Chapter VIII
Ordered Sets, Ordinals and Transfinite Methods

1. Introduction

In this chapter, we will look at certain kinds of ordered sets. If a set X is ordered in a reasonable way, then there is a natural way to define an "order topology" on X. Most interesting (for our purposes) will be ordered sets that satisfy a very strong ordering condition: that every nonempty subset contains a smallest element. Such sets are called well-ordered. The most familiar example of a well-ordered set is $\mathbb{N}$ and it is the well-ordering property that lets us do mathematical induction in $\mathbb{N}$
In this chapter we will see "longer" well ordered sets and these will give us a new proof method called "transfinite induction." But we begin with something simpler.

2. Partially Ordered Sets

Recall that a relation R on a set X is a subset of $X \times X$ (*see Definition I.5.2*). If $(x, y) \in R$, we write xRy. An "order" on a set X is refers to a relation on X that satisfies some additional conditions. Order relations are usually denoted by symbols such as $\leq$, $<$, $\prec$, or $\preceq$.

Definition 2.1 A relation R on X is called:

<u>transitive</u> if :	$\forall\, a, b, c \in X$	$(aRb$ and $bRc) \Rightarrow aRc.$
<u>reflexive</u> if :	$\forall\, a \in X$	aRa
<u>antisymmetric</u> if :	$\forall\, a, b \in X$	$(aRb$ and $bRa\,) \;\Rightarrow\; (a = b)$
<u>symmetric</u> if :	$\forall\, a, b \in X$	$aRb \Leftrightarrow bRa$ (that is, the set R is "symmetric" with respect to the diagonal $\Delta = \{(x, x) : x \in X\} \subseteq X \times X$).

Example 2.2

1) The relation " $=$ " on a set X is transitive, reflexive, symmetric, and antisymmetric. Viewed as a subset of $X \times X$, the relation " $=$ " is the diagonal set $\Delta = \{(x, x) : x \in X\}$.

2) In $\mathbb{R}$, the usual order relation $<$ is transitive and antisymmetric, but not reflexive or symmetric.

3) In $\mathbb{R}$, the usual order $\leq$ is transitive, reflexive and antisymmetric. It is not symmetric.

4) On any set of cardinal numbers $\mathcal{C}$ we have a relation $\leq$. It is transitive, reflexive and antisymmetric (*by the Cantor-Schroeder-Bernstein Theorem I.10.2*), but not symmetric (unless $|\mathcal{C}| = 1$).

Definition 2.3 A relation $\leq$ on a set X is called a partial order if $\leq$ is transitive, reflexive and antisymmetric. The pair $(X, \leq)$ is called a partially ordered set (or, for short, poset).

A relation $\leq$ on a set X is called a linear order if $\leq$ is a partial order and, in addition, any two elements in X are comparable: $\forall\, a, b \in X$ either $a \leq b$ or $b \leq a$. In this case, the pair $(X, \leq)$ is called a linearly ordered set. For short, a linearly ordered set is also called a chain.

We write $a < b$ to mean that $a \leq b$ and $a \neq b$. For any sort of order relation $\leq$ on X, we can invert the order notation and write $b \geq a$ $(b > a)$ to mean the same thing as $a \leq b$ $(a < b)$.

In some books, a partial order is defined as a "strict" relation $<$ which is transitive and irreflexive ($\forall a \in X$, $a \not< a$). In that case, we can define $a \leq b$ to mean "$a < b$ or $a = b$" to get a partial order in the sense defined above. This variation in terminology creates no real mathematical problems: the difference is completely analogous to worrying about whether "$\leq$" or "$<$" should be called the "usual order" on $\mathbb{R}$.

Example 2.4

1) Suppose $A \subseteq X$. For any kind of order $\leq$ on X, we can get an order $\leq_A$ on $A \subseteq X$ by restricting the order $\leq$ to A. More formally, $\leq_A = \leq \cap (A \times A)$. We always assume that a subset of an ordered set has this natural "inherited" ordering unless something else is explicitly stated. With that understanding, we usually omit the subscript and also write $\leq$ for the order $\leq_A$ on A.

If $(X, \leq)$ is a poset (or chain), and $A \subseteq X$, then $(A, \leq)$ is also a poset (or chain). For example, every subset of $(\mathbb{R}, \leq)$ is a chain.

2) For $z, w \in \mathbb{C}$ ($=$ the set of complex numbers), define $z \preceq w$ iff $|z| \leq |w|$, where $\leq$ is the usual order in $\mathbb{R}$. $(\mathbb{C}, \preceq)$ is not a poset. (*Why?*)

3) Let $(X, \mathcal{T})$ be a topological space. For $f, g \in C(X)$, define

$$f \leq g \text{ iff } \forall x \in X,\ f(x) \leq g(x).$$

As a set,

$$\leq\ = \{(f, g) \in C(X) \times C(X) :\ \forall x \in \mathbb{R}, f(x) \leq g(x)\}.$$

Notice that, in contrast to part 2), we are allowing ourselves an ambiguity in the notation here because we are using "$\leq$" with two different meanings: we are defining an order "$\leq$" in $C(X)$, but the comparison "$f(x) \leq g(x)$" refers to the usual order a different ordered set, $\mathbb{R}$. Of course, we could be more careful and write $f \preceq g$ for the new order on $C(X)$, but usually we won't be that fussy when the context makes clear which meaning of "$\leq$" we have in mind.

$(C(X), \leq)$ is a poset but usually not a chain: for example, if $X = \mathbb{R}$ and f, g are given by $f(x) = x$ and $g(x) = x^2$, then $f \not\leq g$ and $g \not\leq f$. When is $(C(X), \leq)$ a chain? (*The answer is not "iff $|X| \leq 1$."*)

4) The following two diagrams represent posets, each with 5 elements. Line segments upward from x to y indicate that $x \leq y$. In Figure (i), for example, $d \leq c$ and $c \leq a$ (so $d \leq a$); in Figure (i), a and b are not comparable.

Figure (ii) shows a chain: $a \leq b \leq c \leq d \leq e$.

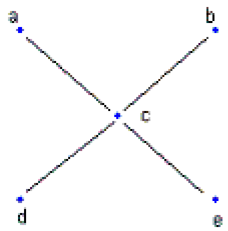


Figure (i)

Figure (ii)

5) Suppose $\mathcal{C}$ is a collection of sets. We can define $\leq$ on $\mathcal{C}$ by $A \leq B$ iff $A \subseteq B$. Then $(\mathcal{C}, \leq)$ is a poset. In this case, we say that $\mathcal{C}$ has been <u>ordered by inclusion</u>. In particular, for any set X, we can order $\mathcal{P}(X)$ by inclusion. What conditions on X will guarantee $(\mathcal{P}(X), \leq)$ is a chain ?

6) Suppose $\mathcal{C}$ is a collection of sets. We can define $\leq$ on $\mathcal{C}$ by $A \leq B$ iff $A \supseteq B$. Then $(\mathcal{C}, \leq)$ is a poset. In this case, we say that $\mathcal{C}$ has been <u>ordered by reverse inclusion</u>. In particular, for any set X, we can order $\mathcal{P}(X)$ by reverse inclusion. What conditions on X will guarantee that $(\mathcal{P}(X), \leq)$ is a chain ?

For a given collection $\mathcal{C}$, Examples 5) and 6) are quite similar: one is a "mirror image" of the other. The identity map $i : \mathcal{C} \to \mathcal{C}$ is an "order-reversing isomorphism" between the posets.

For our purposes, the "reverse inclusion" ordering on a collection of sets will turn out to be more useful. For a point x in a topological space X, we can order the neighborhood system $\mathcal{N}_x$ by <u>reverse inclusion</u> $\leq$. The "order structure" of the poset $(\mathcal{N}_x, \leq)$ reflects some topological properties of X and indicates just how complicated the "neighborhood structure" at x is. For example,

i) If x is isolated, then $\{x\} \in N_x$ and $N \leq \{x\}$ for every $N \in N_x$. So the poset $(\mathcal{N}_x, \leq)$ has a largest element. (*Is the converse true?*)

ii) If X is first countable and $\mathcal{B}_x = \{N_1, N_2, ..., N_k, ...\}$ is a countable neighborhood base at x, then for every $N \in \mathcal{N}_x$ there is a k such that $N \leq N_k$. Thus the poset $(\mathcal{N}_x, \leq)$ contains a countable subset $\{N_1, N_2, ..., N_k, ...\}$ whose members become "arbitrarily large" in the poset.

iii) The poset $(\mathcal{N}_x, \leq)$ is usually not a chain. But it does have interesting order property: for any $N_1, N_2 \in \mathcal{N}_x$, $\exists N_3 \in \mathcal{N}_x$ such that $N_1 \leq N_3$ and $N_2 \leq N_3$ (*let* $N_3 = N_1 \cap N_2$).

It will turn out (in Chapter 9) that this poset $(\mathcal{N}_x, \leq)$ is the inspiration for defining the notion of "convergent nets," a kind of convergence that is more powerful than "convergent sequences." Unlike sequences, convergent nets will be able to determine closures (and therefore the topology) in any topological space.

Definition 2.5 Suppose $(X, \leq)$ is a poset and let $A \subseteq X$. An element $x \in X$ is called

i) the largest (or last) element in X if $y \leq x$ for all $y \in X$
ii) the smallest (or first) element in X if $x \leq y$ for all $y \in X$
iii) a maximal element in X if $(y \geq x \Rightarrow y = x)$ for all $y \in X$
iv) a minimal element in X if $(y \leq x \Rightarrow y = x)$ for all $y \in X$.

It is clear that a largest element in $(X, \leq)$, if it exists, is unique. (*If* z_1 *and* z_2 *were both largest, then* $z_1 \leq z_2$ *and* $z_2 \leq z_1$ *so* $z_1 = z_2$.)

In Figure (i): both a, b are maximal elements and d, e are minimal elements. This poset has no largest or smallest element. *Suppose a poset* $(X, \leq)$ *has a unique maximal element* z. *Must* z *also be the largest element in* $(X, \leq)$ *?*

In Figure (ii): a is the smallest (and also a minimal element); e is the largest (and also a maximal) element.

v) Suppose $A \subseteq X$ and $x \in X$. We say that $x \in X$ is an upper bound for A if $a \leq x$ for all $a \in A$; x is called a least upper bound (sup) for A if x is the smallest upper bound for A. The set A might have many upper bounds, one upper bound, or no upper bounds in X. If A has more than one upper bound, A might or might not have a least upper bound in X. But if X has a least upper bound x, then the least upper bound is unique (*why?*).

vi) An element $x \in X$ is called a lower bound for A if $x \leq a$ for all $a \in A$; x is called a greatest lower bound (inf) for A if x is the largest lower bound for A. The set A might have many lower bounds, one lower bound, or no lower bounds in X. If A has more than one lower bounds, A might or might not have a greatest lower bound in X, but if X has a greatest lower bound x, then the greatest lower bound is unique (*why?*).

vii) If $x < y \in X$ and if $\nexists\, z \in X$ with $x < z < y$, then x is called an immediate predecessor of y and y is called an immediate successor of x. In a poset, an immediate predecessor or successor might not be unique; but if X is a chain, then an immediate predecessor or successor, if it exists, must be unique. (*Why?*)

In Figure (i), the upper bounds on $\{d, e\}$ are a, b, c, and $c = \sup\{d, e\}$; the set $\{d, e\}$ has no lower bounds. Both a, b are immediate successors of c. The elements d, e have no immediate predecessor (in fact, no "predecessors" at all). In Figure (ii), the immediate predecessor of c is b and d is the immediate successor of c.

Example 2.6 If $\leq_1$ is an order on X, then $\leq_1 \subseteq X \times X$. So if $\leq_1$ and $\leq_2$ are orders on X, it makes sense to ask whether $\leq_1 \subseteq \leq_2$, or vice-versa. If we look at the set $\mathcal{P} = \{ \leq : \leq$ is a partial order on $X\}$, then $\mathcal{P}$ is partially ordered by inclusion. A linear order $\leq$ is a maximal element in $(\mathcal{P}, \subseteq)$ (*Why? Is the converse true?*)

3. Chains

Definition 3.1 Let $(X, \leq)$ be a chain. The order topology on X is the topology for which all sets of the form $\{x \in X : x < a\}$ or $\{x \in X : b < x\}$ $(a, b \in X)$ are a subbase. (*As usual, we write $x < y$ as shorthand for "$x \leq y$ and $x \neq y$."*)

It's handy to use standard notation when working with chains: but we need to be careful not to read too much into the notation. For example, if $a, b \in X$:

$$\{x \in X : a < x < b\} = (a, b)$$
$$\{x \in X : a \leq x < b\} = [a, b)$$
$$\{x \in X : x \leq a\} = (-\infty, a]$$

For the chain in Figure (ii), above, we see how the interval notation can be misleading if not used thoughtfully: $(a, b) = \emptyset$, $(d, \infty) = \{e\}$, $(-\infty, b) = \{a\}$, $(a, c) = \{b\}$, and $(a, e) = [b, d]$.

Example 3.2

1) The order topology on the chain in Figure (ii) is the discrete topology.

2) The order topology on $\mathbb{N}$ is the usual (discrete) topology: $\{1\} = \{k \in \mathbb{N} : k < 2\}$ $= (-\infty, 2)$; and for $n > 1$, $\{n\} = (n-1, n+1)$.

Example 3.3 $\mathbb{P}$ and $\mathbb{Q}$ each have an order inherited from $\mathbb{R}$, and their order topologies are the same as the usual subspace topologies. But, in general, we have to be careful about the topology on $A \subseteq X$ when X is a chain with the order topology. There are two possible topologies on A:

a) The order $\leq$ gives an order topology $\mathcal{T}_{\leq}$ on X and we can give A the subspace topology $(\mathcal{T}_{\leq})_A$.

b) A has an ordering $\leq_A$ (inherited from the order $\leq$ on X) and we can use it to give A an order topology. More formally, we could write this topology as $\mathcal{T}_{\leq_A}$.

Unfortunately, these two topologies might not be the same. Let $A = (0, 1) \cup \{2\} \subseteq \mathbb{R}$. The order topology $\mathcal{T}_{\leq}$ on $\mathbb{R}$ is the usual topology on $\mathbb{R}$, and this topology produces a subspace topology for which 2 is isolated in A.

But in the order topology on A, each basic open set containing 2 must have the form $\{x \in A : x > a\} = (a, 2]$ where $a < 1$. So 2 is not isolated in $(A, \mathcal{T}_{\leq_A})$. In fact, the space $(A, \mathcal{T}_{\leq_A})$ is homeomorphic to $(0, 1]$ (*why?*).

Is there any necessary inclusion $\subseteq$ or $\supseteq$ between $\mathcal{T}_{\leq_A}$ and $(\mathcal{T}_{\leq})|_A$? Can you state any hypotheses on X or A that will guarantee that $\mathcal{T}_{\leq_A} = (\mathcal{T}_{\leq})|_A$?

Example 3.4 We defined the order topology only for chains, but the same definition could be used in any ordered set $(X, \leq)$. We usually restrict our attention to chains because otherwise the order topology may not be very nice. For example, let $X = \{a, b, c\}$ with the partial order represented by the following diagram:

X is the only open set containing a (*why?*), so the order topology is not T_1. (*Can you find a poset for which the order topology is not even T_0?*) By contrast, the order topology for any chain $(X, \leq)$ has good separation properties. For example, it is easy to show that the order topology on a chain must be T_2 :

> Suppose $a \neq b \in X$; we can assume $a < b$. If a is the immediate predecessor of b, we can let $U = \{x \in X : x < b\}$ and $V = \{x \in X : x > a\}$. But if there exists a point c satisfying $a < c < b$, then we can define $U = \{x \in X : x < c\}$ and $V = \{x \in X : x > c\}$. Either way, we have a pair of disjoint open sets U and V with $x \in U$ and $y \in V$.

In fact, the order topology on a chain is always T_4, but this is much messier and we will not prove it here. (*Most of our interest in this Chapter will be with ordered sets that are much "better" than chains (well-ordered sets) where it is relatively easy to prove normality.*)

From now on, when the context is clear, we will simply write X, rather than $(X, \leq)$, for an ordered set. We will denote the orders in many different sets all with the same symbol "$\leq$", letting the context settle which order is being referred to. If it is necessary to distinguish carefully between orders, then we will occasionally add subscripts such as $\leq_1, \leq_2, \ldots$.

Definition 3.5 A function $f : X \to Y$ between ordered sets is called an order isomorphism if f is bijection and both f and order preserving – that is, $a \leq b$ if and only if $f(a) \leq f(b)$. (*Since f is one-to-one, it follows that $x < y$ if and only if $f(x) < f(y)$.*) If such an f exists, we say that X and Y are order isomorphic and write $X \simeq Y$.

If f is not onto, then $X \simeq f[X] \subseteq Y$ and f is an order isomorphism of X into Y.

Between ordered sets, "$\simeq$" will always refer to order isomorphism.

Theorem 3.6 Let $X, Y,$ and Z be chains (*parts i-iv) are also true for posets*).

i) $X \simeq X$
ii) $X \simeq Y$ iff $Y \simeq X$
iii) if $X \simeq Y$ and $Y \simeq Z$, then $X \simeq Z$
iv) $X \simeq Y$ implies $|X| = |Y|$.
v) if X and Y are finite chains, then $X \simeq Y$ iff $|X| = |Y|$.

Proof The proof is very easy and is left as an exercise. •

Even though the proof is easy, there are some interesting observations to make.

1) To show that $X \simeq X$, the identity map i might not be the only order isomorphism possible. For example, when $X = \mathbb{R}$, the functions $f(x) = x$ and $f(x) = x^3$ are both order isomorphisms between $\mathbb{R}$ and $\mathbb{R}$. *How many order isomorphisms exist between* $\mathbb{N}$ *and* $\mathbb{N}$*?*

2) A chain can be order isomorphic to a proper subset of itself. For example, $f(n) = 2n$ is an order isomorphism between $\mathbb{N}$ and $\mathbb{E}$ (the set of even natural numbers). Both $\mathbb{N}$ and $\mathbb{E}$ are order isomorphic to the set of all prime numbers. *Must every two countable infinite chains be order isomorphic?*

3) An order isomorphism between X and Y preserves largest, smallest, maximal and minimal elements (if they exist). Therefore $(0,1)$ and $[0,1]$ are not order isomorphic: for example, $[0,1]$ has a smallest element and $(0,1)$ doesn't. Similarly, $\mathbb{N}$ is not order isomorphic to the set of integers $\mathbb{Z}$.

An order isomorphism preserves "betweenness," so $\mathbb{Z}$ is not order isomorphic to $\mathbb{Q}$: in $\mathbb{Q}$, there is a third element between any two elements, but this is false in $\mathbb{Z}$.

4) Let $\mathbb{C}$ be the set of complex numbers. If $f : \mathbb{C} \to \mathbb{R}$ is any bijection, then we can use f to create a chain $(\mathbb{C}, \le)$: simply define $z_1 \le z_2$ iff $f(z_1) \le f(z_2)$. Then $(\mathbb{C}, \le) \simeq (\mathbb{R}, \le)$.

Of course, this chain $(\mathbb{C}, \le)$ is not be very interesting from the point of view of algebra or analysis: we imposed an arbitrary ordering on $\mathbb{C}$ that has nothing to do with the algebraic structure of $\mathbb{C}$. For example, there is no reason to think that $z_1 \le z_2$ and $z_3 \le z_4$, then $z_1 + z_3 \le z_2 + z_4$.

In the same way, a bijection $f : \mathbb{N} \to \mathbb{Q}$ can be used to give $\mathbb{Q}$ a (new) order $\le$ so that the two chains are order isomorphic. In this case, f is just a sequence that enumerates $\mathbb{Q}$: $q_1, q_2, \ldots, q_n, \ldots$ and the new order on $\mathbb{Q}$ is simply defined as $q_1 < q_2 < \ldots < q_n < \ldots$. More generally, if $f : X \to Y$ is a bijection and one of X or Y is ordered, we can use f to "transfer" the order to the other set in such a way that $(X, \le) \simeq (Y, \le)$.

5) Order isomorphic chains are clearly homeomorphic in their order topologies. But the converse is false. Suppose $X = \{-\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ and $Y = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$. The order topologies on X and Y are the usual topologies, and the reflection $f(x) = -x$ is a homeomorphism (in fact, an isometry) between them.

The largest element in Y is 1, and it has an immediate predecessor, $\frac{1}{2}$. But 0 is the largest element in X and it has no immediate predecessor in X. Since an order isomorphism preserves largest elements and immediate predecessors, there is no order isomorphism between X and Y.

The next theorem tells when an ordered set is order isomorphic to the set of rational numbers.

Theorem 3.7 (Cantor) Suppose that $(L, \leq)$ is a nonempty countable chain such that

a) $\forall a \in L, \exists b \in L$ with $a < b$ ("no last element")
b) $\forall a \in L, \exists b \in L$ with $b < a$ ("no first element")
c) $\forall a, b \in L$, if $a < b$ then $\exists c \in L$ such that $a < c < b$.

Then $(L, \leq)$ is order isomorphic to $\mathbb{Q}$.

When a chain that satisfies the third condition – that between any two elements there must exist a third element – we say that X is order-dense. Then Theorem 3.7 can be restated as: a nonempty countable order-dense chain with no first or last element is order isomorphic to $\mathbb{Q}$.

We might attempt a proof in the following way: enumerate $\mathbb{Q} = \{q_1, q_2, ..., q_n, ...\}$ and $L = \{l_1, l_2, .., l_n, ...\}$, and let $f(q_1) = l_1$. Then inductively define $f(q_n) =$ some $l \in L$ chosen so that $f(q_n) \neq f(q_1), ..., f(q_{n-1})$ and so that $f(q_n)$ has the same order relations to $f(q_1), ..., f(q_{n-1})$ as q_n has to $q_1, ..., q_{n-1}$. Such a choice is always be possible since L satisfies a), b), c). In this way we end up with one-to-one order preserving map $g : \mathbb{Q} \to L$. However, g would not necessarily be onto. The "back-and-forth" construction used in the following proof is designed to be sure we end up with an onto order preserving map $g : \mathbb{Q} \to L$.

First we prove a lemma: it states that an order isomorphism between finite subsets of $\mathbb{Q}$ and L can always be extended to include one more point in its domain (or, one more point in its range).

Lemma: Suppose $A \subseteq \mathbb{Q}$, $B \subseteq L$ where A, B are finite. Let g be an order isomorphism from A onto B.

a) If $q \in \mathbb{Q} - A$, there exists an order isomorphism h that extends g and for which $\text{dom}(h) = A \cup \{q\}$ and $\text{ran}(h) \subseteq L$.

b) If $l \in L - B$, there exists an order isomorphism h that extends g and for which $\text{dom}(h) \subseteq \mathbb{Q}$ and $\text{ran}\,(h) = B \cup \{l\}$.

Proof of a) Suppose $A = \{q_1, ..., q_n\}$ and that $g(q_i) = l_i$. Pick an $l \in L - B$ that has the same order relations to $l_1, ..., l_n$ as q does to $q_1, ..., q_n$. (*For example: if q is greater that all of $q_1, ..., q_n$, pick l greater than all of $l_1, ..., l_n$; if i and j are the largest and smallest subscripts for which $q_i < q < q_j$, then choose l between l_i and l_j.*) This choice is always possible since L satisfies a), b), and c). Define h by $h(q) = l$ and $h(x) = g(x)$ for $x \in A$.

Proof for b) The proof is almost identical. •

Proof of Theorem 3.7 L is a countable chain. Since $L \neq \emptyset$ and L has no last element, L must be infinite. Without loss of generality, we may assume that $L \cap \mathbb{Q} = \emptyset$.

We define an order isomorphism g between $\mathbb{Q}$ and L in stages. At each stage, we enlarge the function we have by adding a new point to its domain or range.

Let $M = \mathbb{Q} \cup L = \{m_1, m_2, ..., m_n, ...\}$. Each element of $\mathbb{Q}$ and L appears exactly once in this list.

Define $g_0 = \emptyset$, and continue by induction. Suppose $n > 0$ and that an order isomorphism $g_{n-1} : A_{n-1} \to B_{n-1}$ has been defined, where $A_{n-1} \subseteq \mathbb{Q}$ and $B_{n-1} \subseteq L$.

If $m_n \in \mathbb{Q}$ and $m_n \notin A_{n-1}$, use the Lemma to get an order isomorphism g_n that extends g_{n-1} and for which dom$(g_n) = A_{n-1} \cup \{m_n\} = A_n$. Let $B_n =$ ran$(g_n) \subseteq L$.

If $m_n \in L$ and $m_n \notin B_{n-1}$, use the Lemma to get an order isomorphism g_n that extends g_{n-1} and for which ran$(g_n) = B_{n-1} \cup \{m_n\} = B_n$. Let $A_n =$ dom$(g_n) \subseteq \mathbb{Q}$.

If $m_n \in$ dom$(g_{n-1}) \cup$ ran(g_{n-1}), let $g_n = g_{n-1}$, $A_n = A_{n-1}$ and $B_n = B_{n-1}$.

By induction, g_n is defined for all n, and since g_n extends g_{n-1} and we can define an order isomorphism $g = \bigcup_{n=0}^{\infty} g_n$. The construction guarantees that dom$(g) = \mathbb{Q}$ and ran$(g) = L$. •

"Being order isomorphic" is an equivalence relation among ordered sets so any two chains having the properties in Cantor's theorem are order isomorphic to each other. Since order isomorphic chains have homeomorphic order topologies, we have a topological characterization of $\mathbb{Q}$ in terms of order.

Corollary 3.8 A nonempty countable order-dense chain with no largest or smallest element is homeomorphic to $\mathbb{Q}$.

The following corollary gives a characterization of <u>all</u> order-dense countable chains.

