Chapter IX
Theory of Convergence

1. Introduction

In Chapters II and III, we discussed the convergence of sequences. Sequential convergence is very useful in working with first countable spaces (in particular, with metric spaces).

In any space X, if (a_n) is a sequence in A and $(a_n) \to x$, then $x \in \text{cl}\, A$. But in a first countable space, the converse is also true : $x \in \text{cl}\, A$ iff there is a sequence (a_n) in A with $(a_n) \to x$. Therefore convergent sequences determine the set cl A, so sequences can be used to check whether of not cl $A = A$, that is, whether or not A is closed. we can determine whether of not A is closed using sequences. (*See Theorem III.9.6*). So in a first countable space "sequences determine the topology."

However, sequences are not sufficient in general to describe closures: for example, in $[0, \omega_1]$ we have $\omega_1 \in \text{cl}\,[0, \omega_1)$ but no sequence (α_n) in $[0, \omega_1,)$ converges to ω_1. The basic neighborhoods $(\alpha, \omega_1]$ of ω_1 are very nicely ordered, but there are just "too many" of them: no mere sequence (α_n) in $[0, \omega_1)$ is "long enough" to be eventually inside every neighborhood of ω_1.

A second example (*see Example III.9.8*): $(0,0) \in \text{cl}\,(L - \{(0,0)\})$ but no sequence in $(L - \{(0,0)\})$ converges to $(0,0)$. Here the problem is different. L is a "small" space $(|L| = \aleph_0)$ but the neighborhood system $\mathcal{N}_{(0,0)}$ (ordered either by inclusion or by reverse inclusion) is a very complicated poset – so complicated that a mere sequence (a_n) in $L - \{(0,0)\}$ cannot eventually be in every neighborhood of $(0,0)$.

In this chapter we develop a theory of convergence that is sufficient to describe the topology in any space X. We define a kind of "generalized sequence" called a net. A sequence is a function $f \in X^{\mathbb{N}}$, and we write $f(n) = x_n$. A net is a function $f \in X^{\Lambda}$, where $(\Lambda, \le)$ is a more general kind of ordered set. Informally, we write a sequence f as (x_n); similarly, when f is a net, then we informally write the net as (x_λ).

Thinking of $X = [0, \omega_1]$, we might hope that it would be a sufficient generalization to replace $\mathbb{N}$ with an initial segment of ordinals $\Lambda = [0, \alpha)$: in other words, to replace sequences with "transfinite sequences" with domain some well-ordered set "longer" than $\mathbb{N}$. In fact, this is a sufficient generalization to deal with a case like $[0, \omega_1]$: if (x_α) is the "transfinite sequence" in $[0, \omega_1)$ given by $x_\alpha = \alpha$ $(\alpha < \omega_1)$, then $(x_\alpha) \to \omega_1$ (in the sense that (x_α) is eventually in every neighborhood of ω_1). But as we will see below, such a generalization does not go far enough (s*ee Example 2.9*). We need a generalization that uses some kind of ordered set Λ more complicated than just initial segments $[0, \alpha)$ of ordinals.

The theory of nets turns out to have a "dual" formulation in the theory of filters. In this chapter we will discuss both formulations. It turns out that nets and filters are fully equivalent formulations of convergence, but sometimes one is more natural to use than the other.

2. Nets

Definition 2.1 A nonempty ordered set $(\Lambda, \leq)$ is called a directed set if

1) $\leq$ is transitive and reflexive
2) for all $\lambda_1, \lambda_2 \in \Lambda$, there exists $\lambda_3 \in \Lambda$ such that $\lambda_3 \geq \lambda_1$ and $\lambda_3 \geq \lambda_2$.

Example 2.2

1) Any chain (for example, $\mathbb{N}$) is a directed set.

