normal. (For example, every metric space is completely normal).

a) Prove that X is completely normal if and only if the following condition holds:

whenever $A, B \subseteq X$ and each of A, B is disjoint from the closure of the other (i.e., $(\text{cl } A \cap B) \cup (A \cap \text{cl } B) = \emptyset$), then there exist disjoint open sets U and V with $A \subseteq U$ and $B \subseteq V$.

b) Recall that the “scattered line” (*Exercise IIIE.10*) consist of the set $X = \mathbb{R}$ with the topology $\mathcal{T} = \{U \cup V : U \text{ is open in the usual topology on } \mathbb{R} \text{ and } V \subseteq \mathbb{P}\}$. Prove that the scattered line is completely normal and therefore T_4 .

E21. A T_1 space X is called perfectly normal if whenever A and B are disjoint nonempty closed sets in X , there is an $f \in C(X)$ with $f^{-1}(0) = A$ and $f^{-1}(1) = B$.

a) Prove that every metric space (X, d) is perfectly normal.

b) Prove that X is perfectly normal iff X is T_4 and every closed set in X is a G_δ -set.

Note: Example 3.10 shows a T_4 space X that is not perfectly normal.

c) Show that the scattered line (see *Exercise E20*) is not perfectly normal, even though every singleton set $\{p\}$ is a G_δ -set.

d) Show that the scattered line is T_4 .

Hint: Use the fact that \mathbb{R} , with the usual topology, is normal. Nothing deeper than Urysohn's Lemma is required but the problem is a bit tricky.

E22. Prove that a T_3 space (X, \mathcal{T}) has a locally finite base \mathcal{B} iff \mathcal{T} is the discrete topology. (*Compare to Theorem 6.2.*)

Chapter VII Review

Explain why each statement is true, or provide a counterexample.

1. Suppose (X, \mathcal{T}) is a topological space and let \mathcal{T}_w be the weak topology on X generated by $C(X)$. Then $\mathcal{T} \subseteq \mathcal{T}_w$.
2. If X is regular and $x \in \text{cl } \{y\}$, then $y \in \text{cl } \{x\}$.
3. If F_1, F_2 , and F_3 are pairwise disjoint closed sets in the normal space X , then there exist pairwise disjoint open sets U_1, U_2 , and U_3 such that for each $i = 1, 2, 3$, $U_i \supseteq F_i$.
4. Let X denote the real numbers with the topology \mathcal{T} for which the “right rays” (a, ∞) form a base. \mathcal{T} is the same as the weak topology generated by $C(X)$.
5. If X is infinite and has the cofinite topology, then X is completely regular.
6. Every separable Tychonoff space can be embedded in $[0, 1]^{\aleph_0}$.
7. If $f \in C(\mathbb{Q})$, we say g is a square root of f if $g \in C(\mathbb{Q})$ and $g^2 = f$. If a function f in $C(\mathbb{Q})$ has more than one square root, then it has c square roots.
8. In a Tychonoff space, every closed set is an intersection of zero sets.
9. A subspace of a separable space need not be separable, but every subspace of the Sorgenfrey line is separable.
10. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be given by $f(x) = \sin |x|$ and $g : \mathbb{R} \rightarrow \mathbb{R}$ be given by $g(x) = \begin{cases} \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$. The weak topology on \mathbb{R} induced by the functions f and g is completely regular.
11. Suppose \mathbb{N} has the cofinite topology. If A is closed in \mathbb{N} , then every $f \in C(A)$ can be extended to a function $g \in C(\mathbb{N})$.
12. Let X be the Sorgenfrey plane. Then X^{\aleph_0} is second countable.
13. For $n = 1, 2, \dots$, let $f_n : \mathbb{R} \rightarrow \mathbb{R}$ be given by $f_n(x) = x + n$ and let \mathcal{T} be the weak topology on \mathbb{R} generated by the f_n 's. Then the evaluation map $e : \mathbb{R} \rightarrow \mathbb{R}^{\aleph_0}$ given by $e(x)(n) = f_n(x)$ is an embedding.
14. Let C be the set of points in the Cantor set with the subspace topology from the Sorgenfrey line S . Every continuous function $f : C \rightarrow \mathbb{R}$ can be extended to a continuous function $g : S \rightarrow \mathbb{R}$.
15. The product of two Lindelöf spaces cannot contain an uncountable closed discrete subset.
16. The Sorgenfrey line S is homeomorphic to an open subspace of the cube $[0, 1]^3$.
17. If (X, d) is a metric space, then X is homeomorphic to a dense subspace of some compact Hausdorff space.

18. Suppose $F \subseteq \mathbb{R}^2$. F is closed iff F is a zero set.
19. A closed subspace of a product of two normal spaces is normal.
20. Suppose K is a compact subset of the Hausdorff space $X \times Y$. Let $A = \pi_X[K]$. Then A is T_4 .
21. For any cardinal m , a subspace X of $[0, 1]^m$ is metrizable if and only if X is second countable.
22. Every space is the continuous image of a metrizable space.
23. Let $\mathcal{F} = \{f \in \mathbb{R}^{\mathbb{R}} : f \text{ is not continuous}\}$. The weak topology on \mathbb{R} generated by \mathcal{F} is the discrete topology.
24. Every Lindelöf T_3 space can be embedded in $[0, 1]^m$ for some cardinal m .
25. A compact T_2 space is metrizable if and only if it is second countable.
26. If F is closed and W is open in the Sorgenfrey line S , and $F \subseteq W$, then there is an open set O in S such that $F \subseteq O \subseteq \bar{O} \subseteq W$.
27. Suppose F and K are disjoint subsets of a Tychonoff space X , where F is closed and K is compact. There are disjoint cozero sets U and V with $F \subseteq U$ and $K \subseteq V$.
28. A separable metric space with a basis of clopen sets is homeomorphic to a subspace of the Cantor set.
29. Every T_4 space is homeomorphic to a subspace of some cube $[0, 1]^m$.
30. If every function in $C^*(X)$ is constant, then every function in $C(X)$ must be constant.
31. \mathbb{N}^{\aleph_0} is T_4 .
32. If $f : X \rightarrow Y$ is continuous and onto and X is normal, then Y is normal.
33. If S is the Sorgenfrey line, and let \mathcal{T} be the weak topology on S generated by the functions in $C(S)$. Then (S, \mathcal{T}) is T_4 .
34. There is a closed set in the Sorgenfrey plane that is not a zero set.
35. Every closed set in $\mathbb{N}^{\mathbb{N}}$ is a zero set.