Corollary 3.9 If X is a countable order-dense and $|X| > 1$, then X is order isomorphic to exactly one of the following chains :

a) $\mathbb{Q} \cap (0,1) \simeq \mathbb{Q}$
b) $\mathbb{Q} \cap [0,1)$
c) $\mathbb{Q} \cap (0,1]$
d) $\mathbb{Q} \cap [0,1]$

Proof By looking at largest and smallest elements, we see that no two of these chains are order isomorphic. Therefore no chain is order isomorphic to more than one of them.

If X has no largest or smallest element then, Cantor's theorem gives $X \simeq \mathbb{Q}$ and a second application of Cantor's Theorem gives that $\mathbb{Q} \simeq \mathbb{Q} \cap (0,1)$.

If X has a smallest element, a, but no largest element, then $X - \{a\}$ is nonempty and has no smallest element (*why*?). $X - \{a\}$ clearly satisfies the other hypotheses of the Cantor's theorem so there is an order isomorphism $h : X - \{a\} \to \mathbb{Q} \cap (0,1)$. Then define an order isomorphism $g : X \to \mathbb{Q} \cap [0,1)$ by setting $g(a) = 0$ and $g(x) = h(x)$ for $x \neq a$..

The proofs of the other cases are similar. •

No two of the chains a)-d) mentioned in Corollary 3.9 are order isomorphic, but they are, in fact, all <u>homeomorphic</u> topological spaces. We can see that b) and c) are homeomorphic by using the map $f(x) = 1 - x$, but the other homeomorphisms are not so obvious. Here is a sketch of a proof, contributed by Edward N. Wilson, that $(0,1) \cap \mathbb{Q}$ is homeomorphic to $(0,1] \cap \mathbb{Q}$.

For short, write $(0,1) \cap \mathbb{Q} = (0,1)_{\mathbb{Q}}$ and $(0,1] \cap \mathbb{Q} = (0,1]_{\mathbb{Q}}$.

In $(0,1)$, choose a strictly increasing sequence of irrationals $(p_n) \to 1 \in \mathbb{R}$. Let $p_0 = 0$. For $n \geq 0$: let $O_n = (p_n, p_{n+1}) \cap \mathbb{Q}$, $U_n = (\frac{1}{2}p_n, \frac{1}{2}p_{n+1}) \cap \mathbb{Q}$, and $V_n = (1 - \frac{1}{2}p_{n+1}, 1 - \frac{1}{2}p_n) \cap \mathbb{Q}$. Each of O_n, U_n, and V_n is clopen in $(0,1)_{\mathbb{Q}}$ and in $(0,1]_{\mathbb{Q}}$.

We have $(0,1]_{\mathbb{Q}} = \bigcup_{n=0}^{\infty} O_n \cup \{1\}$ and $(0,1)_{\mathbb{Q}} = \bigcup_{n=0}^{\infty} U_n \cup \{\frac{1}{2}\} \cup \bigcup_{n=0}^{\infty} V_n$

Define a map ϕ: $(0,1]_{\mathbb{Q}} \to \bigcup_{n=0}^{\infty} U_n \cup \{\frac{1}{2}\} \cup \bigcup_{n=0}^{\infty} V_n$ by

$\phi \mid_{O_{2n}}$= an increasing linear map from O_{2n} onto U_n

$\phi \mid_{O_{2n+1}}$= a decreasing linear map from O_{2n+1} onto V_n

$\phi(1) = \frac{1}{2}$

Then ϕ is a homeomorphism. (Since the sets O_n, U_n, V_n are clopen, it is clear that ϕ is continuous at all points except perhaps 1 and that ϕ^{-1} is continuous at all points except perhaps $\frac{1}{2}$. These special cases are easy to check separately.)

There is, in fact, a more general theorem that states that *every infinite countable metric space with no isolated points is homeomorphic to* $\mathbb{Q}$. *This theorem is due to Sierpinski (1920).*

We can also characterize the real numbers as an ordered set.

Theorem 3.10 Suppose X is a nonempty chain which

i) has no largest or smallest element
ii) is order-dense
iii) is separable in the order topology
iv) is Dedekind complete (that is, every nonempty subset of X which has an upper bound in X has a least upper bound in X).

Then X is order isomorphic to $\mathbb{R}$ (and therefore X, with its order topology, is homeomorphic to $\mathbb{R}$).

Proof We will not give all the details of a proof, However, the ideas are completely straightforward and the details are easy to fill in.

Let D be a countable dense set in the order topology on X. Then D satisfies the hypotheses of Cantor's Theorem (*why?*) so there exists an (onto) order isomorphism $f : \mathbb{Q} \to D$. For each irrational $p \in \mathbb{R}$, let $\mathbb{Q}_p = \{q \in \mathbb{Q} : q < p\}$ and extend f to an order isomorphism $g : \mathbb{R} \to X$ by defining $f(p) = \sup f[\mathbb{Q}_p]$. •

Remark In a separable space, any family of disjoint open sets must be countable (*why?*). Therefore we could ask whether condition iii) in Theorem 3.10 can be replaced by iii)$'$ every collection of disjoint open intervals in X is countable.

In other words, can we say that

(**) a nonempty chain satisfying i), ii), iii)$'$, and iv), must X be order isomorphic to $\mathbb{R}$?

The Souslin Hypothesis (SH) states that the answer to (**) is "yes." The status of SH was famously unknown for many years. Work of Jech, Tennenbaum and Solovay in the 1970's showed that SH is consistent with and independent of the axioms ZFC for set theory – that is, SH is undecidable in ZFC. We could add either SH or its negation as an additional axiom in ZFC without introducing an inconsistency. If one assumes that SH is false, then there is a nonempty chain satisfying i), ii), iii)$'$, and iv) but not order isomorphic to $\mathbb{R}$: such a chain is called a Souslin line.

SH was of special interest for a while in connection with the question "if X is a T_4-space, is $X \times [0,1]$ necessarily T_4?" In the 1960's, Mary Ellen Rudin showed that if she had a Souslin line to work with – that is, if SH is false – then the answer to the question was "no." (*See the remarks following Example III.5.7*)

There are lots of equivalent ways of formulating SH—for example, in terms of graph theory. There is a very nice expository article by Mary Ellen Rudin on the Souslin problem in the American Mathematical Monthly, 76(1969), 1113-1119. The article was written before the consistency and independence results of the 1970's and deals with aspects of SH in a "naive" way.

Exercises

E1. Prove or disprove: if R is a relation on X which is both symmetric and antisymmetric, then R must be the equality relation " $=$ ".

E2. State and prove a theorem of the form:

A power set $\mathcal{P}(X)$, ordered by inclusion, is a chain iff ...

E3. Suppose $(X, \leq)$ is a poset in which every nonempty subset contains a largest and smallest element. Prove that $(X, \leq)$ is a finite chain.

E4. Prove that any countable chain $(L, \leq)$ is order isomorphic to a subset of $(\mathbb{Q}, \leq)$.
(*Hint: See the "Caution" in the proof of Cantor's Theorem 3.7.*)

E5. Let $(X, \leq)$ be an infinite poset. A subset C of X is called <u>totally unordered</u> if no two distinct elements of C are comparable, that is:

$$\forall a \in C \ \forall b \in C \quad (a \leq b) \ \Leftrightarrow \ (a = b)$$

Prove that <u>either</u> X has a subset C which is an infinite chain <u>or</u> X has a totally unordered infinite subset C.

E6. Let $(X, \leq)$ be a poset in which the <u>longest</u> chain has length n ($n \in \mathbb{N}$). Prove that X can be written as the union of n totally unordered subsets (*see E5*) and the n is the smallest natural number for which this is true.

.

4. Order Types

In Chapter I, we assumed that we can somehow assign a cardinal number $|X|$ to each set X, in such a way that $|X| = |Y|$ iff there exists a bijection $f : X \to Y$. Similarly, we now assume that we can assign to each chain an "object" called its order type and that this is done in such a way that two chains have the same order type iff the chains are order isomorphic. Just as with cardinal numbers, an exact description of how this can be done is not important here. In axiomatic set theory, all the details can be made precise. Of course, then, the order type of a chain turns out itself to be a certain set (since "everything is a set" in ZFC). For our purposes, it is enough just to take the naive view that "order-isomorphic" is an equivalence relation among chains, and that each equivalence class is an order type.

We will usually denote order types by lower case Greek letters such as $\mu, \nu, \tau, \omega -$ with a few traditional exceptions mentioned in the next example. If μ is the order type of a chain M, we say that M represents μ.

Example 4.1

1) Two chains with the same order type are order isomorphic. Since the order isomorphism is a bijection, chains with the same order type also have the same cardinal number. But the converse is false: $\mathbb{Q}$ and $\mathbb{N}$ have the same cardinal number but they they have different order types because the sets are not order isomorphic.

However, two finite chains have the same cardinality iff they are order isomorphic. Therefore, for finite chains, we will use the same symbol for both the cardinal number and the order type. (*In the precise definitions of axiomatic set theory, the cardinal number and the order type of a finite chain do turn out to be the same set!*)

2) 0 is the order type of $\emptyset$
1 is the order type of $\{0\}$
2 is the order type of $\{0, 1\}$
.
.
.
n is the order type of $\{0, 1, \ldots, n-1\}$
.
.
.
ω_0 is the order type of $\mathbb{N}$ (*The subscript* "0" *hints at bigger things to come.*)
ω_0 is also the order type of $\mathbb{E} = \{2, 4, 6, \ldots\}$ since this chain is order isomorphic to $\mathbb{N}$.

Notice that each order type in this example is represented by "the set of preceding order types."

Definition 4.2 Let μ and ν be order types represented by chains M and N. We say that $\mu \leq \nu$ if there exists an order isomorphism f of M into N. We write $\mu < \nu$ if $\mu \leq \nu$ but $\mu \neq \nu$ (that is, $M \not\simeq N$). (*Check that the definition is independent of the chains M and N chosen to represent μ*

and ν.)

Example 4.3 Let μ be the order type of a chain M. Since $\emptyset$ is order isomorphic to a subset of M we have $0 \leq \mu$. More generally, $0 < 1 < 2 < ... < \omega_0$.

Suppose $(X, \leq)$ has order type μ. With a little reflection, we can create a new chain $(X, \leq^*)$ by defining $x \leq^* y$ iff $y \leq x$. We write μ^* for the order type of $(X, \leq^*)$. For example, ω_0^* is the order type of the chain $\{... , -2, -1\}$ of negative integers. Since $\omega_0 \not\leq \omega_0^*$ and $\omega_0^* \not\leq \omega_0$ (*why?*), we see that <u>two order types may not be comparable</u>.

The relation $\leq$ between order types is reflexive and transitive but it is <u>not antisymmetric</u> – for example, let μ and ν be order types of the intervals $(0,1)$ and $[0,1]$: then $\mu \leq \nu$ and $\nu \leq \mu$ but $\mu \neq \nu$. Therefore $\leq$ is <u>not even a partial ordering</u> among order types.

Definition 4.4 For $\alpha \in A$, let $(M_\alpha, \leq_\alpha)$ be pairwise disjoint chains and suppose that the index set A is also a chain. We define the <u>ordered sum</u> $\sum_{\alpha \in A} M_\alpha$ as the chain $(\bigcup_{\alpha \in A} M_\alpha, \leq)$, where we define

$$x \leq y \quad \text{if} \quad \begin{cases} x, y \in M_\alpha \text{ and } x \leq_\alpha y, \quad \text{or} \\ \alpha < \beta \in A,\ x \in M_\alpha \text{ and } y \in M_\beta \end{cases}$$

We can "picture" the ordered sum as laying the chains M_α "end-to-end" with larger α's further to the right. In particular, for <u>two</u> disjoint chains $(M, \leq_M)$ and $(N, \leq_N)$, the ordered sum $(M + N, \leq)$ is formed by putting N to the right of ("larger than") M and using the old orders inside each of M and N.

Definition 4.5 Suppose A is a chain and that for each $\alpha \in A$, we have an order type μ_α. Let the μ_α's be represented by pairwise disjoint chains M_α. Then $\sum_{\alpha \in A} \mu_\alpha$ as order type of the chain $(\bigcup_{\alpha \in A} M_\alpha, \leq)$. In particular, if μ and ν are order types represented by disjoint chains M and N, then $\mu + \nu$ is the order type of the ordered sum $(M + N, \leq)$. (*Check that sum of order types is independent of the disjoint chains used to represent the order types.*)

Example 4.6 Addition of order types is clearly associative: $(\mu + \nu) + \tau = \mu + (\nu + \tau)$. It is <u>not commutative</u>. For example $\omega_0 + 1 \neq 1 + \omega_0$, since a chain representing the left side has a largest element but a chain representing the right side does not. In general, for $n \in \mathbb{N}$, $n + \omega_0 = \omega_0$ while, if $m \neq n \in \mathbb{N}$, $\omega_0 \neq \omega_0 + n \neq \omega_0 + m$. Of course, chains representing these different order types all have cardinality $\aleph_0$.

The order type $\omega_0 + \omega_0$ is represented by the ordered set $\{0, 1, 2, 3, ... ;\ a_1,\ a_2,\ a_3,\ ...\}$ where ";" indicates that each a_i is larger than every n. Less abstracting, $\omega_0 + \omega_0$ could also be represented by the chain $\{1 - \frac{1}{n} : n \in \mathbb{N}\} \cup \{2 - \frac{1}{n} : n \in \mathbb{N}\} \subseteq \mathbb{Q}$.

$\omega_0^* + \omega_0$ is the order type of the set of integers $\mathbb{Z}$. Is $\omega_0^* + \omega_0 = \omega_0 + \omega_0^*$? (*Why or why not? Give an example of a subset of* $\mathbb{Q}$ *that represents* $\omega_0 + \omega_0^*$.)

Example 4.7 It is easy to prove that every countable chain C is order isomorphic to a subset of $\mathbb{Q}$: just list the elements of C and inductively define a one-to-one mapping into $\mathbb{Q}$ that preserves order at each step (*see the "attempted" argument that precedes that actual proof of Cantor's Theorem 3.7*)

Theorem 3.7). But if every countable order type can be represented by some subset of $\mathbb{Q}$, then there can be at most c different countable order types.

As a matter of fact, we can prove that there are exactly c different countable order types. A sketch of the argument follows: the details are easy to fill in (or see W. Sierpinski, *Cardinal and Ordinal Numbers* – and old "Bible" on the subject with much more information than anybody would want to know.)

$\mathbb{Z}$ has order type $\xi = \omega_0^* + \omega_0$. For each sequence $a = (a_1, a_2, a_3 ...) \in \{0,1\}^{\mathbb{N}}$, we can define an order type

$$\xi_a = \xi + 1 + a_1 + \xi + 1 + a_2 + \xi + 1 + a_3 + \ldots$$

It is not hard to show that the map $a \mapsto \xi_a$ is one-to-one. Here is a sketch of the argument:

We say that two elements of a chain to be in the same component if there are only finitely many elements between them. (*This use of the word "component" has nothing to do with connectedness.*) It is clear all elements in the same component are smaller (or larger) than all elements in a different component; this observation lets us order the components of the chain.

Suppose C is a chain representing ξ_a and let the finite components of C (listed in order of increasing size) be $F_1, F_2, \ldots, F_n, \ldots$ where F_n has order type $1 + a_n$.

Let $a \neq b \in \{0,1\}^{\mathbb{N}}$ and suppose C' is a chain that represents ξ_b. Call the finite components of C' (listed in order of increasing size) $F_1', F_2', \ldots, F_n', \ldots$ where F_n' has order type $1 + b_n$.

An order isomorphism between C and C' would necessarily carry F_n to F_n' for every n. But this is impossible since, for some n, one of $1 + a_n$ and $1 + b_n$ is 1 and the other is 2.

Thus, there are at least as many different countable order types as there are sequences $a \in \{0,1\}^{\mathbb{N}}$, namely c.

Definition 4.8 Let $(M, \leq_M)$ and $(N, \leq_N)$ be chains representing the order types μ and ν. We define the product $\mu\nu$ to be the order type of $(N \times M, \leq)$, where $(n_1, m_1) \leq (n_2, m_2)$ iff $n_1 <_N n_2$ or ($n_1 = n_2$ and $m_1 \leq_M m_2$).

This ordering $\leq$ on $N \times M$ is called the lexicographic (or dictionary) order since the pairs are ordered "alphabetically."

Example 4.9

a) $2 \cdot \omega_0 = \omega_0$. To see this, we can represent ω_0 by $\mathbb{N}$ and 2 by $\{0,1\}$. Then the chain representing $2 \cdot \omega_0$, listed in increasing (lexicographic) order, is $\mathbb{N} \times \{0,1\} = \{(1,0), (1,1), (2,0),$

$(2,1)$, $(3,0)$, $(3,1)$, ..., $(n,0)$, $(n,1)$, ... }, and this chain is order isomorphic to $\mathbb{N}$. More generally, $n \cdot \omega_0 = \omega_0$ for each $n \in \mathbb{N}$.

However, $\omega_0 \cdot 2 \neq \omega_0$. The product $\omega_0 \cdot 2$ is represented by $\{0,1\} \times \mathbb{N}$ ordered as:

$$\{(0,1),\ (0,2), ..., (0,n), ...\ ;\ (1,1), (1,2), ..., (1,n), ...\}$$

This chain is not order isomorphic to $\mathbb{N}$: $\mathbb{N}$ has only one element with no "immediate predecessor," while this set has two such elements. In fact, $\omega_0 \cdot 2 = \omega_0 + \omega_0$. Thus, multiplication of order types is <u>not commutative</u>.

b) $\omega_0 = 2 \cdot \omega_0 = (1+1) \cdot \omega_0 \neq 1 \cdot \omega_0 + 1 \cdot \omega_0 = \omega_0 + \omega_0$. Thus the "right distributive" law fails.

c) ω_0^2 can be represented by the lexicographically ordered chain $\mathbb{N} \times \mathbb{N}$. In order of increasing size, this chain is:

$$\{(1,1),\ (1,2), ...\ ;\ (1,n), ...\ ;\ (2,1),\ (2,2),\ ...,\ (2,n), ...\ ;\ ...\ ;\ ...\ ;\ (n,1),\ (n,2), ...,\ ;\ ...\}$$

The chain looks like countably many copies of $\mathbb{N}$ placed end-to-end, so we can also write: $\omega_0^2 = \omega_0 + \omega_0 + ... + \omega_0 + ... = \sum_{n \in \mathbb{N}} \mu_n$ where each $\mu_n = \omega_0$. A subset of $\mathbb{Q}$ that represents ω_0^2 is $\{k - \frac{1}{n} :\ k, n = 1, 2, 3, ...\}$.

Exercise Prove that

1) multiplication of order types is associative

2) the left distributive law holds for order types: $\mu(\nu + t) = \mu\nu + \mu\tau$.

Note: Other books may define $\mu\nu$ *"in reverse" as the order type of the lexicographically ordered set* $M \times N$*. Under that definition,* $\omega_0 \cdot 2 = \omega_0$ and $2\omega_0 = \omega_0 + \omega_0 \neq \omega_0$*. Also, under the "reversed" definition, the right distributive law holds but the left distributive law fails.*

Which way the definition is made is not important mathematically. The arithmetic of order types under one definition is just a "mirror image" of the arithmetic under the other definition. You just need to be aware of which convention a writer is using.

Exercises

E7. Give $[0,1]^2$ the lexicographic order $\leq$, and let (a,b) represent an open interval in $[0,1]^2$. Describe what a "small" open interval around each of the following points looks like: $(0,\frac{1}{2})$, $(0,1)$, $(\frac{1}{2},0)$, $(1,0)$.

E8. For each $a \in \mathbb{N}$, we can write a uniquely in the form $a = 2^r(2s+1)$ for integers $r, s \geq 0$. Suppose $a' = 2^{r'}(2s'+1) \in \mathbb{N}$. Define $a \leq a'$ if $a = a'$, or $r < r'$, or $r = r'$ and $s < s'$. What is the order type of $(\mathbb{N}, \leq)$? Does a nonempty subset of $(\mathbb{N}, \leq)$ necessarily contain a smallest element?

E9. Show that it is impossible to define an order $\leq$ on the set $\mathbb{C}$ of complex numbers in such a way that all three of the following are true:

i) for all $z, w \in \mathbb{C}$, exactly one of $x = w$, $x < w$, or $z > w$ holds
ii) for all $u, w, z \in \mathbb{C}:$ if $z < w$, then $z + u < w + u$
iii) if x, $y > 0$, then $xy > 0$

(Hint: Begin by showing that if such an order exists, then $-1 > 0$. But notice that this, in itself, is not a contradiction.)

E10. Give explicit examples of subsets of $\mathbb{Q}$ which represent the order types:

a) $\omega_0 + 1 + \omega_0$

b) $\omega_0^2 + \omega_0$

c) $\omega_0^2 + 2 \cdot \omega_0 + 3$

d) $\omega_0^2 + \omega_0 \cdot 2 + 3$

e) $\omega_0^2 + \omega_0^2$

f) $\omega_0^3 + \omega_0$

E11. Let η be the order type of $\mathbb{Q}$.

a) Give an example of a set $B \subseteq \mathbb{Q}$ such that neither B nor $\mathbb{Q} - B$ has order type η.

b) Prove that if A is a set of type η and $B \subseteq A$, then either B or $A - B$ contains a subset of type η.

c) Prove or disprove: $\eta + 1 + \eta = \eta$.

Hint: Cantor's characterization of $\mathbb{Q}$ as an ordered set may be helpful.

E12. A chain $(X, \leq)$ is called an η_1-set if the following condition holds in X :

(*) Whenever A and B are countable subsets of X such that $a < b$ for every choice of $a \in A$ and $b \in B$, then $\exists c \in X$ such that $a < c < b$ for all $a \in A$ and all $b \in B$

More informally, we could paraphrase condition (*) as: for countable subsets A,B of X, "$A < B$" $\Rightarrow$ $\exists c \in X$ such that "$A < c < B$"

a) Show that $\mathbb{R}$ is not an η_1-set.

b) Prove that every η_1-set is uncountable.
Hint: there is a one line argument; note that $\emptyset$ is countable.

c) By b), an η_1-set $(X, \leq)$ satisfies $|X| \geq \aleph_1$, and so $|X| \geq c$ <u>if we assume the continuum hypothesis, CH</u>. Prove that $|X| \geq c$ <u>without</u> assuming CH.
Hint: show how to define a one-to-one function $f : \mathbb{R} \to X$; begin by defining f on $\mathbb{Q}$.

More generally, a chain $(X, \leq)$ is called an η_α-set if, whenever A and B are subsets of X, both of cardinality $< \aleph_\alpha$, then there is a $x \in X$ such that "$A < x < B$". So, for example, an η_0-set is simply an order dense chain.

5. Well-Ordered Sets and Ordinal Numbers

We now look at a much stronger kind of order $\leq$ on a set.

Definition 5.1 A poset $(X, \leq)$ is called <u>well-ordered</u> if every nonempty subset of X contains a smallest element.

The definition implies that a well-ordered set X is automatically a chain: if $a \neq b \in X$, then set $\{a, b\}$ has a smallest element, so either $a \leq b$ or $b \leq a$.

$\mathbb{N}$ and all its subsets are well-ordered. The set of integers, $\mathbb{Z}$, is not well-ordered since, for example, $\mathbb{Z}$ itself contains no smallest element. $\mathbb{R}$ is not well-ordered since, for example, the nonempty interval $(0, 1)$ contains no smallest element.

Since a well-ordered set X is a chain, it has an order type. These special order types are very nicely behaved and have a special name.

Definition 5.2 An <u>ordinal</u> <u>number</u> (or simply <u>ordinal</u>) is the order type of a well-ordered set.

Since we know how to add and multiply order types, we already know how to add and multiply ordinals and get new ordinals. We also have a Definition 4.2 for $<$ and $\leq$ that applies to ordinals.

Theorem 5.3 If α and β are ordinals, so are $\alpha + \beta$ and $\alpha \cdot \beta$.

Proof Let α and β be represented by disjoint well-ordered sets A and B. Then $\alpha + \beta$ is represented by the ordered sum $A + B$. We must show this set is well-ordered. Since $A + B$ is a chain, we only need to check that a nonempty subset C of $A + B$ must contain a smallest element.

B is well-ordered so, if $C \subseteq B$, then C has a smallest element. Otherwise $C \cap A \neq \emptyset$ and, since A is well-ordered, there is a smallest element $c \in C \cap A$. In that case c is the smallest element of C.

Similarly, we need to show that the lexicographically ordered product $B \times A$ is well-ordered. If C is a nonempty subset of $B \times A$, let b_0 be the smallest first coordinate of a point in C: more precisely, let b_0 be the smallest element in $\{b \in B : \text{for some } a \in A, (b, a) \in C\}$. Then let a_0 be the smallest element in $\{a \in A : (b_0, a) \in C\}$. Then (b_0, a_0) is the smallest element in C. (*Intuitively,* (b_0, a_0) *is the point at the "lower left corner" of* C. *The fact that* B *and* A *are well-ordered guarantees that such a point exists.*) $\bullet$

Some examples of ordinals (increasing in size) are

$$0, 1, 2, \ldots, \omega_0, \omega_0 + 1, \omega_0 + 2, \ldots, \omega_0 + n, \ldots, \omega_0 \cdot 2, \omega_0 \cdot 2 + 1, \ldots, \omega_0 \cdot 2 + n, \ldots,$$

$$\ldots \omega_0 \cdot 3, \ldots, \omega_0 \cdot n, \ldots, \omega_0^2, \omega_0^2 + 1, \ldots, \omega_0^2 + \omega_0, \omega_0^2 + \omega_0 + 1, \ldots, \omega_0^2 + \omega_0 \cdot 2, \ldots$$

All these ordinals can be represented by countable well-ordered sets (in fact, by subsets of $\mathbb{Q}$) so we refer to them as "countable ordinals." We will see later (assuming AC), there are well-ordered sets of arbitrarily large cardinality $-$ so that this list of ordinals barely scratches the surface.