2) In $\mathbb{R}$, define $x \leq {}^* y$ if $|x| \geq |y|$. Then $(\mathbb{R}, \leq {}^*)$ is a directed set and $x \leq {}^* y$ means that "x is at least as far from the origin as y." Notice that $-1 \leq {}^* 1$ and $1 \leq {}^* -1$, so that $\leq {}^*$ is not antisymmetric. A directed set might not be a poset.

3) Let $\mathcal{N}_x$ be the neighborhood system at x in X, ordered by "reverse inclusion" – that is, $N_1 \leq N_2$ iff $N_1 \supseteq N_2$. $\mathcal{N}_x \neq \emptyset$ and condition 1) in the definition clearly holds. Condition 2) is satisfied because for $N_1, N_2 \in \mathcal{N}_x$, we have $N_1 \cap N_2 = N_3 \in \mathcal{N}_x$ and $N_3 \geq N_1$ and $N_3 \geq N_2$. Therefore $(\mathcal{N}_x, \leq)$ is a directed set.

This example hints at why replacing $\mathbb{N}$ (in the definition of a sequence) with a directed set Λ (in the definition of net) will give a tool strong enough to describe the topology in any space X: the directed set Λ can chosen to be as complicated as the most complicated system of neighborhoods at a point x.

4) Let $\Lambda = \{F \subseteq [0,1] : 0 \in F, 1 \in F$ and F is finite$\}$. In analysis, F is called a partition of the interval $[0,1]$. Order Λ by inclusion: a "larger" partition is a "finer" one – one with more subdivision points. Then Λ is a directed set.

Definition 2.3 i) A net in a set X is a function $f : (\Lambda, \leq) \to X$, where $(\Lambda, \leq)$ is directed. We write $f(\lambda) = x_\lambda$ and, informally, denote the net by (x_λ).

ii) A net (x_λ) in a space X converges to $x \in X$ if (x_λ) is eventually in every neighborhood of x – that is, $\forall N \in \mathcal{N}_x \; \exists \lambda_0 \in \Lambda$ such that $x_\lambda \in N$ wherever $\lambda \geq \lambda_0$.

iii) A point x in a space X is a cluster point of a net (x_λ) if the net is frequently in every neighborhood of x, that is: for all $N \in \mathcal{N}_x$ and all $\lambda_0 \in \Lambda$ there is a $\lambda \geq \lambda_0$ for which $x_\lambda \in N$.

Clearly, every sequence is a net, and when $\Lambda = \mathbb{N}$, the definitions ii), and iii) are the same as the old definitions for convergence and cluster point of a sequence.

Example 2.4 $\Lambda = \{\lambda_1, \lambda_2, \lambda_3\}$ and define an order in which $\lambda_3 \geq \lambda_1$ and $\lambda_3 \geq \lambda_2$ but λ_1 and λ_2 are not comparable. $(\Lambda, \leq)$ is a directed set. We can define a net $f : \Lambda \to \mathbb{R}$ by $f(\lambda_3) = 0$ and assigning any real values to $f(\lambda_1)$ and $f(\lambda_2)$. Since $x_\lambda \in (-\epsilon, \epsilon)$ for all $\lambda \geq \lambda_3$, we have $(x_\lambda) \to 0$. (*Notice that a net can have a finite domain and can have a "last term."*)

Example 2.5 The following examples indicate how several different kinds of limit that come up in analysis can all be reformulated in terms of net convergence. In other words, many different "limit" definitions in analysis are "unified" by the concept of net convergence.

1) Let $\Lambda = \mathbb{R} - \{a\}$ and define $x \le {}^*y$ (in Λ) iff $\mid x - a \mid \; \ge \; \mid y - a \mid$ (in $\mathbb{R}$). Suppose $f : \Lambda \to \mathbb{R}$ is a net. Then the net $(x_\lambda) \to L \in \mathbb{R}$

iff for all $\epsilon > 0$ there exists $\lambda_0 \in \Lambda$ such that $x_\lambda \in (L - \epsilon, L + \epsilon)$ if $\lambda \ge^* \lambda_0$
iff for all $\epsilon > 0$ there exists $\lambda_0 \in \Lambda$ such that $x_\lambda \in (L - \epsilon, L + \epsilon)$ if $0 < |\lambda - a| \le |\lambda_0 - a|$
iff for all $\epsilon > 0$ there exists $\delta > 0$ such that $|f(\lambda) - L| < \epsilon$ if $0 < |\lambda - a| < \delta$
$\uparrow$ ($\Rightarrow$: *use* $\delta = |\lambda_0 - a|/2$)
iff $\lim\limits_{x \to a} f(x) = L$ (in the usual sense of analysis).