Exercise 5.4 Find a subset of $\mathbb{Q}$ that represents the ordinal $\omega_0^2 + \omega_0 \cdot 3 + 2$.

Here are a few very simple properties of well-ordered sets. Missing details should be checked as exercises.

1) In a well-ordered set X, each element a except the largest (if there is one) has an "immediate successor"— namely, the smallest element of the nonempty set $\{x \in X : x > a\}$. However, an element in a well-ordered set might not have an immediate predecessor: for example in $\{1 - \frac{1}{n} : n \in \mathbb{N}\} \cup \{2 - \frac{1}{n} : n \in \mathbb{N}\} \cup \{2\}$, neither 1 nor 2 has an immediate predecessor. This set represents the ordinal $\omega_0 + \omega_0 + 1$.

2) A subset of a well-ordered set, with the inherited order, is well-ordered.

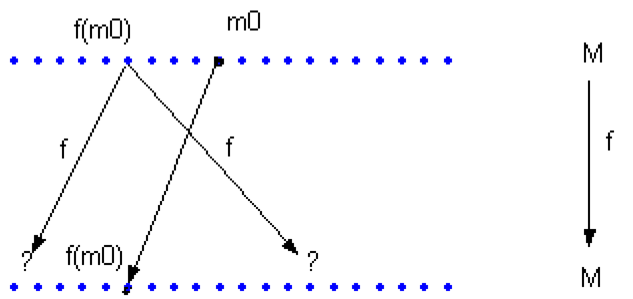
3) Order isomorphisms preserve well-ordering: if a poset is well-ordered, so is any order isomorphic poset. An order isomorphism preserves the smallest element in any nonempty subset.

The following theorems indicate how order isomorphisms between well-ordered sets are much less "flexible" than isomorphisms between chains in general. .

Theorem 5.5 Suppose M is well-ordered. If $f : M \to M$ is a one-to-one, order-preserving map of M into M, then $f(m) \geq m$ for all $m \in M$.

The theorem says that f cannot move an element "to the left." Notice that Theorem 5.5 is false for chains in general: for example, consider $M = \mathbb{R}$ and $f(x) = \frac{1}{2}x$. On the other hand, if you try to construct a counterexample using $M = \mathbb{N}$, you will probably see how the proof of the theorem should go.

Proof Suppose not. Then $A = \{m \in M : f(m) < m\} \neq \emptyset$. Let m_0 be the smallest element in A. We get a contradiction by asking "what is $f(f(m_0))$"?

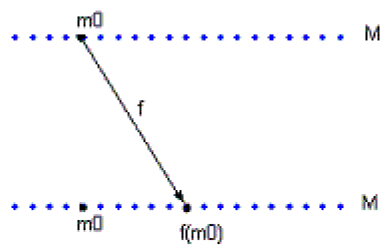


Since $m_0 \in A$, $f(m_0) < m_0$. Since f preserves order and is one-to-one, $f(f(m_0)) < f(m_0)$ which means that $f(m_0) \in A$. But that is impossible because m_0 is the smallest element of A. •

Corollary 5.6 If M is well-ordered, then the only order isomorphism f from M onto M is the identity map $f(m) = m$.

(*Note that the theorem is false for chains in general: if $M = \mathbb{R}$, then $f(x) = x^3$ is an onto order isomorphism. Corollaries 5.6 and 5.7 indicate that well-ordered sets have a very "rigid" structure.*)

Proof Let $f: M \to M$ be an order isomorphism. By Theorem 5.5, $f(m) \geq m$ for all m. If f is not the identity, then $A = \{m : f(m) > m\} \neq \emptyset$. Let m_0 be the least element of A.



Then m_0 cannot be in ran(f) (*why?*). •

Corollary 5.7 Suppose M and N are well-ordered. If $f : M \to N$ and $g : M \to N$ are order isomorphisms from M onto N, then $f = g$. (*So M and N can be order isomorphic "in only one way."*)

Proof If $f \neq g$, then $f^{-1}f$ and $g^{-1}f$ are two different order isomorphisms from M onto M, and that is impossible by the preceding corollary. •

Exercise 5.8 Find two different order isomorphisms between $\mathbb{R}$ and the set of positive reals $\mathbb{R}^+$.

We have already seen that order types, in general, are not very nicely behaved. Therefore, during this the following discussion about well-ordered sets and ordinal numbers, there is a certain amount of fussiness in the notation – to make sure we do not jump to any false conclusions. Much of this fussiness will drop by the wayside as things become clearer.

In a nonempty well-ordered set M, we will often refer to the smallest element as 0. (*In fact, without loss of generality, we can literally assume* 0 *is the smallest element* M.) If we need to carefully distinguish between the first elements in two well-ordered sets M, N we may write them as 0_M and 0_N. (*This might be necessary if, say,* $N \subseteq M$ *and the smallest elements of* N *and* M *are different.*) But usually this degree of care is not necessary.

Definition 5.9 Suppose $m \in M$, where M is well-ordered. The initial segment of M determined by $m = \{x \in M : x < m\}$. We can write this set using the "interval notation" $[0_M, m)_M$.
If a discussion involves only a single well-ordered set M, we may simply write $[0, m)_M$ or even just $[0, m)$.

Notice that:

i) For every $m \in M$, $M \neq [0, m)$, so M is not an initial segment of itself, and we will see in Theorem 5.10 that much more is true: M cannot even be order isomorphic to an initial segment of itself.

ii) Order isomorphisms preserve initial segments: if $f : M \to N$ is an order isomorphism of M onto N, then $f[[0, m)] = [0, f(m))$

iii) Given any two initial segments in a well-ordered set M, one of them is an initial segment of the other. More precisely, if $m < n \in M$, then "the initial segment in M determined by m" $= \{x \in M : x < m\} = \{x \in [0, n) : x < m\} =$ "the initial segment in $[0, n)$ determined by m."

Theorem 5.10 Suppose M is well-ordered and $N \subseteq M$. M is not order isomorphic to an initial segment of N. In particular (*when* $N = M$), M is not order isomorphic to an initial segment of itself.

Proof Suppose $n \in N \subseteq M$. If $f : M \to N \subseteq M$ is one-to-one and order preserving, then Theorem 5.5 gives us that $f(n) \geq n$ for each $n \in N$. Therefore, for each n, $\text{ran}(f) \neq [0_N, n)$. •

Corollary 5.11 No two initial segments of M are order isomorphic (so each initial segment of M, as well as M itself, represents a different ordinal).

Proof One of the two segments is an initial segment of the other, so by Theorem 5.10 the segments cannot be order isomorphic. •

Definition 5.12 Suppose μ and ν are ordinals represented by the well-ordered sets M and N. We say that $\mu < \nu$ if M is order isomorphic to an initial segment of N – that is $M \simeq [0, n)$ for some $n \in N$. If M is order isomorphic to N we write $\mu = \nu$. We write $\mu \leq \nu$ if $\mu < \nu$ or $\mu = \nu$. (*Check that the definition is independent of the choice of well-ordered sets M and N representing μ and ν.*)

Note: We already have a different definition (4.2) for $\mu \leq \nu$ when we think of μ and ν as arbitrary order types. It will turn out – for ordinals – that the two definitions are equivalent, that is:

for well-ordered sets M and N:

M is order isomorphic to a proper subset of N but not to N itself

$$\Updownarrow (*)$$

M is order isomorphic to an initial segment of N.

The equivalence $(*)$ is not true for chains in general: for example, each of $[0, 1]$ and $(0, 1)$ is order isomorphic to a subset of the other, but neither is order isomorphic to an initial segment of the other (why?)

Until Corollary 5.19, where we prove the equivalence $(*)$, we will be using the new definition 5.12 of $\mu \leq \nu$ for ordinals.

The relation $\leq$ among ordinals is clearly reflexive and transitive. The next theorem implies that $\leq$ is antisymmetric $-$ and therefore any set of ordinals is partially ordered by $\leq$.

Theorem 5.13 If μ and ν are ordinals, then at most one of the relations $\mu < \nu$, $\mu = \nu$, and $\mu > \nu$ can be true.

Proof Let M and N represent μ and ν. If $\mu = \nu$, then $M \simeq N$. In this case, $\mu < \nu$ and $\nu < \mu$ are impossible since a well-ordered set cannot be order isomorphic to an initial segment of itself (Theorem 5.10).

If $\mu < \nu$, then M is isomorphic to an initial segment of N. If $\nu < \mu$ were also true, then N would, in turn, be isomorphic to an initial segment of M. By composing these isomorphisms, we would have M order isomorphic to an initial segment of itself $-$ which is impossible. $\bullet$

Notation For an ordinal μ, let ord $(\mu) = \{\alpha : \alpha$ is an ordinal and $\alpha < \mu\}$. If $\mu = 0$, then ord $(\mu) = \emptyset$ and if $\mu > 0$, then $0 \in$ ord(μ), so ord$(\mu) \neq \emptyset$. Like any set of ordinals, we know that ord(μ) is partially ordered by $\leq$.

It turns out that much more is true: every set of ordinals is actually well-ordered by $\leq$, but to see that takes a few more theorems. However, ord(μ) is a very special set of ordinals and, for starters, Theorem 5.14 tells us that ord(μ) is well-ordered by $\leq$. Theorem 5.14 also gives us a very nice "standard" way to pick a well-order that represents an ordinal μ.

Theorem 5.14 If μ is an ordinal represented by the well-ordered set M, then ord$(\mu) \simeq M$. Therefore ord(μ) is a well-ordered set of ordinals and ord(μ) represents μ.

Proof For each ordinal $\alpha \in$ ord(μ), we have $\alpha < \mu$, so α can be represented by some initial segment $[0, m_\alpha)$ of M. Define f : ord$(\mu) \to M$ by $f(\alpha) = m_\alpha$. This function f is one-to-one since different ordinals cannot be represented by the same initial segment of M, and f clearly preserves order.

If $m \in M$, then the initial segment $[0, m)$ in M represents some ordinal $\alpha < \mu$. But α is represented by $[0, m_\alpha)$. Since different initial segments of M are not isomorphic, we get $m = m_\alpha = f(\alpha)$ so f is onto. Therefore ord$(\mu) \simeq M$. $\bullet$

We will often write ord(μ) in "interval" notation:

For an ordinal μ, $[0, \mu) = \{\alpha : \alpha$ is an ordinal and $\alpha < \mu\} =$ ord(μ)

By 5.14, $[0, \mu)$ is well-ordered and represents the ordinal μ; therefore any ordinal μ can be represented by the set of preceding ordinals.

For example,

0 is represented by the set of preceding ordinals, namely $[0, 0) =$ ord$(0) = \emptyset$

1 is represented by $[0,1) = \text{ord}(1) = \{0\}$

2 is represented by $[0,2) = \text{ord}(2) = \{0,1\}$

$$\vdots$$

ω_0 is represented by $[0,\omega_0) = \text{ord}(\omega_0) = \{0,1,2,...,n,...\}$

$\omega_0 + 1$ is represented by $[0,\omega_0 + 1) = \{0,1,2,...,n,...\,;\ \omega_0\}$
(*here, " ; " indicates that ω_0 comes "after" all the natural numbers n*)

$$\vdots$$

α is represented by the set of previously defined ordinals

etc.

$$\vdots$$

Some comments about axiomatics

The informal definition of ordinals is good enough for our purposes, However, the preceding list roughly illustrates how one can define ordinals in axiomatic set theory ZFC. For example, in ZFC the ordinal 2 is defined by $2 = \{0,1\}$ (rather than saying that the set $\{0,1\}$ represents the ordinal 2).

Definition

$$\begin{aligned} 0 &= \emptyset \\ 1 &= \{0\} = \{\emptyset\} \\ 2 &= \{0,1\} = \{\emptyset,\{\emptyset\}\} \\ &\vdots \\ \omega_0 &= \{0,1,2,...,n,...\} \\ \omega_0 + 1 &= \{0,1,2,...,n,...;\omega_0\} \\ &\vdots \\ &\text{etc.} \end{aligned}$$

and in general, an ordinal $\alpha =$ the set of previously defined ordinals. Of course, this presentation is still a little vague: in particular, some sort of "induction" in ZFC is needed to justify the "etc." where an ordinal is defined in terms of ordinals already defined.

Once we have defined ordinals (as sets) in ZFC, we need to say how they are compared, that is, how to define $\leq$. We do this for ordinals α and β by writing $\alpha < \beta$ iff $\alpha \in \beta$. This seems to accomplish what we want. For example:

$1 < 3$ because $1 \in 3$

$0 < 1 < 2 < 3 < ... < \omega_0 < \omega_0 + 1$ because $0 \in 1 \in 2 \in 3 \in ... \in \omega_0 \in \omega_0 + 1 \in ...$

$\omega_0 < \omega_0 + 17$ because $\omega_0 \in \omega_0 + 17$

etc.

If X is <u>any</u> set well-ordered by $\leq$, we can then define its ordinal number ($=$ "the ordinal number associated with X") as follows: from the axioms ZFC one can prove the <u>existence</u> of a function (set) with domain X that is defined "recursively" by :

$$\forall\, y \in X \;\; f(y) = \text{ran}\,(\, f|\{z \in X : z < y\})$$

Then "the ordinal number of X" is defined to be the set ran(f).

For example, for the well-ordered set $X = \{1, 3, 5\}$, what is the function f and what is the ordinal number of X?

$$f(1) = \text{ran}\,(\, f|\{z \in X : z < 1\}) = \text{ran}\,(f|\emptyset\,) = \emptyset$$

$$f(3) = \text{ran}\,(\, f|\{z \in X : z < 3\}) = \text{ran}\,(f|\{1\}) = \{\emptyset\}$$

$$f(5) = \text{ran}\,(\, f|\{z \in X : z < 5\}) = \text{ran}\,(f|\{1, 3\}) = \{\emptyset, \{\emptyset\}\}$$

The ordinal number of X is ran$\,(f) = \{0,\ \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} = \{0, 1, 2\} = 3$

In axiomatic set theory, cardinals are viewed as certain special ordinals: an ordinal α is called a <u>cardinal</u> if for all ordinals $\beta < \alpha$ there is no bijection between β and α $-$ that is a cardinal is an "initial ordinal" $-$ meaning that it's "the first ordinal with a given size." From that point of view ω_0 is a cardinal because there is a bijection between ω_0 and n for any $n < \omega_0$. Earlier, we gave this cardinal the name $\aleph_0$. But $\omega_0 + 1$ is not a cardinal because there is a bijection between ω_0 and $\omega_0 + 1$.

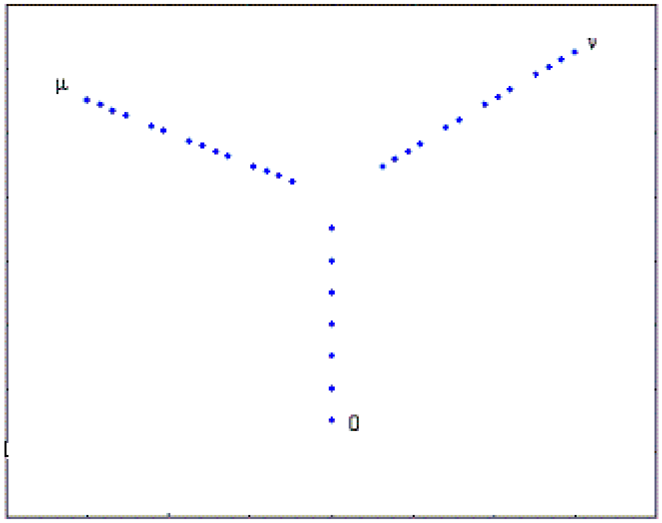
In Theorem I.13.2, we proved that <u>at most</u> one of the relations $<$, $=$, $>$ (as defined for cardinal numbers) can hold between two cardinals. (*Context will make clear whether " $\leq$ " refers to the ordering of cardinals or ordinals.*) We also <u>stated</u> in Chapter I that (assuming AC) <u>at</u> <u>least</u> one of the relations $<$, $=$, $>$ must hold between any two cardinals $-$ that is, for any two sets one must be equivalent to a subset of the other. We are almost ready to prove that statement $-$ in fact, this statement about cardinal numbers follows easily (assuming AC) from the corresponding result about ordinal numbers which we now prove.

Theorem 5.15 (Ordinal Trichotomy Theorem) If μ and ν are ordinals, then <u>at</u> <u>least</u> <u>one</u> of the relations $\mu > \nu$, $\mu = \nu$, $\mu < \nu$ must hold (and so, by Theorem 5.13, <u>exactly</u> <u>one</u> of these relations holds).

Proof We already know that <u>certain</u> <u>special</u> <u>sets</u> of ordinals are well-ordered: for example $[0, \mu) = \{\alpha : \alpha$ is an ordinal and $\alpha < \mu\}$ (Theorem 5.14).

The theorem is certainly true if $\mu = 0$ or $\nu = 0$, so we assume both $\mu > 0$ and $\nu > 0$.

Let $D = [0, \mu) \cap [0, \nu) = \{\alpha : \alpha$ is an ordinal for which $\alpha < \mu$ <u>and</u> $\alpha < \nu\}$. Because we do not yet know that μ and ν are comparable, the situation might look something like the following:



D is well-ordered because $D \subseteq [0, \mu)$, and $D \neq \emptyset$ (since $0 \in D$). Therefore D represents some ordinal $\delta > 0$. We claim that $\delta \leq \mu$ and $\delta \leq \nu$.

1) To show that $\delta \leq \mu$, we assume $\delta \neq \mu$ and prove $\delta < \mu$.

$[0, \mu)$ represents μ. Since $D \subseteq [0, \mu)$ and D represents $\delta \neq \mu$, we conclude $D \neq [0, \mu)$, so $[0, \mu) - D \neq \emptyset$. Let γ be the smallest element in $[0, \mu) - D$. $[0, \gamma)$ is an initial segment of $[0, \mu)$.

We claim $D = [0, \gamma)$. If that is true, then D represents γ; but D represents δ and therefore $\delta = \gamma < \mu$.

$[0, \gamma) \subseteq D$: If $\alpha \in [0, \gamma)$, then $\alpha \in [0, \mu)$ and $\alpha < \gamma =$ smallest element in $[0, \mu) - D$, so $\alpha \in D$.

$D \subseteq [0, \gamma)$: If $\alpha \in D$, then α and γ are comparable since both are in the well-ordered set $[0, \mu)$.

We examine the possibilities:

i) $\gamma = \alpha$: impossible, since $\alpha \in D$ and $\gamma \notin D$

ii) $\gamma < \alpha$: impossible, since that would mean $\gamma < \alpha < \mu$ and $\gamma < \alpha < \nu$, forcing $\gamma \in [0, \mu) \cap [0, \nu)$ $= D$ – which is false.

Therefore $\alpha < \gamma$, so $\alpha \in [0, \gamma)$.

2) A similar argument (interchanging "μ" and "ν" throughout) shows that if $\delta \neq \nu$, then $\delta = \gamma < \nu$.

Since $\delta \leq \mu$ and $\delta \leq \nu$, there are only four possibilities:

a) $\delta < \mu$ and $\delta < \nu$, in which case $\delta \in [0, \mu) \cap [0, \nu) = D = [0, \delta) -$ which is impossible.

Therefore one of the remaining three cases must be true:

b) $\delta = \mu$ and $\delta = \nu$, in which case $\mu = \nu$
c) $\delta < \mu$ and $\delta = \nu$, in which case $\mu > \nu$
d) $\delta = \mu$ and $\delta < \nu$, in which case $\mu < \nu$ $\bullet$

Corollary 5.16 Any set of ordinals is linearly ordered with respect to the ordinal ordering $\leq$.
(W*e shall see in Theorem 5.20 that even more is true: every set of ordinals is well-ordered.*)

We simply state the following theorem. A proof of the equivalences can be found, for example, in *Set Theory and Metric Spaces* (Kaplansky) or *Topology* (Dugundji).

Theorem 5.17 The following statements are equivalent. (Moreover, each is consistent with and independent of the axioms ZF for set theory:)

1) (**Axiom of Choice**) If $\{A_\alpha : \alpha \in A\}$ is a family of pairwise disjoint nonempty sets, there is a set $B \subseteq \bigcup A_\alpha$ such that, for all α, $|A_\alpha \cap B| = 1$.

(*This is clearly equivalent to the statement that* $\prod A_\alpha \neq \emptyset$. *If f is in the product, let $B =$ ran(f); on the other hand, if such a set B exists, define f by $f(\alpha) =$ the unique element of $A_\alpha \cap B$. An element $f \in \prod A_\alpha$ is a function that "chooses" one element $f(\alpha)$ from each A_α.*)

2) (**Zermelo's Theorem**) Every set can be well-ordered, i.e., for every set X there is a subset $\leq$ of $X \times X$ such that $(X, \leq)$ is well-ordered.

3) (**Zorn's Lemma**) Suppose $(X, \leq)$ is a nonempty poset. If every chain in X has an upper bound in X, then X contains a maximal element.

We will look at some powerful uses of Zorn's Lemma later. For now, we are mainly interested in Zermelo's Theorem.

It is tradition to call 2) Zermelo's Theorem and 3) Zorn's Lemma. They appear as "proven" results in the early literature, but the "proofs" used some form of the Axiom of Choice (AC). Generally, we have been casual about mentioning when AC is being used. However in the following theorems, for emphasis, $[AC]$ *indicates that the Axiom of Choice is used in one of these equivalent forms.*

Corollary 5.18 [AC, Cardinal Trichotomy] If m and n are cardinal numbers, at least one (and thus, by Theorem I.13.2 , exactly one) of the relations $m > n$, $m = n$, $m < n$ holds. (*Therefore any set of cardinals is a chain.*)

Proof According to Zermelo's Theorem, we may assume that M and N are well-ordered sets representing the cardinals m and n, so that M and N also represent ordinals μ and ν.

By Theorem 5.15, either $\mu = \nu$, $\mu < n$, or $\mu > \nu$; therefore either M and N are order isomorphic (in which case $m = n$) or one is order isomorphic to an initial segment of the other (so $m < n$ or $n < m$). $\bullet$

In fact, the Cardinal Trichotomy Corollary, is equivalent to the Axiom of Choice. (See Gillman, *Two Classical Surprises Concerning the Axiom of Choice and the Continuum Hypothesis*, Am. Math. Monthly 109(6), 2002, pp. 544-553 for this and other interesting results that do not depend on techniques of axiomatic set theory.) Over 200 equivalents to the Axiom of Choice are given in *Equivalents of the Axioms of Choice* (Rubin & Rubin, North-Holland Publishing, 1963).

The following corollary tells us that, among ordinals, the two definitions of " $\leq$ " (Definition 4.2 and Definition 5.12) are equivalent.

Corollary 5.19 Suppose M and N are well-ordered sets representing μ and ν. If M is order isomorphic to a subset of N (*so $\mu \leq \nu$ in the sense of Definition 4.2*), then M is order isomorphic to N or to an initial segment of N (*so $\mu \leq \nu$ in the sense of Definition 5.12*).

Proof Without loss of generality, we may assume $M \subseteq N$. If M is not order isomorphic to N, then $\mu \neq \nu$. If M is also not isomorphic to an initial segment of N, then $\mu < \nu$ is also false. Therefore the Trichotomy Theorem 5.15 gives $\mu > \nu$. Then N is order isomorphic to an initial segment of a subset M of itself $-$ which violates Theorem 5.10. $\bullet$

Theorem 5.20 Every set W of ordinals is well-ordered. In particular, every nonempty set of ordinals contains a smallest element.

Proof Theorem 5.15 implies that W is linearly ordered by $\leq$. We need to show that if A is a nonempty subset of W, then A contains a smallest element. Pick $\alpha \in A$. If α is itself the smallest in A, we are done. If not, then $\{\beta \in A : \beta < \alpha\}$ is nonempty and well-ordered $-$ because it is a subset of $[0, \alpha)$ $-$ so it contains a smallest element β_0, and β_0 is the smallest element in W. $\bullet$

Corollary 5.21 [AC] Every set C of cardinal numbers is well-ordered. In particular, every nonempty set of cardinal numbers contains a smallest element.

Proof We know that the order $\leq$ (among cardinals) is a linear order. Let D be a nonempty subset of C and, for each cardinal $m \in D$, let M represent m. By Zermelo's Theorem, each set M can be well-ordered, after which it represents some ordinal μ. By Theorem 5.20, the set of all such μ's contains a smallest element μ_0 represented by M_0. Then $m_0 = |M_0|$ is clearly the smallest cardinal in D. $\bullet$

Example 5.22

1) The set $C = \{m : m \text{ is a cardinal and } \aleph_0 < m \leq c\}$ has a smallest element. It is called the immediate successor of $\aleph_0$ and is denoted by $\aleph_0^+$ or $\aleph_1$. The statement $c = \aleph_1$ is the Continuum Hypothesis which, we recall, is independent of the axioms ZFC. If CH is assumed as an additional axiom in set theory, then $C = \{\aleph_1\} = \{c\}$.

2) More generally, for any cardinal m we can consider the smallest element m^+ in the set $\{k : k$ is a cardinal and $m < k \leq 2^m\}$. We call m^+ the <u>immediate successor</u> of m. In particular, we write $\aleph_1^+ = \aleph_2$, $\aleph_2^+ = \aleph_3$, and so on.

The Generalized Continuum Hypothesis is the statement

GCH : "for every infinite cardinal m, $m^+ = 2^m$."

GCH and $\sim$ GCH are equally consistent with the axioms ZFC. (*Curiously, ZF + GCH implies AC. This is discussed in the Gillman article cited after Corollary 5.18.*)

3) In Example VI.4.6, we (provisionally) defined the <u>weight</u> of a topological space $(X, \mathcal{T})$ by

$$w(X) = \aleph_0 + \min\{|\mathcal{B}| : \mathcal{B} \text{ is a base for } \mathcal{T}\}.$$

We now see that the definition makes sense $-$ because there must exist a base of smallest cardinality.

Theorem 5.23 If W is a set of ordinals, then there exists an ordinal greater than any ordinal in W. (*Therefore there is no "set of all ordinals."*)

Proof Let $W^* = \{\mu + 1 : \mu \in W\}$, and represent each ordinal $\mu + 1$ in W^* by a well-ordered set $M_{\mu+1}$. We may assume the $M_{\mu+1}$'s are pairwise disjoint. (*If not, replace each $M_{\mu+1}$ with $M_{\mu+1} \times \{\mu+1\}$, ordered in the obvious way.*) Form the ordered sum: that is, let $S = \bigcup\{M_{\mu+1} : \mu \in W\}$, and order S by

$$x \leq y \text{ if } \begin{cases} x, y \in M_{\mu+1}, \text{ and } x \leq y \text{ in } M_{\mu+1} \\ x \in M_{\mu+1},\ y \in M_{\nu+1} \text{ and } \mu < \nu \end{cases}$$

Clearly, $\leq$ well-orders S, so $(S, \leq)$ represents an ordinal σ. Since each $M_{\mu+1} \subseteq S$, we have $\mu + 1 \leq \sigma$ by Theorem 5.19. Since $\mu < \mu + 1 \leq \sigma$ for each $\mu \in W$, σ is larger than any ordinal in W. $\bullet$

Corollary 5.24 Every set W of ordinals has a least upper bound, denoted $\sup W$ $-$ that is, there is a smallest ordinal $\geq$ every ordinal in W.

Proof Without loss of generality we may assume that if $\alpha < \mu \in W$, then $\alpha \in W$ (*why? and where is this assumption used in what follows?*). If W contains a largest element, then it is the least upper bound. Otherwise, pick an ordinal σ larger than every ordinal in W. Then $[0, \sigma+1) - W \neq \emptyset$ (it contains σ) and the smallest element in $[0, \sigma+1) - W$ is $\sup W$. $\bullet$

Example 5.25

1) $\sup\{0, 1, 2, ...,\} = \omega_0$, and $\sup\{0, 1, 2, ..., \omega_0\} = \omega_0$

2) We say that an ordinal "has cardinal m" if it is represented by a well-ordered set with cardinality m. In particular, countable ordinals are those represented by countable well-ordered sets.

3) sup $\{\alpha : \alpha$ is a countable ordinal$\}$ is called ω_1. Since there is no largest countable ordinal (*why?*), we see that ω_1 is the <u>smallest</u> uncountable ordinal. A set representing ω_1 must have cardinal $\aleph_1$ (*why?*). Since $[0, \omega_1)$ represents ω_1, so there are exactly $\aleph_1$ ordinals $< \omega_1$, that is, exactly $\aleph_1$ countable ordinals. Each countable ordinal can be represented by a subset of $\mathbb{Q}$ (*see Example 4.7*), so there are exactly $\aleph_1$ nonisomorphic well-ordered subsets of $\mathbb{Q}$.

Since ω_1 is the smallest uncountable ordinal, each $\alpha < \omega_1$ is a countable ordinal that is represented by $[0, \alpha)$. Therefore α has only countably many predecessors and ω_1 is the first ordinal with uncountably many ($\aleph_1$) predecessors.

The spaces $[0, \omega_1)$ and $[0, \omega_1 + 1) = [0, \omega_1]$, with the order topology, have some interesting properties that we will look at later. These properties hinge on the fact that ω_1 is the <u>smallest</u> uncountable ordinal.

The ordinals ω_0, ... , $\omega_0 + n$, ... , $\omega_0 \cdot 2$, ... , ω_0^2, ... , ω_0^n, ... are all mere countable ordinals. For ordinals α, β it is possible to define "ordinal exponentiation" α^β. (*The definition is sketched in the appendix at the end of this chapter.*) Then it turns out that $\omega_0^{\omega_0}$, $\omega_0^{\omega_0} + 1$, ... , $\omega_0^{(\omega_0^{\omega_0})}$, ... , are still countable ordinals. If you accept that, then you should also believe that $\epsilon_0 = \sup\{\omega_0, \omega_0^{\omega_0}, \omega_0^{\omega_0^{\omega_0}}, \omega_0^{\omega_0^{\omega_0^{\omega_0}}}, ... \}$ ($=$ "ω_0 to the ω_0 power ω_0 times") is still countable. Roughly, each element in the set has only countably many predecessors and the set has only countable many elements, so the least upper bound of the set still has only countably many predecessors—namely, all the predecessors of its predecessors.

But once you get up to ϵ_0, you can then form $\epsilon_0^{\epsilon_0}$, $\epsilon_0^{(\epsilon_0^{\epsilon_0})}$, ..., take the least upper bound again, and still have only a countable ordinal ϵ_1. And so on. So ω_1, the first ordinal with uncountably many predecessors is way beyond all these: "the longer you look at ω_1, the farther away it gets" (*Robert McDowell*).

We now look at some similar results for cardinals.

Lemma 5.26 If $\{k_\alpha : \alpha \in A\}$ and $\{m_\alpha : \alpha \in A\}$ are sets of cardinals and $k_\alpha \geq m_\alpha$ for each $\alpha \in A$, then $\sum k_\alpha \geq \sum m_\alpha$. (*An infinite sum of cardinals is defined in the obvious way: if the K_α's are pairwise disjoint sets with cardinality k_α, then $\sum k_\alpha = |\bigcup K_\alpha|$.*)

Proof Exercise

Theorem 5.27 If a set of cardinals $C = \{k_\alpha : \alpha \in A\}$ contains no largest element, then $\sum k_\alpha > k_{\alpha_0}$ for each $\alpha_0 \in A$.

Proof For any particular $\alpha_0 \in A$, let $\begin{cases} m_{\alpha_0} = k_{\alpha_0} \\ m_\alpha = 0 \text{ for } \alpha \neq \alpha_0 \end{cases}$.
Then the lemma gives $\sum k_\alpha \geq \sum m_\alpha = k_{\alpha_0}$.

If $\sum k_\alpha = k_{\alpha_0}$ for some α_0, k_{α_0} would be the largest element in C which, by hypothesis, does not exist. Therefore $\sum k_\alpha > k_{\alpha_0}$ for every $\alpha_0 \in A$. $\bullet$

The conclusion may be true even if C has a largest element: for example, suppose $C = \{1, 2\}$. Can you give an example involving an infinite set C of cardinals?

Corollary 5.28 If C is a set of cardinals, then there is a cardinal m larger than every member of C (and therefore there is no "set of all cardinals").

Proof If C has a largest element k, then let $m = 2^k$. Otherwise, use the preceding theorem and let $m = \sum\{k : k \in C\}$. $\bullet$

Corollary 5.29 Every set C of cardinal numbers has a least upper bound, that is, there is a smallest cardinal $\geq$ every cardinal in C.

Proof Without loss of generality, we may assume that if $p \in C$, then all cardinals smaller than p are also in C (*why? and where is this used in what follows?*). If C has a largest element k, then k is the least upper bound. Otherwise, pick a cardinal m greater than all the cardinals in C and let $S = \{p : p$ is a cardinal and $p \leq m\}$. Then $S - C \neq \emptyset$ (it contains m) and the smallest cardinal in this set is the least upper bound for C. $\bullet$

6. Indexing the Infinite Cardinals

By Corollary 5.21, the set of infinite cardinals less than a given cardinal k is well-ordered, so this set is order isomorphic to an initial segment of ordinals. Therefore this set of cardinals can be "faithfully indexed" by that segment of ordinals – that is, indexed in such a way that $k_\alpha < k_\beta$ iff $\alpha < \beta$. When the infinite cardinals are listed in order of increasing size and indexed by ordinals, they are denoted by $\aleph$'s with ordinal subscripts. In this notation, the first few infinite cardinals are

$$\aleph_0,\ \aleph_1\ (=\aleph_0^+),\ \aleph_2\ (=\aleph_1^+),\ \ldots,\ \aleph_n,\ \ldots,\ \aleph_{\omega_0},\ \aleph_{\omega_0+1},\ \ldots,\ \aleph_{\omega_0+\omega_0},\ \ldots,\ \aleph_{\omega_0^2},\ \ldots,\ \aleph_{\epsilon_0},$$
$$\aleph_{\epsilon_0+1},\ \ldots,\ \aleph_{\omega_1},\ \ldots .$$

Thus, $\aleph_{\omega_0} = \sup\{\aleph_n : n < \omega_0\}$ and $\aleph_{\omega_1} = \sup\{\aleph_\alpha : \alpha < \omega_1\}$. $\aleph_{\omega_1}$ is the first cardinal with uncountably many ($\aleph_1$) cardinal predecessors.

In this notation, GCH states that for every ordinal α, $2^{\aleph_\alpha} = \aleph_{\alpha+1}$.

By definition, $c = |\mathbb{R}|$. So where is c is this list of cardinals? The continuum hypothesis states "$c = \aleph_1$." However, it is a fact that for each ordinal α, there exists an ordinal $\beta > \alpha$ such that the assumption "$\aleph_\beta = c$" is consistent with ZFC. (*Perhaps $\mathbb{R}$ is more mysterious than you thought.*) These results imply that not even the simplest exponentiation $2^{\aleph_0}$ involving an infinite cardinal can be "calculated" in ZFC: is $2^{\aleph_0} = \aleph_1$? $= \aleph_{17}$? $= \aleph_{\omega_0+1}$?

On the other hand, one cannot consistently assume "$c = \aleph_\alpha$" for an arbitrary choice of α: even in ZFC, certain $\aleph_\alpha$'s are provably excluded. In fact, if $\aleph_\alpha$ is the least upper bound of a strictly increasing sequence of smaller cardinals, we will prove $\aleph_\alpha^{\aleph_0} > \aleph_\alpha$ (and therefore $\aleph_\alpha \neq c$). *It is also true, but we will not prove it, that these are the only excluded $\aleph_\alpha$'s – it is consistent to assume $2^{\aleph_0} = \aleph_\alpha$ for any cardinal $\aleph_\alpha$ for which $\aleph_\alpha^{\aleph_0} = \aleph_\alpha$ (Solovay, 1965).*

At the heart of what we need is a classical theorem about cardinal arithmetic.

Theorem 6.1 (König) Suppose that for each $\alpha \in A,\ m_\alpha$ and n_α are cardinals with $m_\alpha < n_\alpha$. Then $\sum_\alpha m_\alpha < \prod_\alpha n_\alpha$.

Proof Proving " $\leq$ " is straightforward; proving " $<$ " takes a little more work.

Let sets M_α and N_α represent m_α and n_α. We may assume the N_α's are pairwise disjoint and, since $m_\alpha < n_\alpha$, that each M_α is a proper subset of N_α. For each α, pick and fix an element $n_\alpha \in N_\alpha - M_\alpha$ and define $f : \bigcup M_\alpha \to \prod N_\alpha$ by:

$$\text{for } x \in M_{\alpha_0},\ f(x) = z \in \prod N_\alpha, \text{ where } z(\alpha) = \begin{cases} x & \text{if } \alpha = \alpha_0 \\ n_\alpha & \text{if } \alpha \neq \alpha_0 \end{cases}$$

The M_α's are disjoint so f is well defined, and clearly f is one-to-one, so we conclude that $\sum_\alpha m_\alpha \leq \prod_\alpha n_\alpha$.

We now show that $\sum m_\alpha = \prod n_\alpha$ is impossible. We do this by showing that if $h : \bigcup M_\alpha \to \prod N_\alpha$ is one-to-one, then h cannot be onto.

Let $P = \text{ran}(h) = h[\bigcup M_\alpha] = \bigcup h[M_\alpha]$. Let $h[M_\alpha] = P_\alpha$. Since h is one-to-one, $|P_\alpha| = m_\alpha$ so, for each α, there are at most m_α different α^{th} coordinates of points in P_α – that is, $|\{z_\alpha : z \in P_\alpha\}| \leq m_\alpha < n_\alpha$. Then we can pick a point $w_\alpha \in N_\alpha$ with $w_\alpha \neq z_\alpha$ for every $z \in P_\alpha$, i.e., w_α is not the α-th coordinate of any point in P_α.

Define $w \in \prod N_\alpha$ by $w(\alpha) = w_\alpha$. Then $w \notin P_\alpha$ for all α, so $w \notin P = \text{ran}(h)$. •

Example 6.2 Suppose we have a strictly increasing sequence of cardinals

$$0 \neq m_0 < m_1 < \ldots < m_k < \ldots \ .$$

For each k, let $n_k = m_{k+1}$, so $m_k < n_k$. By König's Theorem, $\sum_{k=0}^{\infty} m_k < \prod_{k=0}^{\infty} n_k = \prod_{k=1}^{\infty} m_k \leq \prod_{k=0}^{\infty} m_k$.

In particular: if $m_k = \aleph_k$, then $\sum_{k=0}^{\infty} \aleph_k < \prod_{k=0}^{\infty} \aleph_k$. Since $\aleph_{\omega_0} \leq \sum_{k=0}^{\infty} \aleph_k$ (*why?*), we have

$$\aleph_{\omega_0} < \prod_{k=0}^{\infty} \aleph_k \ \leq \aleph_{\omega_0}^{\aleph_0}.$$

Since $c = c^{\aleph_0}$, we conclude that $c \neq \aleph_{\omega_0}$.

Note: A similar argument shows that if a cardinal k is the least upper bound of a sequence of strictly increasing cardinals, then $k^{\aleph_0} > k$, so $k \neq c$.

Exercise 6.3 For a cardinal k, there are how many ordinals with cardinality $\leq k$?

7. Spaces of Ordinals

Let X be a set of ordinals with the order topology. Since X is well-ordered, X is order isomorphic to some initial segment of ordinals $[0, \alpha)$. Therefore X and $[0, \alpha)$ are homeomorphic in their order topologies. Therefore to think about "spaces of ordinals" we only need look at initial segments of ordinals $[0, \alpha)$. We will look briefly at some general facts about these spaces. In Section 8, we will consider the spaces $[0, \omega_1)$ and $[0, \omega_1 + 1) = [0, \omega_1]$ in more detail. These particular spaces have some interesting properties that arise from the fact that ω_1 is the first uncountable ordinal.

According to Definition 3.1, a subbase for the order topology on $X = [0, \alpha)$ consists of all sets

$$\{x \in [0, \alpha) : x < \gamma\} = [0, \gamma)\,, \quad \gamma < \alpha$$

and

$$\{x \in [0, \alpha) : x > \beta\} = (\beta, \alpha), \quad \beta \geq 0$$

The set of finite intersections of such sets is a base, so (*check this!*) a basic open set has one of the following forms:

$[0, \gamma)$, where $\gamma \leq \alpha$ ($\gamma = \alpha$ corresponds to the empty intersection)
(β, γ), where $\beta \geq 0$ and $\gamma \leq \alpha$

If $\alpha = 0$, then $X = [0, \alpha) = \emptyset$; so suppose $\alpha > 0$. What does an efficient neighborhood base at a point $\tau \in [0, \alpha)$ look like? .

If $\tau = 0$: $\{0\} = [0, 1)$ is open in $[0, \alpha)$. Therefore 0 is an isolated point in $[0, \alpha)$ and $\{\{0\}\}$ is an open neighborhood base at 0.

If $0 < \tau < \alpha$, then any <u>basic</u> open set containing τ must contain a set of the form $(\sigma, \tau]$:

if $\tau \in [0, \gamma)$, then $\tau \in (0, \tau] \subseteq [0, \gamma)$
if $\tau \in (\beta, \gamma)$, then $\tau \in (\beta, \tau] \subseteq (\beta, \gamma)$

Each set $(\sigma, \tau] = (\sigma, \tau + 1)$ is open; and each set $(\sigma, \tau]$ is also closed because its complement $[0, \sigma + 1) \cup (\tau, \alpha)$ is open. Therefore $\{(\sigma, \tau] : 0 \leq \sigma < \tau\}$ is a <u>neighborhood base of clopen sets</u> at τ.

Putting together these open neighborhood bases, we get that

$$\mathcal{B} = \{\{0\}\} \cup \{(\sigma, \tau] : 0 \leq \sigma < \tau < \alpha\}$$

is a clopen base for the topology.

We noted in Example 3.4 that any chain with the order topology is Hausdorff. Therefore every ordinal space $[0, \alpha)$ is Hausdorff. Since there is a neighborhood base of closed (in fact, clopen) neighborhoods at each point $\tau \in [0, \alpha)$, we know even more: Theorem VII.2.7 tells us that $[0, \alpha)$ is a T_3-space. But still more is true.

Theorem 7.1 For any ordinal α, $[0, \alpha)$ is T_4.

As remarked earlier, every <u>chain</u> *with the order topology is* T_4 *– but the proof is much simpler for well-ordered sets.*

Proof We know that $[0, \alpha)$ is T_1, so need to prove that $[0, \alpha)$ is normal. Suppose A and B are disjoint closed sets in $[0, \alpha)$.

If $\tau \in A$, let $U_\tau =$ a basic open set of form $(\sigma, \tau]$ disjoint from B (or, $U_\tau = \{0\}$ if $\tau = 0$)
If $\tau \in B$, let $V_\tau =$ a basic open set of form $(\sigma, \tau]$ disjoint from A (or, $V_\tau = \{0\}$ if $\tau = 0$)

We claim that if $\tau_1 \in A$ and $\tau_2 \in B$, then $U_{\tau_1} \cap V_{\tau_2} = \emptyset$:

The statement is clearly true if τ_1 or $\tau_2 = 0$ so suppose both are > 0. Then $U_{\tau_1} = (\sigma_1, \tau_1]$ and $V_{\tau_2} = (\sigma_2, \tau_2]$. We can assume without loss of generality that $\tau_1 < \tau_2$.
If $(\sigma_1, \tau_1] \cap (\sigma_2, \tau_2] \neq \emptyset$, then we have $\tau_1 \in (\sigma_2, \tau_2]$ which would mean that $V_{\tau_2} \cap A \neq \emptyset$.

If $U = \bigcup_{\tau \in A} U_\tau$ and $V = \bigcup_{\tau \in B} V_\tau$, then U and V are disjoint open sets with $U \supseteq A$ and $V \supseteq B$. •

(*Why doesn't the same proof work for chains with the order topology?*)

Definition 7.2 An ordinal β is called a <u>limit</u> ordinal if $\beta > 0$ and β has no immediate predecessor; β is called a <u>nonlimit</u> ordinal if $\beta = 0$ or β has an immediate predecessor (that is, $\beta = \gamma + 1$ for some ordinal γ).

Example 7.3

1) If β is a limit ordinal in $[0, \alpha)$, then for all $\sigma < \beta$, $(\sigma, \beta] \neq \{\beta\}$. Therefore β is not isolated in the order topology. If β is a nonlimit ordinal, then $\{\beta\} = \{0\}$ or, for some γ, $\{\beta\} = (\gamma, \beta]$. Either way, $\{\beta\}$ is open so β is isolated. Therefore the isolated points in $[0, \alpha)$ are the exactly the points that are not limit ordinals.

2) $[0, \omega_0)$ is discrete: it is homeomorphic to $\mathbb{N}$.

3) $[0, \omega_0 + 1)$ is homeomorphic to $\{1 - \frac{1}{n} : n \in \mathbb{N}\} \cup \{1\}$.

4) For what α's is $[0, \alpha)$ connected?

Theorem 7.4 Suppose $\alpha > 0$. The ordinal space $[0, \alpha)$ is compact iff $[0, \alpha) = [0, \beta]$ for some ordinal β – that is, iff $[0, \alpha)$ contains a largest element β.

Proof Suppose $[0, \alpha)$ has no largest element. Then $\mathcal{U} = \{[0, \gamma) : \gamma < \alpha\}$ is an open cover of $[0, \alpha)$. The sets in $\mathcal{U}$ are nested so, if there were a finite subcover, there would be a single set $[0, \gamma)$ covering $[0, \alpha)$. That is impossible since $\gamma \in [0, \alpha) - [0, \gamma)$.

Conversely, suppose $[0, \alpha) = [0, \beta]$. If $\beta = 0$, then $[0, \beta] = \{0\}$ is compact, so we assume $\beta > 0$.

Let $\mathcal{U}$ be an open cover of $[0, \beta]$. We can assume $\mathcal{U}$ consists of <u>basic</u> open sets, that is, sets of the form $\{0\}$ or $(\sigma, \tau]$. In that case, necessarily, $\{0\} \in \mathcal{U}$.

Let $\beta_1 = \beta > 0$. For some $\sigma_1 < \beta_1$, we have a set $(\sigma_1, \beta_1] \in \mathcal{U}$. If $\sigma_1 = 0$, then $\{\{0\}, (\sigma_1, \beta_1]\}$ is a finite subcover.

If $\sigma_1 > 0$, then for some β_2, $\sigma_1 \in (\sigma_2, \beta_2] \in \mathcal{U}$. If $\sigma_2 = 0$, then $\{\{0\}, (\sigma_2, \beta_2], (\sigma_1, \beta_1]\}$ is a finite subcover.

We proceed inductively. Having chosen $(\sigma_k, \beta_k]$, if $\sigma_k > 0$ we can choose $(\sigma_{k+1}, \beta_{k+1}] \in \mathcal{U}$ so that $\sigma_k \in (\sigma_{k+1}, \beta_{k+1}]$.

We continue until $\sigma_n = 0$ occurs $-$ and this must happen in a finite number of steps because otherwise, we would generate an infinite descending sequence of ordinals $\sigma_1 > \sigma_2 > \sigma_3 >$ ... $> \sigma_n > \ldots$. This is impossible because then the well-ordered set $\{\sigma_1, \sigma_2, ..., \sigma_n, ... \}$ would have no smallest element.

When we reach $\sigma_n = 0$, we have a finite subcover from $\mathcal{U}$: $\{\{0\}, (\sigma_n, \beta_n], ..., (\sigma_1, \beta_1]\}$. •

Example 7.5 $[0, \omega_0)$ is not compact, but $[0, \omega_0 + 1) = [0, \omega_0]$ is compact. In fact, $[0, \omega_0]$ is homeomorphic to $\{1\} \cup \{1 - \frac{1}{n} : n \in \mathbb{N}\}$.

8. The Spaces $[0, \omega_1)$ and $[0, \omega_1]$

Theorem 8.1 For each $n < \omega_0$, suppose $\alpha_n < \omega_1$. Then $\alpha = \sup \{\alpha_n : n < \omega_0\} < \omega_1$ $-$ that is, the sup of a countable set of countable ordinals is countable.

Proof $[0, \alpha_n)$ is countable so $\bigcup_{n < \omega_0} [0, \alpha_n)$ is countable. Since ω_1 has uncountably many predecessors, there is an ordinal $\gamma \in [0, \omega_1) - \bigcup_{n < \omega_0} [0, \alpha_n)$. Then $\gamma > \alpha_n$ for each n, so $\alpha = \sup \{\alpha_n : n < \omega_0\} \leq \gamma < \omega_1$. •

Corollary 8.2 $[0, \omega_1]$ and $[0, \omega_1)$ are not separable.

Proof If D is a countable subset of $[0, \omega_1]$, then $\alpha = \sup (D - \{\omega_1\}) < \omega_1$. Then $\text{cl}\, D \subseteq [0, \alpha] \cup \{\omega_1\} \neq [0, \omega_1]$.

A dense set in $[0, \omega_1)$ is also dense in $[0, \omega_1]$, so $[0, \omega_1)$ is not separable. •

Corollary 8.3 In $[0, \omega_1]$, no sequence from $[0, \omega_1)$ can converge to ω_1.

Proof Suppose (α_n) is a sequence in $[0, \omega_1)$. Let $\alpha = \sup \{\alpha_n : n \in \mathbb{N}\} < \omega_1$. The α_n's are all in the closed set $[0, \alpha]$, so $(\alpha_n) \not\to \omega_1$. •

Example 8.4 In Example III.9.8, we saw a rather complicated space L in which sequences are not sufficient to describe the topology. Corollary 8.3 gives an example that may be easier to "see" : $\omega_1 \in$ cl $[0, \omega_1)$, but no sequence (α_n) in $[0, ,\omega_1)$ can converge to ω_1.

This implies that $[0, \omega_1]$ is not first countable. Of course, $\{(\sigma, \tau] : \sigma < \tau\}$ is a countable neighborhood base at each point $\tau < \omega_1$ – so the "problem" point is ω_1. The neighborhood poset $\mathcal{N}_{\omega_1}$ (ordered by reverse inclusion) is very nicely ordered (in fact, well-ordered) but the chain of neighborhoods is just "too long" and we cannot "thin it out" enough to get a countable neighborhood base at ω_1.

In contrast, the basic neighborhoods of $(0, 0)$ in the space L were very badly "entangled". The neighborhood system $\mathcal{N}_{(0,0)}$ had a very complicated order structure – too complicated for us to find a countable subset of $\mathcal{N}_{(0,0)}$ that "goes arbitrarily far out" in the poset. (*See discussion in Example 2.4.6*).

In Theorem IV.8.11 we proved that certain implications hold between various "compactness-like" properties in a topological space X.

$$(*) \begin{cases} X \text{ is compact} \\ \qquad \text{or} \\ X \text{ is sequentially compact} \end{cases} \Rightarrow X \text{ is countably compact} \Rightarrow X \text{ is pseudocompact.}$$

We asserted that, in general, no other implications are valid. The following corollary shows that "sequentially compact" $\not\Rightarrow$ "compact" (and therefore "countably compact" $\not\Rightarrow$ "compact" and "pseudocompact" $\not\Rightarrow$ "compact").

Corollary 8.5 $[0, \omega_1)$ is sequentially compact.