2) Let $\Lambda = \mathbb{R}$ with the usual order $\le$. If $f : \mathbb{R} \to \mathbb{R}$, we can think of f as a net, and write $f(r) = x_r$. Then the net $(x_r) \to L \in \mathbb{R}$

iff for all $\epsilon > 0$ there exists $r_0 \in \Lambda = \mathbb{R}$ such that $x_r \in (L - \epsilon, L + \epsilon)$ if $r \ge r_0$
iff for all $\epsilon > 0$ there exists $r_0 \in \mathbb{R}$ such that $|f(r) - L| < \epsilon$ if $r \ge r_0$
iff $\lim\limits_{x \to \infty} f(x) = L$ (in the usual sense of analysis).

Parts 1) and 2) show how two different limits in analysis can each be expressed as convergence of a net. In fact each of the limits $\lim\limits_{x \to a} f(x) = L$, $\lim\limits_{x \to a^+} f(x) = L$, $\lim\limits_{x \to a^-} f(x) = L$, $\lim\limits_{x \to \infty} f(x) = L$, and $\lim\limits_{x \to -\infty} f(x) = L$ can be expressed as the convergence of some net. In each case, the trick is to choose the proper directed set.

3) Integrals can also be defined in terms of the convergence of nets.

Let g be a bounded real valued function defined on $[0,1]$ and let $\Lambda = \{F \subseteq [0,1] : F$ is finite and $0 \in F,\, 1 \in F\}$. Order Λ by inclusion: $F_1 \le F_2$ iff $F_1 \subseteq F_2$ iff F_2 is a "finer" partition than F_1.

For $F \in \Lambda$, enumerate F as $0 = x_0 < x_1 < \ldots < x_{n-1} < x_n = 1$. Let $\Delta x_i = x_i - x_{i-1}$ and

$$f(F) = \sum_{i=1}^{n} \Big(\inf_{[x_{i-1}, x_i]} g\Big) \cdot \Delta x_i = x_F \in \mathbb{R}.$$

$f : \Lambda \to \mathbb{R}$ is a net in $\mathbb{R}$ and we can write $f(F) = x_F$. Then

$(x_F) \to L$ iff for all $\epsilon > 0$ there is a partition F_0 such that
for all partitions F finer than F_0, $|x_F - L| < \epsilon$

One can show that such an L always exists: L is called the <u>lower integral of</u> g <u>over</u> $[0,1]$, denoted $L = \underline{\int}_0^1 g$.

If we replace "inf" by "sup" in defining f, then the limit U of the net is called the <u>upper integral of</u> g <u>over</u> $[0,1]$, denoted $U = \overline{\int}_0^1 g$. If $L = U$, we say that g is (Riemann) integrable on $[0,1]$ and write $\int_0^1 g = L \, (= U)$.

A sequence can have more than one limit: for example, if $|X| > 1$ and X has the trivial topology, every sequence (x_n) in X converges to every point in X. Since a sequence is a net (with $\Lambda = \mathbb{N}$), a net can have more than one limit. We also proved that a sequence in a Hausdorff space can have at most one limit (Theorem III.9.3). This theorem still holds for nets, but then even more is true: uniqueness of net limits actually characterizes Hausdorff spaces.

Theorem 2.6 A space X is Hausdorff iff every net in X has at most one limit.