Proof Suppose (α_n) is a sequence in $[0, \omega_1)$. We need to show that (α_n) has a convergent subsequence in $[0, \omega_1)$. Without loss of generality, we may assume that all the α_n's are distinct (*why?*) The sequence (α_n) has either an increasing subsequence $\alpha_{n_1} < \alpha_{n_2} < ... < \alpha_{n_k} < ...$ or a decreasing subsequence $\alpha_{n_1} > \alpha_{n_2} > ... > \alpha_{n_k} > ...$

> The argument is completely parallel to the one in Lemma IV.2.10 showing that a sequence in $\mathbb{R}$ has a monotone subsequence. Call α_n a peak point of the sequence if $\alpha_n \geq \alpha_k$ for all $k \geq n$.
>
> If the sequence has only finitely many peak points, then after some α_{n_1} there are no peak points and we can choose an increasing subsequence $\alpha_{n_1} < \alpha_{n_2} < ... < \alpha_{n_k} < ...$
>
> If (α_n) has infinitely many peak points, then we can choose a subsequence of peak points $\alpha_{n_1}, ...\alpha_{n_k}, ...$ and for these, $\alpha_{n_1} \geq \alpha_{n_2} \geq ... \geq \alpha_{n_k} \geq ...$. Since the α_n's are distinct, we have $\alpha_{n_1} > \alpha_{n_2} > ... > \alpha_{n_k} > ...$.

However, a strictly decreasing sequence of ordinals $\alpha_{n_1} > \alpha_{n_2} > ... > \alpha_{n_k} > ...$ is impossible. Therefore (α_n) has a subsequence of the form $\alpha_{n_1} < \alpha_{n_2} < ... < \alpha_{n_k} < ...$. Setting $\alpha = \sup\{\alpha_{n_k} : k = 1, 2, ...\} < \omega_1$, we see that then $(\alpha_n) \to \alpha \in [0, \omega_1)$. ●

An example of a pseudocompact space that is not countably compact is given in Exercise E32. In Chapter X, we will discuss a space $\beta\mathbb{N}$ that is compact (therefore countably compact and pseudocompact) but not sequentially compact (see Example X.6.5). That will complete the set of examples showing that "no other implications exist" other than those stated in (*).

Corollary 8.6 In $[0, \omega_1]$, the intersection of a countable collection of neighborhoods of ω_1 is again a neighborhood of ω_1 – that is, the intersection must contain a "tail" $(\alpha, \omega_1]$. Therefore $\{\omega_1\}$ is not a G_δ-set (and therefore not a zero set) in $[0, \omega_1]$.

Proof Let $\{N_k : k = 1, 2, ...\}$ be a collection of neighborhoods of ω_1. For each k, there is a α_k for which $\omega_1 \in (\alpha_k, \omega_1] \subseteq \text{int}\, N_k \subseteq N_k$. Let $\alpha = \sup\{\alpha_k : k \in \mathbb{N}\} < \omega_1$. Then $\bigcap_{k=1}^{\infty} N_k \supseteq \bigcap_{k=1}^{\infty} \text{int}\, N_k \supseteq (\alpha, \omega_1]$. In particular, $\bigcap_{k=1}^{\infty} N_k \neq \{\omega_1\}$. •

Since $[0, \omega_1]$ is compact, we know that each $f \in C([0, \omega_1])$ is bounded. In fact, something more is true. (*Why does the corollary state something "more" ?*)

Corollary 8.7 If $f \in C([0, \omega_1])$, then f is constant on the "tail" $[\alpha, \omega_1]$ for some $\alpha < \omega_1$.

Proof Suppose $f(\omega_1) = r$. By Corollary 8.6, $f^{-1}[\{r\}] = \bigcap_{n=1}^{\infty} f^{-1}[(r - \frac{1}{n}, r + \frac{1}{n})]$ contains a tail $(\alpha, \omega_1]$. Therefore $f|[\alpha, \omega_1] = r$. •

Proving Corollary 8.7 was relatively easy because we can see immediately what the constant value would have to be for the theorem to be true: $r = f(\omega_1)$. A more remarkable thing is that the same result holds for $[0, \omega_1)$. But to prove that fact, we have no "initial guess" about what constant value f might have on a tail, so we have to work harder.

Theorem 8.8 If $f \in C([0, \omega_1))$, then f is constant on the "tail" (α, ω_1) for some $\alpha < \omega_1$.

Proof Let $T_\alpha = [\alpha, \omega_1) =$ "the α^{th} tail". By Corollary 8.5, $[0, \omega_1)$ is countably compact so the closed set T_α is also countably compact. It is easy to see that a continuous image of a countably compact space is countably compact, so $f[T_\alpha]$ is a countably compact subset of $\mathbb{R}$. Since countable compactness and compactness are equivalent for subsets of $\mathbb{R}$ (Theorem IV.8.17), each $f[T_\alpha]$ is a nonempty compact set: $f[T_\alpha] \subseteq f[T_0] \subseteq \mathbb{R}$.

The T_α's are nested : $f[T_\alpha] \subseteq f[T_\beta] \subseteq f[T_0]$ if $\beta < \alpha$. Therefore $f[T_\alpha]$'s have the finite intersection property, and by compactness $\bigcap_{\alpha < \omega_1} f[T_\alpha] \neq \emptyset$ (*see Theorem IV.8.4*). In fact, we claim the intersection contains a single number : $\bigcap_{\alpha < \omega_1} f[T_\alpha] = \{r\}$.

> If $r, s \in \bigcap_{\alpha < \omega_1} f[T_\alpha]$, then f assumes each of the values r, s for arbitrarily large values of α. Therefore we can pick an increasing sequence $\alpha_1 < \beta_1 < \alpha_2 < \beta_2 < ... < \alpha_n < \beta_n < ...$ such that $f(\alpha_n) = r$ and $f(\beta_n) = s$. Let $\gamma = \sup\{\alpha_n, \beta_n : n \in \mathbb{N}\} < \omega_1$. Then $(\alpha_n) \to \gamma$ and $(\beta_n) \to \gamma$. By continuity $(f(\alpha_n)) = (r) \to \gamma$ and $(f(\beta_n)) = (s) \to \gamma$. and we conclude $\gamma = r = s$.

We claim that $f \equiv r$ on some tail of $[0, \omega_1)$. First notice that if $\beta < \omega_1$, then $\exists\, \gamma_n > \beta$ for which $f[T_{\gamma_n}] \subseteq (r - \frac{1}{n}, r + \frac{1}{n})$.

> If not, then $\{\, f[T_\gamma] : \ \beta < \gamma < \omega_1 \,\} \cup \{\, f[[0, \omega_1)] - (r - \frac{1}{n}, r + \frac{1}{n}) : n \in \mathbb{N} \,\}$ would be a family of closed subsets of $f[T_0]$ with the finite intersection property, so this family would have a nonempty intersection. That is impossible since $\bigcap_{\beta < \gamma < \omega_1} f[T_\gamma] = \{r\}$ and
> $r \notin \bigcap_{n=1}^{\infty} f[[0, \omega_1)] - (r - \frac{1}{n}, r + \frac{1}{n})$.

Pick γ_1 so that $f[T_{\gamma_1}] \ \subseteq (r - 1, r + 1)$. Pick $\gamma_2 > \gamma_1$ so that $f[T_{\gamma_2}] \subseteq (r - \frac{1}{2}, r + \frac{1}{2})$ and continue inductively to pick $\gamma_{n+1} > \gamma_n$ so that $f[T_{\gamma_{n+1}}] \subseteq (r - \frac{1}{n+1}, r + \frac{1}{n+1})$.

Let $\tau = \sup\{\gamma_n : n \in \mathbb{N}\} < \omega_1$. Then $f[T_\tau] \subseteq f[\bigcap T_{\gamma_n}] \subseteq \bigcap f[T_{\gamma_n}] \subseteq \bigcap_{n=1}^{\infty} (r - \frac{1}{n}, r + \frac{1}{n}) = \{r\}$, so $f|T_\tau = r$. $\bullet$

(*Since f is bounded on the compact set $[0, \tau]$, we see in a different way that $[0, \omega_1)$ is pseudocompact.*)

Corollary 8.9 Every continuous function $f : [0, \omega_1) \to \mathbb{R}$ can be extended in a unique way to a continuous function $F : [0, \omega_1] \to [0, \omega_1]$.

Proof For some $r \in \mathbb{R}$, $f = r$ on a tail $[\alpha, \omega_1)$. Let $F|[0, \omega_1) = f$ and define $F(\omega_1) = r$.
Any continuous extension G of f must agree with F since F and G agree on the dense set $[0, \omega_1)$.$\bullet$

Note: $[0, \omega_1]$ *is a compact* T_2 *space which contains* $[0, \omega_1)$ *as a dense subspace. We call* $[0, \omega_1]$ *a* <u>*compactification*</u> *of* $[0, \omega_1)$. *The property stated in Corollary 8.9 is a very special property for a compactification to have : in fact, it characterizes* $[0, \omega_1]$ *as the so-called* <u>*Stone*</u>*-*<u>*Cech compactification*</u> *of* $[0, \omega_1)$. *We will discuss compactifications in Chapter 10.*

<u>*By way of contrast*</u>*, notice that* $[-1, 0]$ *is compactification of* $[-1, 0)$ *which, just as above, is obtained by adding a single point to the original space. However, the continuous function* $f : [-1, 0) \to \mathbb{R}$ *defined by* $f(x) = \sin(\frac{1}{x})$ <u>*cannot*</u> *be continuously extended to the point* 0.

We saw in Example VII.5.10 that a subspace of a normal space need not be normal: the Sorgenfrey plane X is not normal however it can be embedded in the T_4-space $[0, 1]^m$ for some m. Any space X that is $T_{3\frac{1}{2}}$ but not T_4 works just as well. However, these examples are not very explicit – it is hard to "picture why" the normality of $[0, 1]^m$ isn't inherited by the subspace X. The "picturing" may be easier in the following example.

Example 8.10 Let $T^* = [0, \omega_1] \times [0, \omega_0]$, sometimes called the "Tychonoff plank." T^* is compact T_2 and therefore T_4. Discarding the "upper right corner point," we are left with the (open) subspace $T = T^* - \{(\omega_1, \omega_0)\}$ We claim that T is not normal. Let

> $A = \{(\omega_1, n) \in T \ : n < \omega_0\}$ = "the right edge of T" and
> $B = \{(\alpha, \omega_0) \in T : \alpha < \omega_1\}$ = "the top edge of T"

A and B are disjoint sets and closed in T (although not, of course, in T^*).

Suppose U is an open set in T containing the "right edge" A. For each point $(\omega_1, n) \in A \subseteq U$, we can choose a basic open set $(\alpha_n, \omega_1] \times \{n\} \subseteq U$. Let $\alpha = \sup\{\alpha_n : n = 0, 1, 2...\} < \omega_1$. Then $(\alpha, \omega_1] \times \{n\} \subseteq U$ for all n – that is, A is contained in a "vertical strip" $(\alpha, \omega_1] \times [0, \omega_0)$ inside U.

Suppose V is an open set in T containing the "top edge" B. Since $(\alpha + 1, \omega_0) \in B$, there is a basic open set $\{\alpha + 1\} \times (n, \omega_0] \subseteq V$. But then $(\alpha + 1, n + 1) \in U \cap V$, so $U \cap V \neq \emptyset$.

Exercise 8.11 Show that every continuous function $f : T \to \mathbb{R}$ can be continuously extended to a function $F : T^* \to \mathbb{R}$.
Hint: $f|[0, \omega_1) \times \{\omega_0\}$ has constant value r on some tail, and for each $n < \omega_0$, $f|\,[0, \omega_1] \times \{n\}$ has a constant value r_n on a tail. Define $F((\omega_1, \omega_0)) = r$. Prove that $(r_n) \to r$ and then show that the extension F is continuous at (ω_1, ω_0).

As in the remark following Corollary 8.9, T^* is a compactification of T and the functional extension property in the exercise characterizes T^* as the so-called Stone-Cech compactification of T. Since T^* is compact, F must be bounded – so, in retrospect, f must have been bounded in the first place. Therefore T is pseudocompact. But T is not countably compact – because the "right edge" A is an infinite set that has no limit point in T. T is an example of <u>a pseudocompact space that is not countably compact</u>.

Exercises

E13. Suppose $C \subseteq \mathbb{R}$ and that C is well-ordered (in the usual order on $\mathbb{R}$). Prove that C is countable.

E14. Let ω_0^* denote the order type of the set of <u>nonpositive</u> integers, with its usual ordering.

a) Prove that a chain $(X, \leq)$ is well-ordered iff X contains no subset of order type ω_0^*.

b) Prove that if $(X, \leq)$ is a chain in which every countable subset is well-ordered, then X is well-ordered.

c) Prove that every infinite chain either has a subset of order type ω_0^* or one of order type ω_0.

E15. Prove the following facts about ordinal numbers α, β, γ :

a) if $\beta > 0$, then $\alpha + \beta > \alpha$

b) if $\alpha > \beta$, then there exists a unique γ such that $\alpha = \beta + \gamma$
We might try using b) to define subtraction of ordinals: $\alpha - \gamma = \beta$ *if* $\alpha = \beta + \gamma$. *However this is perhaps not such a good idea.* (*Consider* $\omega_0 = \alpha$, $\beta = 1$). *Problems arise because ordinal addition is not commutative.*

c) $1 + \alpha = \alpha$ iff $\alpha \geq \omega_0$.

E16. Let $B \subseteq A$, where $(A, \leq)$ be a chain. B is called <u>inductive</u> if

$$\text{for all } t \in A, \quad \{a \in A : a < t\} \subseteq B \Rightarrow t \in B.$$

Prove that if A is the only inductive subset of A, then A is well-ordered.

E17. Let X be a first countable space. Suppose that for each $\alpha < \omega_1$, F_α is a closed subset of X and that $F_{\alpha_1} \subseteq F_{\alpha_2}$ whenever $\alpha_1 \leq \alpha_2 < \omega_1$. Prove that $\bigcup\{F_\alpha : \alpha < \omega_1\}$ is closed in X.

E18. Let A be well-ordered. Order $L = A \times [0,1)$ lexicographically and give the set the order topology.

a) What does a "nice" neighborhood base look like at each point in L? Discuss some other properties of this space.

b) If $A = [0, \omega_1)$, then $L - \{(0,0)\}$ is called is the "long line." Show that L is path connected and locally homeomorphic to $\mathbb{R}$ but it cannot be embedded in $\mathbb{R}$. (See *Topology*, J. Munkres, 2nd edition, p. 159 for an outline of a proof.)

c) Each point in L homeomorphic to $\mathbb{R}$. L is normal but not metrizable.

E19. a) Let A and B be disjoint closed sets in $[0, \omega_1]$. Prove that at least one of A and B is compact and bounded away from ω_1. (*A set C is <u>bounded away from</u>* ω_1 *if* $C \subseteq [0, \alpha]$ *for some* $\alpha < \omega_1$.)

b) Characterize the closed sets in $[0, \omega_1)$ that are zero sets.

c) Prove that $[0, \omega_1)$ and $[0, \omega_1]$ are not metrizable.

E20. a) Suppose (x_n) and (y_n) are sequences in $X = [0, \omega_1)$ such that, for all n, $x_n \leq y_n \leq x_{n+1}$. Show that both sequences converge and have the same limit.

b) Show that if $f : X \to X$ is such that $f(x) \geq x$ for every x, then there is an x such that (x, x) is a limit point of the graph of f in $X \times X$.

c) Prove that $X \times [0, \omega_1]$ is not normal.

(*Hint: Let Δ be the diagonal of $X \times X$ and B = $X \times \{\omega_1\}$. Show that if U and V are open with $\Delta \subseteq U$ and $B \subseteq V$, then $U \cap V \neq \emptyset$. To do this: if any point (x, ω_1) is in U, we're done. So suppose this is false and define $f(x)$ to be the least ordinal $> x$ such that $(x, f(x)) \notin U$. Use part b).*)

E21. A space X is called σ-compact if X can be written as a countable union of compact sets.

a) Prove $[0, \omega_1)$ is not σ-compact.
b) Using part a) (or otherwise), prove $[0, \omega_1)$ is not Lindelöf.
c) State and prove a theorem of the form: an ordinal space $[0, \alpha)$ is σ-compact iff ...

(*You might begin by thinking about the spaces $[0, \omega_1)$, $[0, \omega_2)$, and $[0, \omega_{\omega_0})$.*)

E22. A space X is called <u>functionally</u> <u>countable</u> if every continuous $f\colon X \to \mathbb{R}$ has a countable range.

a) Show that $X = [0, \omega_1)$ is functionally countable.
b) Let $Y = D \cup \{p\}$ where D is uncountable and $p \notin D$. Give Y the topology for which all the points of D are isolated and for which the basic neighborhoods of p are those cocountable sets containing p. Show Y is functionally countable.
c) Prove that $X \times Y$ is not functionally countable.

(*Hint: Let $g\colon X \to D$ be one-to-one. Consider the set $H = \{(\alpha, g(\alpha) : \alpha$ is isolated in $X\} \subseteq X \times Y$*)

E23. A space X is called <u>Y-compact</u> if X is homeomorphic to a closed subspace of the product Y^m for some cardinal m. For example, X is $[0,1]$-compact iff X compact and T_2. An $\mathbb{R}$-compact space is called <u>realcompact</u>.

a) Prove that if X is both realcompact and pseudocompact, then X is compact.

(*Note: the converse is clear*)

b) Suppose that X is Tychonoff and that however X is embedded in $[0,1]^m$, its projection in every direction is compact. (More precisely, suppose that for <u>all</u> possible embeddings $h : X \to [0,1]^m$, we have that $\pi_\alpha[h[X]]$ is compact for all projections π_α. This statement is certainly true, for example, if X is compact.) Prove or disprove that X must be compact.

E24. An infinite cardinal k is called <u>sequential</u> if $k = \sum_{n=1}^{\infty} k_n$ for some sequence of cardinals $k_n < k$.

a) Prove that if k is sequential, then $k^{\aleph_0} > k$.
b) Assume GCH. Prove that if k is infinite and $k^{\aleph_0} > k$, then $k^{\aleph_0} = 2^k$.
c) Assume GCH. Prove that an infinite cardinal k is sequential iff $k^{\aleph_0} > k$.

9. Transfinite Induction and Transfinite Recursion

"...to understand recursion, you have to understand recursion...."

Suppose we have a sequence of propositions P_n that depend on $n = 0, 1, 2, ...$ We would like to show that all the P_n's are true. (*The initial value* $n = 0$ *is not important. We might want to prove the propositions* P_n *for, say,* $13 \le n < \omega_0$.) For example, we might have in mind the propositions

$$P_n : \quad 0 + 1 + 2 + ... + n = \tfrac{n(n+1)}{2}$$

As you should already know, the proof can be done by induction. Induction in elementary courses takes one of two forms (stated here using P_0 as the initial proposition):

1) (Principle of Induction) If P_0 is true and if (P_{k-1} is true $\Rightarrow$ P_k is true), then P_n is true for all $n < \omega_0$.

2) (Principle of Complete Induction) If P_0 is true and if (P_j is true for all $j < k \Rightarrow P_k$ is true), then P_n is true for all $n < \omega_0$.

Formally, 2) looks weaker than 1), because it has a stronger hypothesis. But in fact the versions 1) and 2) are equivalent statements about $[0, \omega_0)$. (*Why?*) Sometimes form 2) is more convenient to use. For example, try using both versions of induction to prove that every natural number greater than 1 has a factorization into primes.

The Principle of Induction works because $[0, \omega_0)$ is well-ordered:

If $\{n \in (0, \omega_0] : P_n$ is false$\} \neq \emptyset$, then it would contain a smallest element $k > 0$. This is impossible: since P_{k-1} is true, P_k must be true.

You might expect a principle analogous to 1) could be used in every well-ordered sets $[0, \alpha)$, not just in $[0, \omega_0)$. But an ordinal β might not have an immediate predecessor, so version 1) might not make sense. So we work instead with 2): we can generalize "ordinary induction" if we state it in the form of complete induction.

Theorem 9.1 (Principle of Transfinite Induction) Let α be an ordinal and $T \subseteq [0, \alpha)$. If

1) $0 \in T$ and
2) $\forall \beta \in [0, \alpha)$ $[\, [0, \beta) \subseteq T \Rightarrow \beta \in T\,]$,

then $T = [0, \alpha)$.

Proof If $T \neq [0, \alpha)$, then there is a smallest $\beta \in [0, \alpha) - T$. By definition of β, $[0, \beta) \subseteq T$. But then 2) implies $\beta \in T$, contrary to the choice of β.

Using Theorem 9.1 is completely analogous to using complete induction in $[0, \omega_0)$. For each $\gamma < \alpha$, we have a proposition P_γ and we want to show that all the P_γ's are all true. (*For example, we might have a set* $K_\gamma \subseteq X$ *somehow defined for each* $\gamma < \alpha$*, and* P_γ *might be the proposition "*K_γ *is compact."*) Let $T = \{\gamma < \alpha : P_\gamma \text{ is true}\} \subseteq [0, \alpha)$. If we show that P_0 is true, and if, assuming that assume P_γ is true for all $\gamma < \beta$, we can then prove that P_β must be true, then Theorem 9.1 implies that $P_{\gamma.}$ is true for all $\gamma < \alpha$. We will look at several examples in Section 10.

Note: In the statement of Theorem 9.1, part 1) is included only for emphasis. In fact, 1) is automatically true if we know 2) is true: for if we let $\beta = 0$ in 2), then $[0, 0) = \emptyset \subseteq T$ is true, so $0 \in T$ – that is, P_0 is true. But in actually <u>using</u> Theorem 9.1 (as described in the preceding paragraph) and trying to prove that 2) <u>is</u> true, the first value $\beta = 0$ requires us to show P_0 is true with "no induction assumptions" since there are <u>no</u> P_γ's with $\gamma < \beta = 0$. Doing that is just verifying that 1) is true.

We can also <u>define</u> objects in a similiar way – by transfinite <u>recursion</u>. Elementary definitions by recursion should be familiar – for example, we might say :

let $f(0) = 13$ and,
for each $n > 0$, let $f(n) = 2f(n-1)$ (**)

We than say that "f is defined for all $n = 0, 1, 2, ...$" We draw that conclusion by arguing in the following informal way: if not, then there is a smallest $k \in (0, \omega_0)$ for which f is not defined: this is impossible because then $f(k-1)$ <u>is</u> defined, and therefore (by **) so is $f(k)$.

This argument depends only on the fact that $[0, \omega_0)$ is well-ordered, so it generalizes to the following principle.

Informal Principle 9.2 (Transfinite Recursion) For each $\beta < \alpha$, suppose a rule is given that defines an object P_β in terms of objects P_γ already defined (that is, in terms of P_γ's with $\gamma < \beta$). Then P_β is defined for all $\beta < \alpha$. (*The principle implies that* P_0 *is defined "absolutely" – that is, without any previous* P_γ*'s to work with – since there are no* P_γ*'s with* < 0}.

This "informal" statement is a reasonably accurate paraphrase of a precise theorem in axiomatic set theory, and the informal proof is virtually identical to the one given above for simple recursion on $[0, \omega_0)$. As stated in 9.2, this principle is strong and clear enough for everyday use, and we will consider it "proven" and useable.

Principle 9.2 is "informal" because it does seem a little vague in spots – "some object", "a rule is given", "if P_γ is defined ... , then P_β is defined ..." Without spending a lot of time on the set theoretic issues, we digress to show how the statement can be made a little more precise.

> In axiomatic set theory, the "objects" P_β will, of course, be sets (since everything is a set). We can think of "<u>defining</u> P_β" to mean <u>choosing</u> P_β from some specified set $\mathcal{E}$. In the context of some problem, $\mathcal{E}$ is the "universal set" in which the objects P_β will all live. For example, we might have $\mathcal{E} = C(X)$ and want to define continuous functions $P_\beta \in C(X)$ for each $\beta < \alpha$.
>
> The sets P_γ already chosen ("defined") in $\mathcal{E}$ for $\gamma < \beta$ can be described efficiently by a function $\psi_\beta \in \mathcal{E}^{[0,\beta)}$: for $\gamma < \beta$, $\psi_\beta(\gamma) = P_\gamma \in \mathcal{E}$.
>
> To define the set P_β in terms of the preceding P_γ's means that we need to define P_β using ψ_β. We need a function ("rule") R_β so that $R_\beta(\psi_\beta)$ gives the new set $P_\beta \in \mathcal{E}$. In other words, we want $R_\beta : \mathcal{E}^{[0,\beta)} \to \mathcal{E}$.

The conclusion that we have "completed" the process and that P_β is defined for all $\beta < \alpha$ means that there is a function $F \in \mathcal{E}^{[0,\alpha)}$ where, for each $\beta < \alpha,\ F(\beta) =$ the P_β selected at the earlier stage by R_β – that is, $F(\boldsymbol{\beta}) = R_\beta(F|\,[0,\beta))$.

This leads us to the following formulation. The "full" formulation of the standard theorem about transfinite recursion in axiomatic set theory needs to be a little stronger still, so we call this version – which we state without proof – the "weak" version. It is more than adequate for our purposes here.

Theorem 9.3 (Transfinite Recursion, Weak Form) Suppose α is an ordinal. Let $\mathcal{E}$ be a set and suppose that, for each $\beta < \alpha$, we have a function $R_\beta : \mathcal{E}^{[0,\beta)} \rightarrow \mathcal{E}$. Then there exists a unique function $F \in \mathcal{E}^{[0,\alpha)}$ such that, for each $\beta < \alpha,\ F(\beta) = R_\beta(F|\,[0,\beta))$.

Proof See, for example, *Topology (J. Dugundji)*

The following example illustrates what the Recursion Theorem 9.3 in a concrete example. When all is said and done, it looks just the way an informal, simple definition by recursion (Principle 9.2) should look.

Example 9.4 We want to define numbers E_n for every $n = 0, 1, \ldots$. Informally we might say:

Let $E_0 = 1$ and for $n > 0$, let $E_n = {E_0}^2 + \ldots + E_{n-1}^2$.

The informal Principle 9.2 lets us conclude that E_n is defined for all $n < \omega_0$.

In terms of the more formal Theorem 9.3, we can describe what is "really" happening as follows:

Let $[0,\alpha) = [0,\omega_0)$ and $\mathcal{E} = \mathbb{N}$. For each $n < \omega_0$, define $R_n : \mathbb{N}^{[0,n)} \rightarrow \mathbb{N}$ as follows:

For $n = 0$: $\mathbb{N}^{[0,0)} = \mathbb{N}^{\emptyset} = \{\emptyset\}$ and define $R_0(\emptyset) = 1$
For $n > 0$: if $\psi \in \mathbb{N}^{[0,n)}$, define $R_n(\psi) = \psi^2(0) + \ldots + \psi^2(n-1)$

(*Note that the R_n's are defined explicitly for each n, not recursively.*).

The theorem states that there is a unique function $F \in \mathbb{N}^{[0,\omega_0)}$ such that

$$
\begin{aligned}
F(0) &= R_0(\,F|[0,0)\,) \;=\; R_0(\emptyset) = 1 \\
F(1) &= R_1(F|[0,1)\,) \;=\; F(0)^2 \;=\; 1^2 \;=\; 1 \\
F(2) &= R_2(F\,|[0,2)\,) \;=\; F(0)^2 \;+\; F(1)^2 \;=\; 2
\end{aligned}
$$

... etc. ...

which is just what we wanted: dom $F = [0,\omega_0)$, so $E_n = F(n)$ is defined for all $n < \omega_0$.

10. Using Transfinite Induction and Recursion

This section presents a number of examples using recursion and induction in an essential way. Taken together, they are a miscellaneous collection, but each example has some interest in itself.

Borel Sets in Metric Spaces

The classical theory of Borel sets is developed in metric spaces (X, d). The collection of Borel sets in a metric space is important in analysis and also in set theory. Roughly, Borel sets are the sets that can be generated from open sets by the operations of countable union and countable intersection "applied countably many times." Therefore a Borel set is only a "small" number of operations "beyond" the open sets, and Borel sets are fairly well behaved. We will use transfinite recursion (the informal version) to define the Borel sets and prove a few simple theorems. When objects are defined by recursion, proofs about them often involve induction.

We begin with a simple lemma about ordinals.

Lemma 10.1 Every ordinal α can be written uniquely in the form $\alpha = \beta + n$ where β is either 0 or a limit ordinal, and $n < \omega_0$.

Proof If α is finite, then $\alpha = 0 + n$ where $n < \omega_0$.

Suppose α is infinite. If α is a limit ordinal, we can write $\alpha = \alpha + 0$. If α is not a limit ordinal, then α has an immediate predecessor which, for short, we denote here as "$\alpha - 1$." If $\alpha - 1$ is not a limit ordinal, then it has an immediate predecessor $\alpha - 2$. Continuing in this way, we get to a limit ordinal after a finite number of steps, n – for otherwise $\alpha - 1 > \alpha - 2 > ... > \alpha - n > ...$ would be an infinite decreasing sequence of ordinals. If $\beta = \alpha - n$ is a limit ordinal, then $\alpha = \beta + n$.

To prove uniqueness, suppose $\alpha = \beta + n = \beta' + n'$ where each of β, β' is 0 or a limit ordinal and n, n' are finite. If β or $\beta' = 0$: say $\beta = 0$. Then $\beta + n$ is finite, so $\beta' = 0$ and $n = n'$. So suppose β and β' are both limit ordinals. We have an order isomorphism f from $[0, \beta + n)$ onto $[0, \beta' + n')$. Since $[0, \beta + n)$ contains $n - 1$ ordinals after its largest limit ordinal β, the same must be true in the range $[0, \beta' + n')$, and therefore $n = n'$. Let $g = f|[0, \beta)$. Then $g : [0, \beta) \to [0, \beta')$ is an order isomorphism, so $\beta = \beta'$. •

Definition 10.2 Suppose $\alpha = \beta + n$, where β is a limit ordinal or 0, and n is finite. We say that α is <u>even</u> if n is even, and that α is <u>odd</u> if n is odd.

For example, every limit ordinal $\alpha = \alpha + 0$ is even.

Definition 10.3 Suppose (X, d) is a metric space. Let $\mathcal{G}_0 = \mathcal{T}_d$, the collection of open sets. For each $0 < \alpha < \omega_1$, and suppose that $\mathcal{G}_\beta$ has been defined for all $\beta < \alpha$. Then let

$$\begin{cases} \mathcal{G}_\alpha = \{G : G \text{ is a countable intersection of sets from } \bigcup\{\mathcal{G}_\beta : \beta < \alpha\} & \text{(if } \alpha \text{ is odd)} \\ \mathcal{G}_\alpha = \{G : G \text{ is a countable union of sets from } \bigcup\{\mathcal{G}_\beta : \beta < \alpha\} & \text{(if } \alpha \text{ is even)} \end{cases}$$

$\mathcal{B} = \bigcup\{\mathcal{G}_\alpha : \alpha < \omega_1\}$ is the family of <u>Borel sets</u> in (X, d).

The sets in $\mathcal{G}_1$ are the G_δ sets; the sets in $\mathcal{G}_2$ are countable unions of G_δ sets and are traditionally called $G_{\delta\sigma}$ sets; the sets in $\mathcal{G}_3$ are called $G_{\delta\sigma\delta}$-sets, etc.

Theorem 10.4 Suppose (X, d) is a metric space.

1) If $\alpha < \beta < \omega_1$, then $\mathcal{G}_\alpha \subseteq \mathcal{G}_\beta$, so

$$\mathcal{G}_0 \subseteq \mathcal{G}_1 \subseteq \mathcal{G}_2 \subseteq ... \subseteq \mathcal{G}_\alpha \subseteq ... \subseteq \mathcal{G}_\beta \subseteq ... \qquad (\alpha < \beta < \omega_1)$$

2) $\mathcal{G}_\alpha$ is closed under countable unions if α is even, and $\mathcal{G}_\alpha$ is closed under countable intersections if α is odd.

3) $\mathcal{B}$ is closed under countable unions, intersections and complements. Also, if B_1 and B_2 are in $\mathcal{B}$, so is $B_1 - B_2$.

Proof 1) Suppose $\alpha < \beta < \omega_1$. $\mathcal{G}_\beta$ is defined as the collection of all countable unions (if β is even) or intersections (if is odd) of sets in the preceding families $\mathcal{G}_\alpha$ $(\alpha < \beta)$. In particular, any one set from $\mathcal{G}_\alpha$ is in $\mathcal{G}_\beta$.

2) Suppose α is even and $G_1, G_2, ..., G_n, ... \in \mathcal{G}_\alpha$. Each G_n is a countable union of sets from $\bigcup\{\mathcal{G}_\beta : \beta < \alpha\}$. Therefore $\bigcup_{n=1}^\infty G_n$ is also a countable union of sets from $\bigcup\{\mathcal{G}_\beta : \beta < \alpha\}$, so $\bigcup_{n=1}^\infty G_n \in \mathcal{G}_\alpha$. The proof is similar if α is odd.

3) Suppose $B_1, B_2, ..., B_n, ... \in \mathcal{B}$. For each n, $B_n \in \mathcal{G}_{\alpha_n}$ for some $\alpha_n < \omega_1$. If $\alpha = \sup\{\alpha_n : n = 1, 2, ...\} < \omega_1$, then $\mathcal{G}_{\alpha_n} \subseteq \mathcal{G}_\alpha \subseteq \mathcal{G}_{\alpha+1}$ for every n. By part 2), one of the collections $\mathcal{G}_\alpha$ or $\mathcal{G}_{\alpha+1}$ is closed under countable intersections and the other under countable unions. Therefore $\bigcup_{n=1}^\infty B_n$ and $\bigcap_{n=1}^\infty B_n$ are both in $\mathcal{G}_{\alpha+1} \subseteq \mathcal{B}$.

To show that $\mathcal{B}$ is closed under complements, we first prove, using transfinite induction, that if $G \in \mathcal{G}_\alpha$, then $X - G \in \mathcal{G}_{\alpha+1}$.

$\alpha = 0$: If $G \in \mathcal{G}_0$, then G is open so $X - G$ is closed. But a closed set in a metric space is a G_δ set, so $X - G \in \mathcal{G}_1$.

Suppose the conclusion holds for all $\beta < \alpha < \omega_1$. We must show it holds also for $\mathcal{G}_\alpha$.

Let $G \in \mathcal{G}_\alpha$. If α is odd, then $G = \bigcap_{n=1}^\infty G_n$, where $G_n \in \mathcal{G}_{\beta_n}$ $(\beta_n < \alpha)$. By the induction hypothesis, $X - G_n \subseteq \mathcal{G}_{\beta_n+1} \subseteq \mathcal{G}_{\alpha+1}$ for all n. Since $\alpha + 1$ is even, we have that $\bigcup_{n=1}^\infty (X - G_n) = X - \bigcap_{n=1}^\infty G_n = X - G \in \mathcal{G}_{\alpha+1}$. (*The case when α is even is similar*).

If $B_2 \in \mathcal{B}$, then $B_2 \in \mathcal{G}_\alpha$ for some $\alpha < \omega_1$, so $X - B_2 \in \mathcal{G}_{\alpha+1} \subseteq \mathcal{B}$. So if $B_1 \in \mathcal{B}$, then $B_1 \cap (X - B_2) = B_1 - B_2 \in \mathcal{B}$. •

Part 3) of the preceding theorem shows why the definition of the Borel sets only uses ordinals $\alpha < \omega_1$. Once we get to $\mathcal{B} = \bigcup\{\mathcal{G}_\alpha : \alpha < \omega_1\}$, the process "closes off" $-$ that is, continuing with additional countable unions and intersections produces no new sets.

Definition 10.3 presents the construction of the Borel sets "from the bottom up." It has the advantage of exhibiting how the sets in $\mathcal{B}$ are constructed step by step. However, it is also possible to define $\mathcal{B}$ "from the top down." This approach is neater, but it gives less insight into which sets are Borel.

Definition 10.5 A family $\mathfrak{S}$ of subsets of X is called a <u>σ-algebra</u> if $X \in \mathfrak{S}$ and $\mathfrak{S}$ is closed under complements, countable intersections, and countable unions.

Suppose $\mathcal{A}$ is a collection of subsets of X. Then $\mathcal{P}(X)$ is (the largest) σ-algebra containing $\mathcal{A}$. It is also clear that the intersection of a collection of σ-algebras is a σ-algebra. Therefore the smallest σ-algebra containing $\mathcal{A}$ exists: it is the intersection of all σ-algebras containing $\mathcal{A}$.

The following theorem could be taken as the definition of the family of Borel sets.

Theorem 10.6 The family $\mathcal{B}$ of Borel sets in (X, d) is the smallest σ-algebra containing all the open sets of X.

Proof The rough idea is that our previous construction puts into $\mathcal{B}$ all the sets that need to be there to form a σ-algebra, but no others.

We have already proven that $\mathcal{B}$ <u>is</u> a σ-algebra containing the open sets. We must show $\mathcal{B}$ is the smallest – that is, if $\mathcal{B}'$ is a σ-algebra containing the open sets, then $\mathcal{B} \subseteq \mathcal{B}'$. We show this by transfinite induction.

We are given that $\mathcal{G}_0 \subseteq \mathcal{B}'$.

Suppose that $\mathcal{G}_\beta \subseteq \mathcal{B}'$ for all $\beta < \alpha < \omega_1$. We must show $\mathcal{G}_\alpha \subseteq \mathcal{B}'$.

Assume α is odd. If $G \in \mathcal{G}_\alpha$ then $G = \bigcap_{n=1}^{\infty} G_n$ where $G_n \in \mathcal{G}_{\beta_n}$ for some $\beta_n < \alpha$. By hypothesis each $\mathcal{G}_{\beta_n} \in \mathcal{B}'$ and $\mathcal{B}'$ is closed under countable intersections so $G \in \mathcal{B}'$. Therefore $\mathcal{G}_\alpha \subseteq \mathcal{B}'$. (*The case when α is even is entirely similar.*)

Since $\mathcal{G}_\alpha \subseteq \mathcal{B}'$ for all $\alpha < \omega_1$, we get that $\mathcal{B} = \bigcup_{\alpha < \omega_1} \mathcal{G}_\alpha \subseteq \mathcal{B}'$. •

The next theorem gives us an upper bound for the number of Borel sets in a separable metric space.

Theorem 10.7 If (X, d) is separable metric space, then $|\mathcal{B}| \leq c$.

Proof We prove first that for each $\alpha < \omega_1$, $|\mathcal{G}_\alpha| \leq c$.

$\alpha = 0$: A separable metric space has a countable base $\mathcal{C}$ for the open sets. Since every open set is the union of a subfamily of $\mathcal{C}$, we have $|\mathcal{T}_d| = |\mathcal{G}_0| \leq |\mathcal{P}(\mathcal{C})| \leq 2^{\aleph_0} = c$.

Assume that $|\mathcal{G}_\beta| \leq c$ for all $\beta < \alpha < \omega_1$. Since α has only countably many predecessors, $|\bigcup_{\beta<\alpha} \mathcal{G}_\beta| \leq c$. Since each set in $\mathcal{G}_\alpha$ is a countable intersection or union of a <u>sequence</u> of sets from $\bigcup_{\beta<\alpha} \mathcal{G}_\beta$, we have $|\mathcal{G}_\alpha| \leq |\bigcup_{\beta<\alpha} \mathcal{G}_\beta|^{\mathbb{N}} \leq c^{\aleph_0} = c$.

Therefore $|\mathcal{B}| = |\bigcup_{\alpha<\omega_1} \mathcal{G}_\alpha| \leq \aleph_1 \cdot c = c$. •

(*Can you see a generalization to arbitrary metric spaces?*)

Corollary 10.8 There are non-Borel sets in $\mathbb{R}$.

For those who know a bit of measure theory: every Borel set in $\mathbb{R}$ is Lebesgue measurable. Since a subset of a set of measure 0 is measurable, all 2^c subsets of the Cantor set C are measurable. Therefore there are Lebesgue measurable subsets of C that are not Borel sets.

Example 10.9

1) If (X, d) is discrete, then every subset is open, so every subset is Borel: $\mathcal{B} = T_d = \mathcal{P}(X)$.

2) If $X = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$, then every subset is a G_δ, but X is not discrete. Therefore

$$\mathcal{G}_0 \underset{\neq}{\subseteq} \mathcal{G}_1 = \mathcal{G}_2 = \ldots = \mathcal{G}\alpha = \ldots = \mathcal{B}.$$

3) The following facts are true but harder to prove:

a) For each $\alpha < \omega_1$, there exists a metric space (X, d) for which

$$\mathcal{G}_0 \underset{\neq}{\subseteq} \mathcal{G}_1 \underset{\neq}{\subseteq} \ldots \underset{\neq}{\subseteq} \mathcal{G}_\alpha = \mathcal{G}_{\alpha+1} = \ldots = \mathcal{G}_\beta = \ldots = \mathcal{B}$$

In other words, the Borel construction continually adds fresh sets until the α^{th} stage but not thereafter.

b) In $\mathbb{R}$, $\mathcal{G}_\alpha \neq \mathcal{G}_\beta$ for all $\alpha < \beta < \omega_1)$ $-$ that is, new Borel sets appear at every stage in the construction.

<u>A New Characterization of Normality</u>

Definition 10.10 A family $\mathcal{F}$ of subsets of a space X is called <u>point-finite</u> if $\forall x \in X$, x is in only finitely many sets from $\mathcal{F}$.

Definition 10.11 An open cover $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ of X is called <u>shrinkable</u> if there exists an open cover $\mathcal{V} = \{V_\alpha : \alpha \in A\}$ of X such that, for each α, cl $V_\alpha \subseteq U_\alpha$. $\mathcal{V}$ is called a <u>shrinkage</u> of $\mathcal{U}$.

Theorem 10.12 X is normal iff every point-finite open cover of X is shrinkable. (*In particular, every finite open cover of a normal space is shrinkable: see Exercise VII.E18.*)

Proof Suppose every point-finite open cover of X is shrinkable and let A and B be disjoint closed sets in X. The open cover $\mathcal{U} = \{X - A, X - B\}$ has a shrinkage $\mathcal{V} = \{V_1, V_2\}$ and the sets $U = X -$ cl V_1 and $V = X -$ cl V_2 are disjoint open sets containing A and B. Therefore X is normal.

Conversely, suppose X is normal and let $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ be a point-finite open cover of X. Without loss of generality, we may assume that the index set A is a segment of ordinals $[0, \gamma)$ (*why?*) so that $\mathcal{U} = \{U_\alpha : \alpha < \gamma\}$.

Let $F_0 = X - \bigcup_{\alpha > 0} U_\alpha$. Since $F_0 \subseteq U_0$, we can use normality to choose an open set V_0 such that $F_0 \subseteq V_0 \subseteq \text{cl}\, V_0 \subseteq U_0$ and $\{V_0\} \cup \{U_\alpha : \alpha > 0\}$ covers X.

Suppose $0 < \alpha < \gamma$ and that for all $\beta < \alpha$ we have defined an open sets V_β such that $\text{cl}\, V_\beta \subseteq U_\beta$ and such that $\{V_\beta : \beta < \alpha\} \cup \{U_\beta : \beta \geq \alpha\}$ covers X. Letting $F_\alpha = X - (\bigcup_{\beta < \alpha} V_\beta \cup \bigcup_{\beta > \alpha} U_\beta) \subseteq U_\alpha$, we can use normality to choose an open set V_α with $F_\alpha \subseteq V_\alpha \subseteq \text{cl}\, V_\alpha \subseteq U_\alpha$.

Clearly, $\{V_\beta : \beta < \alpha + 1\} \cup \{U_\beta : \beta \geq \alpha + 1\}$ covers X.

By transfinite recursion, the V_α's are defined for all $\alpha < \gamma$, and we claim that $\mathcal{V} = \{V_\alpha : \alpha < \gamma\}$ is a cover of X.

> Notice that there <ins>is</ins> something here that needs to be checked: we know that we have a cover $\{V_\beta : \beta < \alpha\} \cup \{U_\beta : \beta \geq \alpha\}$ at each step in the process, but do we still have a cover when we're finished? To see explicitly that there <ins>is</ins> an issue, consider the following example.
>
> Let X be the set of reals with the "left-ray" topology (a normal space) and consider the open cover $\mathcal{U} = \{(-\infty, n) : n < \omega_0\}$. If we go through the procedure described above, we get $F_0 = X - \bigcup_{n > 0}(-\infty, n) = \emptyset$ so we might have chosen $V_0 = \emptyset$: that would give $F_0 \subseteq V_0 \subseteq \text{cl}\, V_0 \subseteq U_0$ and $\{V_0\} \cup \{U_n : n > 0\}$ would still be a cover. Continuing, we can see that at every stage we could choose $V_n = \emptyset$ and that $\{V_0, V_1, ...V_n\} \cup \{U_k : k > n\}$ is still a cover. But when we're done, the collection $\mathcal{V} = \{V_n : n < \omega_0\}$ is not a cover! Of course, the cover $\mathcal{U}$ is <ins>not point-finite</ins>.

Suppose $x \in X$. Then x is in only finitely many sets of $\mathcal{U}$ – say $U_{\alpha_1}, ..., U_{\alpha_n}$. Let α be the <ins>largest</ins> of these indices so that $x \notin \bigcup_{\beta > \alpha} U_\beta$. If x is in one of the V_β's with $\beta < \alpha$, we're done. Otherwise $x \in X - (\bigcup_{\beta < \alpha} V_\beta \cup \bigcup_{\beta > \alpha} U_\beta) = F_\alpha \subseteq V_\alpha$. Either way x is in a set in $\mathcal{V}$, so $\mathcal{V}$ is a cover. •

Question: what happens if "point-finite" is changed to "point-countable" in the hypothesis? Could the "max" in the argument be replaced by a "sup" ?

Definition 10.13 A cover of X is called <ins>locally</ins> <ins>finite</ins> if every point $x \in X$ has a neighborhood that intersects at most finitely many of the sets in the cover.

Corollary 10.14 Suppose $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ is a locally finite open cover of a normal space X. Then there exist continuous functions $f_\alpha: X \to [0,1]$ such that

i) $f_\alpha \mid X - U_\alpha = 0$ for every $\alpha \in A$
ii) $\sum\{f_\alpha(x) : \alpha \in A\} = 1$ for every $x \in X$.

The collection of functions $\{f_\alpha\}$ is called a <ins>partition</ins> <ins>of</ins> <ins>unity</ins> <ins>subordinate</ins> <ins>to</ins> $\mathcal{U}$.

Proof It is easy to see that a locally finite open cover of X is point-finite. By Theorem 10.12, $\mathcal{U}$ has a shrinkage $\mathcal{V} = \{V_\alpha : \alpha \in A\}$. For each α, we can use Urysohn's Lemma (VII.5.2) to pick a continuous function $g_\alpha : X \to [0,1]$ such that $g_\alpha | \text{cl}\, V_\alpha = 1$ and $g_\alpha | X - U_\alpha = 0$.

Each point x is in only finitely many U_α's, so $f_\alpha(x) = 0$ for all but finitely many α's and therefore $g(x) = \sum_{\alpha \in A} g_\alpha(x) \in \mathbb{R}$. Each point x has a neighborhood N_x which intersects only finitely many U_α's so $g|N_x$ is essentially a finite sum of continuous functions and g is continuous (see Exercise III.E21).

Since each x is in some set $\operatorname{cl} V_{\alpha_0}$, so $g_{\alpha_0}(x) = 1$. Therefore $g(x)$ is never 0 so each $f_\alpha(x) = \frac{g_\alpha(x)}{g(x)}$ is continuous. The f_α's clearly satisfy both i) and ii). •

A Characterization of Countable Compact Metric Spaces

In 1920, the journal *Fundamenta Mathematicae* was founded by Zygmund Janiszewski, Stefan Mazurkiewicz and Waclaw Sierpinski. It was a conscious attempt to raise the profile of Polish mathematics thorough a journal devoted primarily to the exciting new field of topology. To reach the international community, it was agreed that published articles would be in one of the most popular scientific languages of the day: French, German or English. *Fundamenta Mathematicae* continues today as a leading mathematical journal with a scope broadened somewhat to cover set theory, mathematical logic and foundations of mathematics, topology and its interactions with algebra, and dynamical systems.

An article by Sierpinski and Mazurkiewicz appeared in the very first volume of this journal characterizing compact, countable metric spaces in a rather vivid way. We will prove only part of this result: our primary purpose here is just to illustrate the use of transfinite recursion and induction and the omitted details are messy.

Suppose (X, d) is a nonempty compact, countable metric space.

Definition 10.15 For $A \subseteq X$, define $A' = \{x \in X : x \text{ is a limit point of } A\}$. A' is called the derived set of A. If A is closed, it is easy to see that $A' \subseteq A$ and that A' is also closed.

We will use the derived set operation $(\,'\,)$ repeatedly in a definition by transfinite recursion.

Let $A_0 = X$ and, for each $\alpha < \omega_1$, define

$$\begin{cases} A_\alpha = A_\beta' & \text{if } \alpha = \beta + 1 \\ A_\alpha = \bigcap\{A_\beta : \beta < \alpha\} & \text{if } \alpha \text{ is a limit ordinal} \end{cases}$$

A_α is closed for all $\alpha < \omega_1$ and $A_0 \supseteq A_1 \supseteq \ldots \supseteq A_\alpha \supseteq \ldots$. This sequence is called the derived sequence of X.

For some $\alpha < \omega_1$, we have $A_{\alpha+1} = A_\alpha$ – because otherwise, for each α, we could choose a point $x_\alpha \in A_\alpha - A_{\alpha+1}$ and $\{x_\alpha : \alpha < \omega_1\}$ would be a subset of X with cardinal $\aleph_1$. Let α_0 be the smallest α for which $A_{\alpha_0} = A_{\alpha_0+1}$. It follows that $A_\beta = A_{\alpha_0}$ for all $\beta > \alpha_0$.

Since X is compact metric, the closed set A_{α_0} is complete, and because $A_{\alpha_0} = A_{\alpha_0+1}$, every point in A_{α_0} is a limit point. Therefore $A_{\alpha_0} = \emptyset$, since a nonempty complete metric space with no isolated points contains at least c points (Theorem IV.3.6).

We know $\alpha_0 = 0$ is impossible (because that would mean $A_{\alpha_0} = A_0 = X = \emptyset$). If α_0 were a limit ordinal, then $A_{\alpha_0} = \bigcap\{A_\beta \colon \beta < \alpha_0\} \neq \emptyset$ (because $\{A_\beta : \beta < \alpha_0\}$ is a family of nonempty compact sets with the finite intersection property). Therefore α_0 must have an immediate predecessor β_0.

The definition of α_0 implies that $A_{\beta_0} \neq \emptyset$. In fact, A_{β_0} must be finite (if it were an infinite set in the compact space X, A_{β_0}would have a limit point and then $A'_{\beta_0} = A_{\alpha_0} \neq \emptyset$). Let $n = |A_{\beta_0}| < \omega_0$.

In this way, we arrive at a pair (β_0, n), where

A_{β_0} is the last nonempty derived set of X $(\beta_0 < \omega_1)$, and
A_{β_0} contains n points $(0 < n < \omega_0)$.

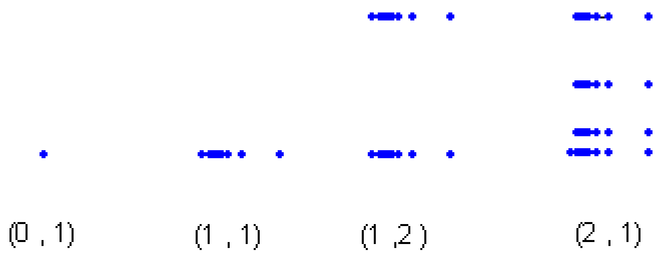
Since homeomorphisms preserve limit points and intersections, it is clear that the construction in any space homeomorphic to X will produce the same pair (β_0, n).