Proof Suppose X is Hausdorff and $x \neq y \in X$. Choose disjoint open sets U, V with $x \in U$ and $y \in V$. If a net $(x_\lambda) \to x$, then (x_λ) is eventually in U and therefore (x_λ) is eventually outside V. Therefore (x_λ) does not also converge to y.

Conversely, suppose X is not Hausdorff. Then there are points $x \neq y \in X$ such that $U \cap V \neq \emptyset$ whenever $U \in \mathcal{N}_x$ and $V \in \mathcal{N}_y$. Let $\Lambda = \mathcal{N}_x \times \mathcal{N}_y$ and order Λ by defining $(U, V) \leq (U', V')$ iff $U' \subseteq U$ and $V' \subseteq V$ (reverse inclusion in both coordinates). For each $(U, V) \in \Lambda$, let $f((U, V)) =$ any point $x_{(U,V)} \in U \cap V$. We claim that $(x_{(U,V)}) \to x$ and $(x_{(U,V)}) \to y$.

> If $V \in \mathcal{N}_y$, let $\lambda_0 = (X, V) \in \Lambda$. Then if $\lambda = (U', V') \geq \lambda_0$, we have $x_\lambda = x_{(U',V')} \in U' \cap V' \subseteq X \cap V = V$. Therefore $(x_\lambda) \to y$.
> The proof that $(x_\lambda) \to x$ is similar. •

Example 2.7 The implication ($\Leftarrow$) in Theorem 2.6 is false for sequences. Let $X = [0, \omega_1] \cup \{\omega_1^*\}$ where $\omega_1^* \notin [0, \omega_1]$. Let points in $[0, \omega_1]$ have their usual neighborhoods and let a basic neighborhood of ω_1^* be a set of the form $(\alpha, \omega_1] - \{\omega_1\} \cup \{\omega_1^*\}$, where $\alpha < \omega_1$. (*In effect, the basic neighborhoods of ω_1 and ω_1^* are identical except for replacing ω_1 by ω_1^* or vice versa; it is as if ω_1^* is a "shadow" of ω_1 which can't be separated from ω_1).*

This space is not Hausdorff, but we claim that a sequence (α_n) in X has at most one limit.

> If (α_n) contains infinitely many terms $\alpha_{n_k} < \omega_1$, then let $\alpha = \sup\{\alpha_{n_k} : k = 1, 2, ...\} < \omega_1$, so that the subsequence (α_{n_k}) is in $[0, \alpha]$. If $(\alpha_n) \to \beta$ and $(\alpha_n) \to \gamma$, then $(\alpha_{n_k}) \to \beta$ and $(\alpha_{n_k}) \to \gamma$. Therefore β, γ are both in the closed set $[0, \alpha]$. But $[0, \alpha]$ is T_2 so a limit for (α_{n_k}) must be unique: $\beta = \gamma$.
>
> If only finitely many α_n's are less than ω_1, we can assume without loss of generality that for all n, $\alpha_n \in \{\omega_1, \omega_1^*\}$. It is easy to see that if $\alpha_n = \omega_1$ for only finitely many n, then $(\alpha_n) \to \omega_1^*$ only. Similarly, if $\alpha_n = \omega_1^*$ for only finitely many n, then $(\alpha_n) \to \omega_1$ only. If $\alpha_n = \omega_1$ and $\alpha_n = \omega_1^*$ each for infinitely many n, then (α_n) has no limit.

The next theorem tells us that nets are sufficient to describe the topology in any space X.

Theorem 2.8 Suppose $A \subseteq (X, \mathcal{T})$. Then $x \in \text{cl}\, A$ iff there is a net (a_λ) in A for which $(a_\lambda) \to x$.

Proof Suppose $(a_\lambda) \to x$, where (a_λ) is a net in A. For each $N \in \mathcal{N}_x$, (a_λ) is eventually in N so $N \cap A \neq \emptyset$. Therefore $x \in \text{cl}\, A$.