Theorem 10.16 (Sierpinski-Mazurkiewicz) Let $\beta_0 < \omega_1$ and $n < \omega_0$. Two nonempty compact, countable metric spaces X and Y are homeomorphic iff they are associated with the same pair (β_0, n). (*Therefore* (β_0, n) *is a "topological invariant" that characterizes nonempty compact countable metric spaces*.) For any such pair (β_0, n), there exists a nonempty compact countable metric space associated with this pair.

Corollary 10.17 There are exactly $\aleph_1$ nonhomeomorphic compact countable metric spaces.

Proof The number of different compact countable metric spaces is the same as the number of pairs (β_0, n), namely $\aleph_1 \cdot \aleph_0 = \aleph_1$. •

Example 10.18 In the following figure, each column contains a compact countable subspace of $\mathbb{R}^1$ with the invariant pair listed. Each figure is built up using sets order-isomorphic to $\{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$.



Alexandroff's Problem

A compact metric space (X, d) is second countable and therefore satisfies $|X| \leq c$ (Theorem II.5.21). In 1923, Alexandroff and Urysohn conjectured that a stronger result is true:

A first countable compact Hausdorff space X *satisfies* $|X| \leq c$.

This conjecture was not settled until 1969, in a rather famously complicated proof by Arhangel'skii.

Here is a proof of an even stronger result $-$ "compact" is replaced by "Lindelöf" $-$ that comes from a few years after Arhangel'skii's work.

Theorem 10.16 (**Pol, Šapirovski**) If X is first countable, Hausdorff and Lindelöf, then $|X| \leq c$.

Proof For each $p \in X$, choose a countable open neighborhood base $\mathcal{V}_p$ at p and, for each $A \subseteq X$, let $\mathcal{V}_A = \bigcup\{\mathcal{V}_p : p \in A\} =$ the collection of all the basic neighborhoods of all the points in A.

For each countable family of sets $\mathcal{V} \subseteq \mathcal{V}_A$ for which $X - \bigcup \mathcal{V} \neq \emptyset$, pick a point $q_{\mathcal{V}} \in X - \bigcup \mathcal{V}$. Define $P(A) = \text{cl}\,(A \cup \{\text{all such } q_{\mathcal{V}}\text{'s chosen for } A\})$.

Notice that if $|A| = c$, then $|P(A)| = c$.

Since $|A| = c$, we have $|\mathcal{V}_A| = c$. There are at most $c^{\aleph_0} = c$ countable families $\mathcal{V} \subseteq \mathcal{V}_A,$ so $|A \cup \{$all such $q_{\mathcal{V}}$'s chosen for $A\}| = c$. Since X is first countable, sequences suffice to describe the topology, and since X is Hausdorff, sequential limits are unique. Therefore $|P(A)|$ is no larger than the number of sequences in $A \cup \{$all such $q_{\mathcal{V}}$'s chosen for $A\}$ – namely $c^{\aleph_0} = c$.

Now fix, once and for all, a set $A \subseteq X$ with $|A| = c$. (*If no such A exists, we are done!*) By recursion, we now "build up" some new sets A_α from A. The idea is that the new A_α's always have cardinality $\leq c$ and that the new sets eventually include all the points of X.

Let $A_0 = A$.
For each ordinal $\alpha < \omega_1$, define

$$A_\alpha = \begin{cases} P(A_\beta) & \text{if } \alpha = \beta + 1 \\ \bigcup\{A_\beta \colon \beta < \alpha\} & \text{if } \alpha \text{ is a limit ordinal} \end{cases}.$$

For each $\alpha < \omega_1,\ |A_\alpha| = c$:

$|A_0| = c$.
Suppose $|A_\beta| = c$ for all $\beta < \alpha$.

If $\alpha = \beta + 1$, then $A_\alpha = P(A_\beta)$, so $|A_\alpha| = c$.
If α is a limit ordinal, then $|A_\alpha| = |\bigcup\{A_\beta :\ \beta < \alpha\}| = c \cdot \aleph_0 = c$.

Let $B = \bigcup\{A_\alpha : \alpha < \omega_1\}$. Then $c \leq |B| = |\bigcup\{A_\alpha : \alpha < \omega_1\}| \leq c \cdot \aleph_1 = c$.

We claim that $B = X$ and, if so, we are done.

B is closed in X: if $x \in \text{cl}\, B$, then (using first countability) there is a sequence (x_n) in B with $(x_n) \to x$. If $x_n \in A_{\alpha_n}$ and we let $\alpha = \sup\{\alpha_n\} < \omega_1$, then every x_n is in A_α. Therefore $x \in \text{cl}\, A_\alpha = A_{\alpha+1} \subseteq B$.

Since B is a closed subspace of the Lindelöf space X, B is Lindelöf.

If $B \neq X,$ then we can pick a point $q \in X - B$. For each $p \in B$, choose an open neighborhood $V_p \in \mathcal{V}_p$ such that $q \notin V_p$. The V_p's form an open cover of B, so a countable collection of these sets, say $\mathcal{V} = \{V_p : p \in C$, for some countable $C \subseteq B\}$ covers B. Thus $B \subseteq \bigcup\mathcal{V}$ and $q \notin \bigcup\mathcal{V}$. But this is impossible:

Since C is countable, we have that $C \subseteq A_\alpha$ for some $\alpha < \omega_1$ and $\mathcal{V}$ is a countable subfamily of $\mathcal{V}_{A_\alpha}$ for which $X - \bigcup\mathcal{V} \neq \emptyset$. By definition of $P(A_\alpha) = A_{\alpha+1}$), a point $q_{\mathcal{V}} \in X - \bigcup\mathcal{V}$ was put into the set $A_{\alpha+1} \subseteq B$. This contradicts the fact that $\mathcal{V}$ covers B. ●

The Mazurkiewicz 2-set

The circle S^1 is a subset of the plane which intersects every straight line in at most two points. We use recursion to construct something more bizarre.

Theorem 10.17 (Mazurkiewicz) There exists a set $A \subseteq \mathbb{R}^2$ such that $|A \cap L| = 2$ for every straight line L.

Proof Let δ be the first ordinal of cardinal c. Since the well-ordered segment $[0, \delta)$ represents δ, there are c ordinals $\alpha < \delta$.

In fact, there are c limit ordinals $< \delta$. There are certainly infinitely many (say m) limit ordinals $< \delta$ (*why?*) and, by Lemma 10.1, every ordinal $\alpha < \delta$ can be written uniquely in the form $\alpha = \beta + n$, where β is a limit ordinal (or 0) and n is finite. Therefore there are $m \cdot \aleph_0 = m$ ordinals $< \delta$. But δ has c predecessors. Therefore $m = c$.

Since $\mathbb{R}^2$ has exactly c points and exactly c straight lines, we can index both $\mathbb{R}^2$ and the set of straight lines using the ordinals less than δ: $\mathbb{R}^2 = \{p_\xi : \xi < \delta\}$ and $\{L_\xi : \xi < \delta\}$.

We will define points a_α for each $\alpha < \delta$, and the set $A = \{a_\alpha : \alpha < \delta\}$ will be the set we want.

Let a_0 be the first point of $\mathbb{R}^2$ (as indexed above) not on L_0.

Suppose that we have defined points a_ξ for all $\xi < \alpha < \delta$. We need to define a_α. Let

$A_\alpha = \{a_\xi : \xi < \alpha\}$ *Note* $|A_\alpha| < c$
$T_\alpha = \{L : L$ is a straight line containing ≥ 2 points of $A_\alpha\}$ *Note* $|T_\alpha| < c$
$\beta_\alpha =$ least ordinal so that $L_{\beta_\alpha} \notin T_\alpha$
(that is, L_{β_α} is the first line listed that is not in T_α)
$S_\alpha = \{p : p$ is a point of intersection of L_{β_α} with a line in $T_\alpha\}$ *Note* $|S_\alpha| < c$

Since $|A_\alpha \cup S_\alpha| < c$, there are points on L_{β_α} not in $A_\alpha \cup S_\alpha$: let a_α be the first p_ξ listed that is on L_{β_α} but not in $A_\alpha \cup S_\alpha$.

By recursion, we have now defined a_α for all $\alpha < \delta$. Let $A = \{a_\alpha : \alpha < \delta\}$.

We claim that for each straight line L, $|A \cap L| \leq 2$.

If $|A \cap L| > 2$, then we could pick 3 points $a_\beta, a_\gamma, a_\alpha \in L \cap A$, where, say, $\beta < \gamma < \alpha$. Then $a_\beta, a_\gamma \in A_\alpha$, so that $L \in T_\alpha$. Since $a_\alpha \in L_{\beta_\alpha}$(by definition) and $a_\alpha \in L$, we have $a_\alpha \in L \cap L_{\beta_\alpha}$. Therefore $a_\alpha \in S_\alpha$ – which contradicts the definition of a_α.

To complete the proof, we will show that for each straight line L , $|A \cap L| \geq 2$.

We begin with a series of observations:

a) If $\alpha_1 \leq \alpha_2 < \delta$, then $\beta_{\alpha_1} \leq \beta_{\alpha_2}$. [Clearly $T_{\alpha_1} \subseteq T_{\alpha_2}$. But $L_{\beta_{\alpha_1}}$is the first line not in T_{α_1} and $L_{\beta_{\alpha_2}}$is the first line not in T_{α_2}, so $\beta_{\alpha_1} \leq \beta_{\alpha_2}$.]

b) If $\alpha_1 < \alpha_2 < \alpha_3 < \delta$, then $\beta_{\alpha_1} < \beta_{\alpha_3}$. [Otherwise, by a), $\beta_{\alpha_1} = \beta_{\alpha_2} = \beta_{\alpha_3}$. Then $a_{\alpha_1}, a_{\alpha_2}, a_{\alpha_3}$ are on $L_{\beta_{\alpha_1}} = L_{\beta_{\alpha_2}} = L_{\beta_{\alpha_3}}$. Then $L_{\beta_{\alpha_3}}$contains 3 points of A which is impossible.] In particular, if α_1, α_3 are distinct limit ordinals, there is a third ordinal α_2 between them so β_{α_1} and β_{α_3} must be distinct.

c) For any $\gamma < \delta$, there is $\alpha < \delta$ such that $\beta_\alpha > \gamma$. [Since there are c limit ordinals $< \delta$, there are c distinct values for β_α. They cannot all be $\leq \gamma$, since γ has fewer than c predecessors.]

Finally, if $L =$ some L_γ, pick an α so that $\beta_\alpha > \gamma$. Since L_{β_α} is the first line $\notin T_\alpha$, we get that $L = L_\gamma \in T_\alpha$. Therefore L contains ≥ 2 points of $A_\alpha \subseteq A$. •

More generally, it can be shown more that if for each line L we are given a cardinal number m_L with $2 \leq m_L \leq c$, then there exists a set $A \subseteq \mathbb{R}^2$ such that for each L, $|A \cap L| = m_L$.

11. Zorn's Lemma

Zorn's Lemma (ZL) states that if every chain in a nonempty poset $(X, \leq)$ has an upper bound in X, then X contains a maximal element. We remarked in Theorem 5.17 that the Axiom of Choice (AC) and Zermelo's Theorem are equivalent to Zorn's Lemma.

As a first example using Zorn's Lemma, we prove part of Theorem 5.17, in two different ways.

Theorem 11.1 ZL $\Rightarrow$ AC

Proof 1 Let $\{A_\alpha : \alpha \in A\}$ be a collection of pairwise disjoint nonempty sets. Consider the poset set $\mathcal{P} = \{S \subseteq \bigcup_{\alpha \in A} A_\alpha$: for all α, $|S \cap A_\alpha| \leq 1\}$, ordered by inclusion. $\mathcal{P}$ is a nonempty poset because $\emptyset \in \mathcal{P}$.

Suppose $\{S_\beta : \beta \in I\}$ is a chain in $\mathcal{P}$ and let $B = \bigcup\{S_\beta : \beta \in I\}$. We claim that $B \in \mathcal{P}$.

Suppose $|B \cap A_{\alpha_0}| \geq 2$ for some α_0. Consider two points $x \neq y \in B \cap A_{\alpha_0}$ Then $x \in S_{\beta_1}$ and $y \in S_{\beta_2}$ for some $\beta_1, \beta_2 \in I$. Since the S_β's are a chain, either $S_{\beta_1} \subseteq S_{\beta_2}$ or $S_{\beta_2} \subseteq S_{\beta_1}$. Therefore one of these sets, say S_{β_1}, contains both x, y. Therefore $|S_{\beta_1} \cap A_{\alpha_0}| > 1$. But this is impossible since $S_{\beta_1} \in \mathcal{P}$. Therefore B contains at most one point from each A_α so $B \in \mathcal{P}$.

Since B is an upper bound for the chain in $\mathcal{P}$, Zorn's Lemma says that there is a maximal element $M \in \mathcal{P}$.

Since $M \in \mathcal{P}$, $|M \cap A_\alpha| \leq 1$ for every α. If $M \cap A_{\alpha_0} = \emptyset$ for some α_0, then we could choose an $a \in A_{\alpha_0}$ form the set $M' = M \cup \{a\} \underset{\neq}{\supseteq} M$. Then $|M \cap A_\alpha| \leq 1$ would still be true for every α, so we would have $M' \in \mathcal{P}$, which is impossible because M is maximal. Therefore $|M \cap A_\alpha| = 1$ for every α. •

Notice that the function $f : A \to \bigcup A_\alpha$ given by $f = \{(\alpha, y) \in A \times \bigcup_{\alpha \in A} A_\alpha : y \in M \cap A_\alpha\}$ is in the product $\prod\{A_\alpha : \alpha \in A\}$, so we see that $\prod\{A_\alpha : \alpha \in A\} \neq \emptyset$.

Proof 2 Let $\{A_\alpha : \alpha \in A\}$ be a collection of nonempty sets. Let

$$\mathcal{P} = \{g : B \to \bigcup A_\alpha : B \subseteq A \text{ and } f(\beta) \in A_\beta \text{ for each } \beta \in B\}$$

If $g_1, g_2 \in \mathcal{P}$, then the function g_1, g_2 are sets of ordered pairs. So we can order $\mathcal{P}$ by inclusion: $g_1 \leq g_2$ iff $g_1 \subseteq g_2$. (*This relation is just "functional extension": $g_1 \leq g_2$ iff dom$(g_1) \subseteq$ dom(g_2) and $g_2|$dom$(g_1) = g_1$.*) $(\mathcal{P}, \leq)$ is a nonempty poset because $\emptyset \in \mathcal{P}$.

Suppose $\{g_i : i \in I\}$ is a chain in $\mathcal{P}$. Define $g = \bigcup\{g_i : i \in I\}$. Since the g_i's form a chain, their union is a function $g : B \to \bigcup A_\alpha$, where $B = \bigcup_{i \in I} \text{dom}(g_i)$. Moreover, if $\beta \in B$, then $\beta \in \text{dom}(g_i)$ for some i, so $g(\beta) = g_i(\beta) \in A_\beta$. Therefore $g \in \mathcal{P}$, and g is an upper bound on the chain. By Zorn's Lemma, $\mathcal{P}$ has a maximal element, f.

By maximality, the domain of f is $A -$ if not, we could extend the definition of f by adding to its domain a point α from $A - \text{dom}(f)$ and defining $f(\alpha)$ to be a point in the nonempty set A_α. Therefore $f \in \prod\{A_\alpha : \alpha \in A\}$, so $\prod\{A_\alpha : \alpha \in A\} \neq \emptyset$. $\bullet$

In principle, it should be possible to rework any proof using transfinite induction into a proof that uses Zorn's Lemma and vice-versa. However, sometimes one is much more natural to use than the other.

We now present several miscellaneous examples that further illustrate how Zorn's Lemma is used.

The Countable Chain Condition and ϵ-discrete sets in (X, d)

Definition 11.2 Suppose (X, d) is a metric space and $\epsilon > 0$. A set $A \subseteq X$ is called ϵ-discrete if $d(x, y) \geq \epsilon$ for every pair $x \neq y \in A$.

Theorem 11.3 For every $\epsilon > 0$, (X, d) has a maximal ϵ-discrete set.

For example, $\mathbb{Z}$ is a maximal 1-discrete set in $\mathbb{R}$.

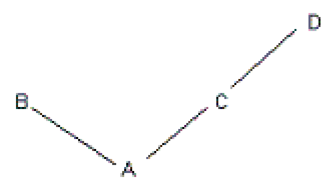
Proof Let $\epsilon > 0$. The theorem is clearly true if $X = \emptyset$, so we assume $X \neq \emptyset$.
Let $\mathcal{P} = \{A \subseteq X : A \text{ is } \epsilon\text{-discrete}\}$ and partially order $\mathcal{P}$ by inclusion $\subseteq$.
If $a \in X$, then $\{a\} \in \mathcal{P}$, so $\mathcal{P} \neq \emptyset$.

Suppose $\{C_\alpha : \alpha \in A\}$ is a chain in $(\mathcal{P}, \leq)$. We claim $\bigcup_{\alpha \in A} C_\alpha$ is ϵ-discrete.

If $x, y \in \bigcup_{\alpha \in A} C_\alpha$, then C_{α_1} and $y \in C_{\alpha_2}$ for some α_1, α_2. Since the C_α's form a chain, so $C_{\alpha_1} \subseteq C_{\alpha_2}$ or $C_{\alpha_2} \subseteq C_{\alpha_1}$. Without loss of generality, $C_{\alpha_1} \subseteq C_{\alpha_2}$. Then $x, y \in C_{\alpha_2}$, and this set is ϵ-discrete. Therefore $d(x, y) \geq \epsilon$.

Therefore $\bigcup_{\alpha \in A} C_\alpha \in \mathcal{P}$. Clearly, $\bigcup_{\alpha \in A} C_\alpha$ is an upper bound for the chain $\{C_\alpha : \alpha \in A\}$. By Zorn's Lemma $(\mathcal{P}, \leq)$ has a maximal element. $\bullet$

Notice that when we use Zorn's Lemma, an upper bound that we produce for a chain in $(\mathcal{P}, \leq)$ $-$ for example, $\bigcup_{\alpha \in A} C_\alpha$ in the preceding paragraph $-$ might not itself be a maximal element in $\mathcal{P}$. For example, suppose $\mathcal{P}$ is a poset that contains exactly four sets A, B, C, D ordered by inclusion. A particular $\mathcal{P}$ is indicated in the following diagram:



One chain in $\mathcal{P}$ is $\{A, C\}$. Then $\bigcup\{A, C\} = C$ is an upper bound for the chain in $\mathcal{P}$. However, $C \subseteq D$ so C is not a maximal element in $\mathcal{P}$.

> *It is always true that an upper bound for a maximal chain in $\mathcal{P}$ (such as $\{A, C, D\}$, above) is a maximal element in $\mathcal{P}$ (why?). The statement that "in any poset, every chain is contained in a maximal chain" is called the Hausdorff Maximal Principle $-$ and it is yet another equivalent to the Axiom of Choice.*

Definition 11.4 A space X satisfies the countable chain condition (CCC) if every family of disjoint open sets in X is countable.

We already know that every separable space satisfies CCC. The following theorem shows that CCC is equivalent to separability among metric spaces.

Theorem 11.5 Suppose (X, d) is a metric space satisfying CCC. Then (X, d) is separable.

Proof For each $n \in \mathbb{N}$, we can use Theorem 11.3 to get a maximal $\frac{1}{n}$-discrete subset D_n. For $x \neq y \in D_n$, $B_{\frac{1}{2n}}(x) \cap B_{\frac{1}{2n}}(y) = \emptyset$, so, by CCC, the family $\{B_{\frac{1}{2n}}(x) : x \in D_n\}$ must be countable. Therefore D_n is countable, and we claim that the countable set $D = \bigcup_{n=1}^{\infty} D_n$ is dense.

> Suppose $z \in X - D$ and, for $\epsilon > 0$, choose n so that $\frac{1}{n} < \epsilon$. Since D_n is a maximal $\frac{1}{n}$-discrete set, $D_n \cup \{z\}$ is not $\frac{1}{n}$-discrete, so there is a point $x \in D_n \subseteq D$ with $d(x, z) < \frac{1}{n} < \epsilon$. •

Sets of cardinals are well-ordered

We proved this result earlier (Corollary 5.21) using ordinals. The following proof, due to Metelli and Salce, avoids any mention of ordinals $-$ but it makes heavy use of Zorn's Lemma and the Axiom of

Choice. The statement that any set of cardinals is well-ordered is clearly equivalent to the following theorem.

Theorem 11.6 If $\{X_\alpha : \alpha \in A\}$ is a nonempty collection of sets, then $\exists\, \overline{\alpha} \in A$ such that $|X_{\overline{\alpha}}| \leq |X_\alpha|$ for every $\alpha \in A$ $-$ in other words, for each $\alpha \in A$ there exists a one-to-one map $\phi_{\overline{\alpha}\alpha} : X_{\overline{\alpha}} \to X_\alpha$.

Proof Assume all the X_α's are nonempty (otherwise the theorem is obviously true). Then by [AC], $\prod_{\alpha \in A} X_\alpha \neq \emptyset$. Let $\mathcal{P} = \{B \subseteq \prod X_\alpha :$ for all $\alpha \in A$, $\pi_\alpha | B$ is one-to-one$\}$, ordered by inclusion. $(\mathcal{P}, \subseteq)$ is a <u>nonempty</u> poset since $\emptyset \in \mathcal{P}$.

If $\{B_i : i \in I\}$ is any chain in $\mathcal{P}$, we claim that $B = \bigcup B_i \in \mathcal{P}$. Otherwise there would be points $x \neq y \in B$ and an α for which $\pi_\alpha(x) = \pi_\alpha(y)$. Since the B_i's form a chain, we would have both $x, y \in B_i$ for some i, and this would imply that $\pi_\alpha | B_i$ is not one-to-one.

B is an upper bound for the chain $\{B_i : i \in I\}$ in $\mathcal{P}$, so by Zorn's Lemma $\mathcal{P}$ contains a maximal element, M.

We claim that for some $\overline{\alpha} \in A$, $\pi_{\overline{\alpha}} | M : M \to X_{\overline{\alpha}}$ is <u>onto</u>.

> Otherwise we would have $X_\alpha - \pi_\alpha[M] \neq \emptyset$ for <u>every</u> α. Using [AC] again, we could choose a point $y = (y_\alpha) \in \prod_{\alpha \in A} (X_\alpha - \pi_\alpha[M]) \neq \emptyset$. Since $y_\alpha \notin \pi_\alpha[M]$, $\pi_\alpha | (M \cup \{y\})$ would be one-to-one for all α so that $M \cup \{y\} \in \mathcal{P}$. Since $y \notin M$, $M \cup \{y\}$ is strictly larger than M and that is impossible because M is maximal.

Therefore $\pi_{\overline{\alpha}} | M : M \to X_{\overline{\alpha}}$ is a bijection and the map $\phi_{\overline{\alpha}\alpha} = \pi_\alpha \circ (\pi_{\overline{\alpha}} | M)^{-1} : X_{\overline{\alpha}} \to X_\alpha$ is $1-1$ for every $\alpha \in A$. ∙

<u>Maximal ideals in a commutative ring with unit 1</u> Suppose K is a commutative ring with a unit element. (*If "commutative ring with unit" is unfamiliar, then just let $K = C(X)$ throughout the whole discussion. In that case, the "unit" is the constant function* **1**.).

Definition 11.7 Suppose $I \subseteq K$, where K is a commutative ring with unit. A subset I of K is called an <u>ideal</u> in K if

> 1) $I \neq K$
> 2) $a, b \in I \ \Rightarrow\ a + b \in I$
> 3) $a \in I$ and $k \in K \ \Rightarrow\ ka \in I$

In other words, an ideal in K is a proper subset of K which is closed under addition and "superclosed" under multiplication.

An ideal I in K is called a <u>maximal ideal</u> if: whenever J is an ideal and $I \subseteq J$, then $I = J$.
A maximal ideal is a maximal element in the poset of all ideals of K, ordered by inclusion.

For example, two ideals in $K = C(\mathbb{R})$ are:

> $I = \{g : g = fi$ where $f, i \in C(\mathbb{R})$ and i is the identity function $i(x) = x\}$
> For example, $g(x) = xe^x \in I$ (using $f(x) = e^x$).

$M = \{f : f(0) = 0\}$

In fact, M is a maximal ideal in $C(\mathbb{R})$ (T*his take a small amount of work to verify; see Exercise E33.*)

Maximal ideals, M, are important in ring theory – for example, if M is a maximal ideal in K, then the quotient ring K/M is actually a field.

Using Zorn's Lemma, we can prove

Theorem 11.8 Let K be a commutative ring with unit. Every ideal I in K is contained in a maximal ideal M. (M *might not be unique.)*

Proof Let $\mathcal{P} = \{J : J$ is an ideal and $I \subseteq J\}$. $(\mathcal{P}, \subseteq)$ is a nonempty poset since $I \in \mathcal{P}$. We want to show that $\mathcal{P}$ contains a maximal element.

Suppose $\{J_\alpha : \alpha \in A\}$ is a chain in $\mathcal{P}$. Let $J = \bigcup\{J_\alpha : \alpha \in A\}$. Since $I \subseteq J$, we only need to check that J is an ideal to show that $J \in \mathcal{P}$.

If $a, b \in J$, then $a \in J_\gamma$ and $b \in J_\beta$ for some $\gamma, \beta \in A$. Since the J_α's form a chain, either $J_\gamma \subseteq J_\beta$ or $J_\beta \subseteq J_\gamma$: say $J_\gamma \subseteq J_\beta$. Then a, b are both in the ideal J_β, so $a + b \in J_\beta \subseteq J$. Moreover, if $k \in K$, $ka \in J_\beta \subseteq J$. Therefore J is closed under addition and superclosed under multiplication.

Finally, $J \neq K$: If $J = K$, then $1 \in J$ so $1 \in J_\alpha$ for some α. Then, for all $k \in K$, $k = k \cdot 1 \in J_\alpha$. So $J_\alpha = K$, which is impossible since J_α is an ideal.

By Zorn's Lemma, we conclude that $\mathcal{P}$ contains a maximal element M. •

Basis for a Vector Space It is assumed here that you know the definition of a vector space V over a field K. (K *is the set of "scalars" which can multiply the vectors in* $\mathcal{V}$. *If "field" is unfamiliar, then you may just assume* $K = \mathbb{R}$, $\mathbb{Q}$, *or* $\mathbb{C}$ *in the result.*) Beginning linear algebra courses usually only deal with finite-dimensional vector spaces: the number of elements in a basis for V is called the dimension of V. But some vector spaces are infinite dimensional. Even then, a basis exists, as we now show.

Definition 11.9 Suppose V is a vector space over a field K. A collection of vectors $B \subseteq V$ is called a basis for V if each nonzero $v \in V$ can be written in a unique way as a finite linear combination of elements of B using nonzero coefficients from K. More formally, B is a basis if for each $v \in V$, $v \neq 0$, there exist unique nonzero $\alpha_1, ..., \alpha_n \in K$ and unique $b_1, ..., b_n \in B$ such that $v = \sum_{i=1}^{n} \alpha_i b_i$. V is called finite dimensional if a finite basis B exists.

Definition 11.10 A set of vectors $C \subseteq V$ is called linearly independent if, whenever α_1, ..., $\alpha_n \in K$ and c_1, ..., $c_n \in C$ and $\sum_{i=1}^{n} \alpha_i c_i = 0$, then $\alpha_1 = ... = \alpha_n = 0$ (in other words, the only linear combination of elements of C adding to 0 is the trivial combination).

Theorem 11.11 Every vector space $V \neq \{0\}$ over a field K has a basis. (*The trivial vector space* $\{0\}$ *cannot have a basis under our definition: the only nonempty subset* $\{0\}$ *is not linearly independent.*)

Proof We will use Zorn's Lemma to show that there is a <u>maximal</u> linearly independent subset of $\mathcal{V}$ and that it must be a basis.

Let $\mathcal{P} = \{C \subseteq V : C$ is linearly independent$\}$, ordered by $\subseteq$. For any $0 \neq v \in V$, we have $\{v\} \in \mathcal{P}$, so $(\mathcal{P}, \subseteq)$ is a nonempty poset.

Let $\{C_\alpha : \alpha \in I\}$ be a chain in $\mathcal{P}$. We claim $C = \bigcup C_\alpha$ is linearly independent, so that $C \in \mathcal{P}$.

> If $\alpha_1, ..., \alpha_n \in K$ and $c_1, ..., c_n \in C$ and $\sum_{i=1}^n \alpha_i c_i = 0$, then each c_i is in some C_{α_i}. The C_α's form a chain with respect to $\subseteq$, so one of the C_{α_i}'s $-$ call it C_{α^*} $-$ contains all the others. Then $0 = \sum_{i=1}^n \alpha_i c_i$ is a linear combination of elements from C_{α^*}, and C_{α^*} is linearly independent. So all the α_i's must be 0.

Therefore, $C \in \mathcal{P}$ and C is an upper bound for the chain $\{C_\alpha : \alpha \in I\}$. By Zorn's Lemma, $\mathcal{P}$ has a maximal element B.

We claim that B is a basis for V.

> First we show that if $v \in V$, then v <u>can</u> be written as a finite linear combination of elements of $\mathcal{B}$.
>
> > If $v \in B$, then $v = 1 \cdot v$. If $v \notin B$, then $B \subsetneq B \cup \{v\}$ so, by maximality, $B \cup \{v\}$ is not linearly independent. That means there is a nontrivial linear combination of elements of $B \cup \{v\}$ (*necessarily involving* v) with sum 0 :
> >
> > $$\exists\, b_1, ..., b_n \text{ and } \exists\, \alpha_1, ..., \alpha_n,\ \beta \in K \text{ with } \beta \neq 0 \text{ and } \textstyle\sum_{i=1}^n \alpha_i b_i + \beta v = 0.$$
> >
> > Since $\beta \neq 0$, we can solve the equation and write $v = \sum_{i=1}^n \frac{-\alpha_i}{\beta}\, b_i$.
>
> We complete the proof by showing that such a representation for v is <u>unique</u>.
>
> > Suppose that we have $\sum_{i=1}^n \alpha_i b_i = v = \sum_{i=1}^m \alpha_i' b_i'$ with $\alpha_i,\ \alpha_i' \in K$ and $b_i,\ b_i' \in B$. By allowing additional b's with 0 coefficients as necessary, we may assume that the same elements of B are used on both sides of the equation, so that $\sum_{i=1}^k \beta_i b_i'' = v = \sum_{i=1}^k \gamma_i b_i''$. Then $\sum_{i=1}^k (\beta_i - \gamma_i) b_i'' = 0$ and B is linearly independent, so $\beta_i = \gamma_i$ for $i = 1, ..., k$.

Example 11.12 In Example II.5.14 we showed that the <u>only</u> <u>continuous</u> functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the functional equation

$$f(x+y) = f(x) + f(y) \text{ for all } x, y \in \mathbb{R} \qquad (*)$$

are the linear functions $f(x) = cx$ for some $c \in \mathbb{R}$. Now we can see that there are other functions, necessarily discontinuous, that satisfy (*).

Consider $\mathbb{R}$ as a vector space over the field $\mathbb{Q}$, and let $B \subseteq \mathbb{R}$ be a basis. Choose any $b_1 \in B$. Then for each $x \in \mathbb{R}$, there is a unique expression $x = q_x b_1 + \sum_{i=2}^{n} q_i b_i$ for some $b_2, ..., b_n \in B$ and $q_x, q_2, ..., q_n \in \mathbb{Q}$. (We can insist that b_1 be part of this sum by allowing $q_x = 0$ when necessary.). Define $f : \mathbb{R} \to \mathbb{Q} \subseteq \mathbb{R}$ by $f(x) = q_x$. Clearly $f(x+y) = f(x) + f(y)$.

Although the definition of f looks complicated, perhaps it could happen, for a cleverly chosen $c \in \mathbb{R}$, that $f(x) = cx$ for all x? No: we show that $f(x) = cx$ is not possible (and therefore f is not continuous).

1) If $f(x) = cx$ for some constant c, we would have:

$f(\sqrt{2}) = c \cdot \sqrt{2} \in \mathbb{Q}$, and
$f(1) = c \cdot 1 = c \in \mathbb{Q}$. But $c \neq 0$, or else $f(x) = 0$ for every x, whereas $f(b) = 1 \cdot b = 1$. So we can divide by c getting
$c\sqrt{2}/c = \sqrt{2} \in \mathbb{Q}$, which is false.

2) Here is a different argument to the same conclusion: For every $b \in \mathcal{B}$ where $b \neq b_1$, we have $b = 0 \cdot b_1 + 1 \cdot b$, so $f(b) = 0$. Therefore the equation $f(x) = 0$ has infinitely many solutions so $f(x)$ is not linear.

As we remarked earlier in Example II.5.14, it can be shown that discontinuous solutions f for (*) must be "not Lebesgue measurable" (a nasty condition that implies that f must be "extremely discontinuous.")

A "silly" example from measure theory (optional) It is certainly possible for an uncountable union of sets of measure 0 to have measure 0. For example, let $I_p = \{x \in \mathbb{Q} \cap [0,1] : x < p\}$ for each irrational $p \in [0,1]$. There are uncountably many I_p's and each one has measure 0. In this case, $\bigcup I_p \subseteq \mathbb{Q}$, so $\bigcup I_p$ also has measure 0.

Might it be true that every union of sets of measure 0 (say, in $[0,1]$) must have measure 0? (It is easy to answer this question: how?) What follows is an "unnecessarily complicated" answer using Zorn's Lemma.

If every such union had measure zero, we could apply Zorn's Lemma to the poset consisting of all measure-0 subsets of $[0,1]$ and get a maximal subset M with measure 0 in $[0,1]$. Since $[0,1]$ does not have measure 0, there is a point $p \in [0,1] - M$. Then $M \cup \{p\}$ also has measure 0, contradicting the maximality of M.

Exercises

E24. Let (X, d) be a metric space. In Definition 10.3, we defined collections $\mathcal{G}_\alpha$ $(\alpha < \omega_1)$ and the collection of Borel sets $\mathcal{B} = \bigcup\{\mathcal{G}_\alpha : \alpha < \omega_1\}$.

a) Let $\mathcal{F}_0$ be the family of closed sets in X. For $\alpha < \omega_1$, define families

$$\mathcal{F}_\alpha = \begin{cases} \{\bigcap_{n=1}^{\infty} F_n : F_n \in \mathcal{F}_{\beta_n}, \beta_n < \alpha\} & \text{if } \alpha \text{ is even} \\ \{\bigcup_{n=1}^{\infty} F_n : F_n \in \mathcal{F}_{\beta_n}, \beta_n < \alpha\} & \text{if } \alpha \text{ is odd} \end{cases}$$

$\mathcal{F}_1$ is the collection of countable unions of closed sets (called F_σ-sets) and $\mathcal{F}_2$ is the family of countable intersections of F_σ sets (called $F_{\sigma\delta}$-sets).

Prove that $\mathcal{F}_\alpha \subseteq \mathcal{G}_\beta$ and $\mathcal{G}_\alpha \subseteq \mathcal{F}_\beta$ for all $\alpha < \beta < \omega_1$.

It follows that $\mathcal{B} = \bigcup\{\mathcal{F}_\alpha : \alpha < \omega_1\}$. We can build the Borel sets "from the bottom up" beginning with either the open sets or the closed sets. Would this be true if we defined Borel sets the same way in an arbitrary topological space?)

b) Suppose X and Y are separable metric spaces. A function $f : X \to Y$ is called <u>Borel-measurable</u> (or <u>B-measurable</u>, for short) if $f^{-1}[B]$ is a Borel set in X whenever B is a Borel set in Y. Prove that f is B-measurable iff $f^{-1}[O]$ is Borel in X whenever O is open in Y.

c) Prove that there are $\leq c$ B-measurable maps f from X to Y.

E25. a) Is a locally finite cover of a space X necessarily point finite? Is a point finite cover necessarily locally finite (*see Definitions 10.10, 10.13*)? Give an example of a space X and an open cover $\mathcal{U}$ that satisfies one of these properties but not the other.

b) Suppose F_i $(i = 1, ..., n)$ are closed sets in the normal space X with $\bigcap_{i=1}^{n} F_i = \emptyset$. Prove that there exist open sets V_i $(i = 1, ..., n)$ such that $F_i \subseteq V_i$ and $\bigcap_{i=1}^{n} \overline{V}_i = \emptyset$.
Hint: Use the characterization of normality in Theorem 10.12.

E26. Prove that the continuum hypothesis ($c = \aleph_1$) is true iff $\mathbb{R}^2$ can be written as $A \cup B$ where A has countable intersection with every horizontal line and B has countable intersection with every vertical line.
Hints: $\Rightarrow$: See Exercise I.E44. If CH is true, $\mathbb{R}$ can be indexed by the ordinals $< \omega_1$.
$\Leftarrow$: If CH is false, then $c > \aleph_1$. Suppose A meets every horizontal line only countably often and that $A \cup B = \mathbb{R}^2$. Show that B meets some vertical line uncountably often by letting Q be the union of any $\aleph_1$ horizontal lines and examining $\pi_X[Q \cap A]$.)

E27. A cover $\mathcal{U}$ of the space X is called irreducible if it has no proper subcover.

a) Give an example of an open cover of a noncompact space which has no irreducible subcover.

b) Prove that X is compact iff every open cover has an irreducible subcover.

Hint: Let $\mathcal{U}$ be any open cover and let $\mathcal{A}$ be a subcover with the smallest possible cardinality m. Let γ be the least ordinal with cardinality m. Index $\mathcal{A}$ using the ordinals less than γ, so that $\mathcal{A} = \{U_\alpha : \alpha < \gamma\}$. Then consider $\{V_\beta : \beta < \gamma\}$, where $V_\beta = \bigcup\{U_\alpha : \alpha < \beta\}$.

E28. Two set theory students, Ray and Debra, are arguing.

> Ray: "There must be a maximal countable set of real numbers. Look: partially order the countable infinite subsets of $\mathbb{R}$ by inclusion. Now every chain of such sets has an upper bound (remember, a countable union of countable sets is countable), so Zorn's Lemma gives us a maximal element."
>
> Debra: "I don't know anything about Zorn's Lemma but it seems to me that you can always add another real to any countable set of real numbers and still have a countable set. So how can there be a largest one?"
>
> Ray: "I didn't say largest! I said maximal!"

Resolve their dispute.

E29. Suppose $(P, \leq)$ is a poset. Prove that $\leq$ can be "enlarged" to a relation $\leq^*$ such that $(P, \leq^*)$ is a chain. (" $\leq^*$ enlarges $\leq$ " means that " $\leq \; \subseteq \; \leq^* \subseteq P \times P$")
Hint: Suppose $\leq$ is a linear ordering on P. Can $\leq$ be enlarged?

E30. Let m be a cardinal. A space X has caliber m if, whenever $\mathcal{U}$ is a family of open sets with $|\,\mathcal{U}\,| = m$, there is a family $\mathcal{V} \subseteq \mathcal{U}$ such that $|\,\mathcal{V}\,| = m$ and $\bigcap\{V : V \in \mathcal{V}\} \neq \emptyset$.

a) Prove that every separable space has caliber $\aleph_1$.

b) Prove that any product of separable spaces has caliber $\aleph_1$.
Hint: Recall that a product of c separable spaces is separable (Theorem VI.3.5).

c) Prove that if X has caliber $\aleph_1$, then X satisfies the countable chain condition (*see Definition 11.4*).

d) Let X be a set of cardinal $\aleph_1$ with the cocountable topology. Is X separable? Does X satisfy the countable chain condition? Does X have caliber $\aleph_1$? (*For notational convenience, you can assume, without loss of generality, that $X = [0, \omega_1)$.*)

E31. a) Prove that there exists an infinite maximal family $\mathcal{E}$ of infinite subsets of $\mathbb{N}$ with the property that the intersection of any two sets from $\mathcal{E}$ is finite.

b) Let $D = \{\omega_E : E \in \mathcal{E}\}$ be a set of distinct points such that $D \cap \mathbb{N} = \emptyset$. Let $Z = \mathbb{N} \cup D$, with the following topology:

i) points of $\mathbb{N}$ are isolated
ii) a basic neighborhood of ω_E is any set containing ω_E and all but at most finitely many points of E.

Prove Z is Tychonoff.

c) Prove that Z is not countably compact. *Hint: Consider the set* D

d) Prove that Z is pseudocompact. *Hint: This proof uses the maximality of* $\mathcal{E}$.

E32. According to Theorem V.5.10, the closed interval $[0,1]$ cannot be written nontrivially as a countable union of pairwise disjoint nonempty closed sets. Of course, $[0,1]$ can certainly be written as the union of c such sets: for example, $[0,1] = \bigcup_{x \in [0,1]} \{x\}$.

Prove that $[0,1]$ can be written as the union of uncountably many pairwise disjoint closed sets each of which is countably infinite.

Hint: Use Zorn's Lemma to choose a maximal family $\mathcal{F}$ *of subsets of* $[0,1]$ *each homeomorphic to* $\{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$. *Let* $A = [0,1] - \bigcup \mathcal{F}$. A *is relatively discrete and therefore countable. For each* $x \in A$, *choose a different* $C_x \in \mathcal{F}$ *and replace* C_x *by* $C_x \cup \{x\}$.)

E33. Suppose X is Tychonoff. For $p \in X$, let $M_p = \{f \in C(X) \colon f(p) = 0\}$. Clearly, M_p is an ideal in $C(X)$.

a) Prove that M_p is a maximal ideal in $C(X)$.

b) Prove that is X is compact iff every maximal ideal in $C(X)$ is of the form M_p for some $p \in X$.

E34. Prove that there exists a subset A of $\mathbb{R}$ that has only countably many distinct “translates” $A_r = \{a + r : a \in A\}$ – that is only countably many of the sets $A_r = \{a + r : a \in A\}$, $r \in \mathbb{R}$ are distinct).

Hint: Consider $\mathbb{R}$ *as a vector space over the field* $\mathbb{Q}$, *and pick a basis* B. *Pick a point* $b \in B$ *and consider all reals whose expression as a finite linear combination of elements of* B *does not involve* b.

Appendix
Exponentiation of Ordinals: A Sketch

The appendix gives a brief sketch about exponentiation of ordinals. Some of the details are omitted. The main point is to explain why ordinals like $\omega_0^{\omega_0}$ are still countable ordinals. (See Example 5.25, part 3)

If μ and α are ordinals, let

$$\mu^\alpha = \{f \in [0,\mu)^{[0,\alpha)} : f(\xi) = 0 \text{ for all but finitely many } \xi < \alpha\}.$$

We can think of a point f in μ^α as a "transfinite sequence" in $[0,\mu)$, where f has well-ordered domain $[0,\alpha)$ rather than $\mathbb{N}$. Using "sequence-like" notation, we can write:

$$\mu^\alpha = \{(a_\xi) : 0 \le \xi < \alpha, 0 \le a_\xi < \mu, \text{ and } a_\xi = 0 \text{ for all but finitely many } \xi\}$$

We put an ordering on μ^α by:

> Given $(a_\xi) \neq (b_\xi)$ in μ^α, let ν be the largest index for which $a_\nu \neq b_\nu$. We write $(a_\xi) < (b_\xi)$ if $a_\nu < b_\nu$.

Example For $\alpha = \mu = 2$, μ^α consists of 4 pairs ordered as follows: $(0,0) < (1,0) < (0,1) < (1,1)$.

We also use μ^α to denote the order type associated with $(\mu^\alpha, \le)$. It turns out that $(\mu^\alpha, \le)$ is well-ordered, so this order type is actually an ordinal number.

Example $\omega_0^{\omega_0}$ is represented by the set of all sequences in $\{0,1,2,...\}$ which are eventually 0. (*The sequences in the set $\omega_0^{\omega_0}$ turn out to be those which are "eventually 0" because each element of ω_0 has has only finitely many predecessors. In general, the condition that members of μ^α be 0 for all but finitely many $\xi < \alpha$ is much stronger than merely saying "eventually 0."*)

The order relation on $\omega_0^{\omega_0}$ between two sequences is determined by comparing the largest term at which the sequences differ. For example,

$$(5,0,0,1,2,0,0,0,0,...) < (107,12,1,3,5,0,0,0,0,...) < (103,7,0,0,6,0,0,0,0...,)$$

The initial segment of $\omega_0^{\omega_0}$ representing ω_0 consists of:

$$(0,0,0,...,...) < (1,0,0,...,...) < (2,0,0,...,...) < ... < (n,0,0,...,...) < ...$$

$\omega_0^{\omega_0}$ is clearly a countable ordinal. A little thought shows that $\omega_0^{\omega_0} = \omega_0 + \omega_0^2 + \omega_0^3 + ...$

Once μ^α is defined, it is easy to see that the ordinals $\omega_0^{\omega_0}, \omega_0^{\omega_0^{\omega_0}}, \omega_0^{\omega_0^{\omega_0^{\omega_0}}}, ...$ are all countable ordinals. (*Here, α^{β^γ} is understood, as usual, to mean $\alpha^{(\beta^\gamma)}$*).

We can then define $\epsilon_0 = \sup\{\omega_0^{\omega_0}, \omega_0^{\omega_0^{\omega_0}}, \omega_0^{\omega_0^{\omega_0^{\omega_0}}}, ...\}$. Roughly, ϵ_0 is "ω_0 raised to the ω_0 power ω_0 times". As noted earlier, the sup of a countable set of countable ordinals is still countable: ϵ_0 is a countable ordinal, that is, $\epsilon_0 < \omega_1$!

Exercise Prove that $\mu^1 = \mu$. Prove that for ordinals μ and α, μ^α is also an ordinal, i.e., $(\mu^\alpha, \leq)$ is a well-ordered set (not trivial!). For ordinals $\mu, \alpha, \beta > 0$, and $\mu^{\alpha+\beta} = \mu^\alpha \cdot \mu^\beta$.

For further information, see Sierpinski, Cardinal and Ordinal Numbers, pp. 309 ff.

Chapter VIII Review

Explain why each statement is true, or provide a counterexample.

1. Let $X = [0, \omega_1)$. For each $\alpha \in X$, let $\alpha \sim \alpha + 1$. In the quotient space $X/\sim$, exactly one point is isolated.

2. $\aleph_{\omega_0}^{\aleph_0}$ can be written as a sum of countably many smaller cardinals.

3. Let $n \in \mathbb{N}$. A continuous function $f : [0, \omega_1) \to \mathbb{R}^n$ is constant on a tail of $[0, \omega_1)$.

4. Every order-dense chain with more than 1 point contains a subset order isomorphic to $\mathbb{Q}$.

5. If $x \in O$, where O is an open set in $X = [0, \omega_1] \times [0, \omega_0] - \{(\omega_0 + 1, \omega_0)\}$, then there must be a continuous function $f\colon X \to \mathbb{R}$ such that $x \in \operatorname{coz}(f) \subseteq O$.

6. Let C be a non-Borel subset of the unit circle S^1 and let $f : S^1 \to \mathbb{R}$ be the characteristic function of C. Then f is not Borel measurable but f is Borel measurable in each variable separately (that is, for each $a \in [-1, 1]$, the functions ϕ and ψ defined by $\phi(x) = f(x, a)$ and $\psi(a, y) = f(a, y)$ are Borel measurable.

7. Let C be a dense subset of $\mathbb{R}$. Then C is not well-ordered (in the usual order on $\mathbb{R}$).

8. Let $D = \{f \in \mathbb{Q}^{\mathbb{N}} : f(n) = 0$ for all but finitely many $n\}$, with the lexicographic ("dictionary") ordering $\leq$. Then $(D, \leq)$ is order isomorphic to $\mathbb{Q}$.

9. Suppose $\leq_1$ and $\leq_2$ are linear orders on a set X. If $\leq_1 \subseteq \leq_2$, then $\leq_1 = \leq_2$.

10. Consider the ordinals 1, ω_0 and $\omega_0 \cdot 2$. Considering all possible sums of these ordinals (in the six different possible orders) produces exactly 3 distinct values.

11. If α and β are nonzero ordinals and $a + \beta = \omega_0$, then $\alpha\beta = \omega_0$.

12. The order topology on $(1, 3) \cup (3, 5)$ is the same as the subspace topology from $\mathbb{R}$.

13. If $(X, \mathcal{T})$ is a finite topological space, then there exists a partial ordering $\leq$ on X for which $\mathcal{T}$ is the order topology.

14. If O is open and F is closed in $T = [0, \omega_1] \times [0, \omega_0] - \{(\omega_1, \omega_0)\}$ and $F \subseteq O$, then there must exist an open set V such that $F \subseteq V \subseteq \operatorname{cl} V \subseteq O$.

15. There are exactly $\aleph_\alpha$ limit ordinals $< \omega_\alpha$.

16. If α denotes the order type of the irrationals, then $\alpha^2 = \alpha$.

17. If $\alpha > \omega_1$, then the ordinal space $X = [0, \alpha]$ is not metrizable because X is not normal.

18. Suppose $\mathcal{P}$ is a nonempty collection containing all subsets of X with a certain property (*). Suppose $\{P_\alpha : \in A\}$ is a chain of subsets of $\mathcal{P}$ and that $\bigcup P_\alpha \in \mathcal{P}$. By Zorn's Lemma, $\bigcup P_\alpha$ is a maximal subset of X (with respect to $\subseteq$) having property (*).

19. For any infinite cardinal k, there are exactly k^+ ordinals with cardinality k.

20. Suppose α is an infinite order type. If $1 + \alpha = \alpha + 1$, then there is an order type ξ (possibly 0) such that $\alpha = \omega_0 + \xi + \omega_0^*$.

21. If $\leq$ a linear order on X, then the "reversed relation" $\leq^*$ defined by $x \leq^* y$ iff $y \leq x$ is also a linear order on X.

22. For any infinite cardinal k, $2^k = k^k$ (without GCH).

23. If $\leq$ is a linear ordering on a finite set S and $|S| = n$, then $|\leq| = \frac{n(n+1)}{2}$.

24. If A and B are disjoint closed sets in $[0, \omega_1)$, then at least one of them is countable.

25. A locally finite open cover (by nonempty sets) of a compact space must be finite.

26. $[0, \omega_1]$ is homeomorphic to a subspace of the Cantor set.

27. Let $\mathbb{N}^{\aleph_0}$ have the lexicographic order $\leq$. $(\mathbb{N}^{\aleph_0}, \leq)$ is not well-ordered, because the set $A = \{x \in \mathbb{N}^{\aleph_0} : \exists k \ \forall n > k \ \ x(n) = 2\}$ contains no smallest element.

28. Every subset of $\mathbb{Q}$ is a Borel set in $\mathbb{R}$.

29. With the order topology, the set $\{0\} \cup \{x \in \mathbb{R} : |x| > 1\}$ is homeomorphic to $\mathbb{R}$.

30. If $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ is a point-finite open cover of the Sorgenfrey plane $S \times S$, then there must be an open cover $\mathcal{V} = \{V_\alpha : \alpha \in A\}$ of $S \times S$ such that, for each $\alpha \in A$, $\operatorname{cl}(V_\alpha) \subseteq U_\alpha$.

31. There is a countably infinite compact connected metric space.

32. If $\leq$ is a linear ordering on X, then there can be linear ordering $\leq'$ for which $\leq \ \underset{\neq}{\subseteq} \ \leq' \subseteq X \times X$