We can prove the converse because we can choose a directed set "as complicated as" the neighborhood system of x: Suppose $x \in \text{cl}\, A$ and let $\Lambda = \mathcal{N}_x$, ordered by reverse inclusion. For $\lambda = N \in \Lambda$, let $f(\lambda) = a_\lambda =$ a point chosen in $N \cap A$. If $\lambda_0 = N_0 \in \Lambda$ and $\lambda = N \geq \lambda_0$ then $a_\lambda \in N \cap A \subseteq N_0 \cap A \subseteq N_0$. Therefore $(a_\lambda) \to x$. •

Example 2.9 In general, "sequences aren't sufficient" to describe the topology of a space X. However we might ask whether something simpler than nets will do. For example, suppose we consider "transfinite sequences" $-$ that is, very special nets $f : [0, \alpha) \to X$, where α is some ordinal. It turns out that these are not sufficient.

Suppose $X = [0, \omega_1) \times [0, \omega_0) \subseteq T^* = [0, \omega_1] \times [0, \omega_0]$. Let $p =$ the "upper right corner point" (ω_1, ω_0). Then $p \in \text{cl}\, X$, but we claim that no transfinite sequence in X can converge to $p \in T^*$.

Let $f : [0, \alpha) \to X$ and write $f(\lambda) = (x_\lambda, y_\lambda)$. We claim that $((x_\lambda, y_\lambda)) \not\to p$ (*no matter how large an* α *we use!*). So $((x_\lambda, y_\lambda)) \to p$:

ran(f) must be uncountable.

If ran(f) were countable, then the set $C = \{x_\lambda : \lambda < \alpha\}$ would be countable and $\beta = \sup \{x_\lambda : \lambda < \alpha\} < \omega_1$. Then ran$(f) \subseteq [0, \beta] \times [0, \omega_0)$ and so (x_λ, y_λ) could not converge to p.

For every countable set $E \subseteq [0, \alpha)$, sup $E < \alpha$:

Since E well-ordered, E represents an ordinal $\beta \leq \alpha$, so there is an order isomorphism $g : [0, \beta) \to E$ for some $\beta \leq \alpha$.

Certainly sup $E \leq \alpha$.

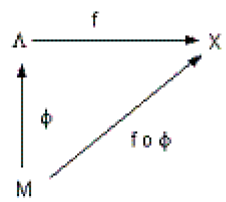
If sup$(E) = \alpha$, then $f \circ g : [0, \beta) \to X$ would be a transfinite sequence with countable range converging to p (since f does). But we have shown that a net converging to p must have uncountable range. Therefore sup $E < \alpha$.

For each $m < \omega_0$, let $E_m = \{\lambda < \alpha : y_\lambda = m\}$ and $\lambda_m =$ sup $E_m \leq \alpha$ (" $\leq$ " is the most we can say since E_m might be uncountable). For each $\beta < \alpha$, β is in some E_m $-$ so, for that m, $\lambda_m \geq \beta$. Therefore sup $\{\lambda_m : m < \omega_0\} = \alpha$. By the preceding paragraph, we conclude $\lambda_{m_0} = \alpha$ for some m_0.

Since $(x_\lambda, y_\lambda) \to p$, we conclude that $(x_\lambda, y_\lambda)_{\lambda \in E_{m_0}} \to p$. But $(x_\lambda, y_\lambda)_{\lambda \in E_{m_0}} \to (\omega_1, m)$, so this is impossible. •

The definition of a subnet is analogous to the definition of a subsequence (*see Definition III.10.1*).

Definition 2.10 Let Λ and M be directed sets Suppose $f : \Lambda \to X$, $\phi\colon M \to \Lambda$. Suppose that for each $\lambda_0 \in \Lambda$, there exists a $\mu_0 \in M$ such that $\phi(\mu) \geq \lambda_0$ whenever $\mu \geq \mu_0$,



then $f \circ \phi$ is called a <u>subnet</u> of f. We write $\phi(\mu) = \lambda_\mu$ and $f(\phi(\mu)) = x_{\lambda_\mu}$. Informally, the subnet is (x_{λ_μ}). The definition of subnet guarantees that $\phi(\mu) = \lambda_\mu \geq \lambda_0$ whenever $\mu \geq \mu_0$.

Note: an alternate definition for subnet is used in some books. It requires that

1) ϕ is <u>increasing</u>: if $\mu_1 \leq \mu_2$, then $\phi(\mu_1) \leq \phi(\mu_2)$ and

2) ϕ is <u>cofinal</u> in Λ: for each $\lambda_0 \in \Lambda$, there is a $\mu_0 \in M$ for which $\phi(\mu_0) \geq \lambda_0$.

A subnet in the sense of this definition <u>is</u> also a subnet in the sense of Definition 2.10, but the definitions are <u>not equivalent</u>. Definition 2.10 is "more generous" – it allows <u>more</u> subnets because Definition 2.10 does not require ϕ to be increasing. For most purposes, the slight disagreement in the definitions doesn't matter. However, the full generality of Definition 2.10 is required to develop the full duality between nets and filters that we discuss later.

We will state the following theorem, for now, without proof. The proof will be easier after we talk about filter convergence. For now, we simply want to use the theorem to highlight an observation in Example 2.12.

Theorem 2.11 In $(X, \mathcal{T})$, if x is a cluster point of the net (x_λ), then there is a subnet $(x_{\lambda_\mu}) \to x$.

Proof See Corollary 4.8 later in this chapter.

Example 2.12 (*See Example III.9.8*) In $L = \{(m,n) : m, n$ are nonnegative integers$\}$, all points except $(0,0)$ are isolated. Basic neighborhoods of $(0,0)$ are sets containing $(0,0)$ and "most of the points from most of the columns" (where "most" means "all but finitely many").

Let (x_n) be an enumeration of $L - \{(0,0)\}$. Although $(0,0)$ is a cluster point of (x_n), we proved in Example III.9.8 that no sequence in $L - \{(0,0)\}$ can converge to $(0,0)$. In particular, no subsequence of (x_n) can converge to $(0,0)$. However, Theorem 2.11 implies that there is a subnet of (x_n) that converges to $(0,0)$.

This might seem surprising, but it simply highlights something is actually clear from the definition of subnet: even if a net is a sequence ($\Lambda = \mathbb{N}$), the directed set M in the definition of subnet need not be $\mathbb{N}$. Therefore a subnet of a sequence might not be a sequence.

3. Filters

$X^{\mathbb{N}}$ is the set of all sequences in X, but the "set of all nets in X" makes no sense in ZFC. We can put an order $\leq$ on any set Λ to create a directed set $(\Lambda, \leq)$: for example (using Zermelo's Theorem) we could let $\leq$ be a well-ordering of Λ. Therefore there at least as many directed sets $(\Lambda, \leq)$ as there are sets Λ. The "set" of all nets in X = "$\bigcup\{X^{(\Lambda,\leq)} : (\Lambda, \leq)$ is a directed set$\}$" is "too big" to be a set in ZFC. This is not only an aesthetic annoyance; it is also a serious set-theoretic disadvantage for certain purposes.

Therefore we look at an equivalent way to describe convergence, one which is strong enough to describe the topology of any space X but which doesn't have this set-theoretic drawback. The theory of filters does this, and it turns out to be a theory "dual" to the theory of nets $-$ that is, there is a natural "back-and-forth" between nets and filters that converts each theorem about nets to a theorem about filters and vice-versa.

Definition 3.1 A filter $\mathcal{F}$ in a set X is a nonempty collection of subsets of X such that

i) $\emptyset \notin \mathcal{F}$
ii) $\mathcal{F}$ is closed under finite intersections
iii) If $F \in \mathcal{F}$ and $G \supseteq F$, then $G \in \mathcal{F}$.

A nonempty family $\mathcal{B} \subseteq \mathcal{F}$ is called a filter base for $\mathcal{F}$ if $\mathcal{F} = \{F \subseteq X : F \supseteq B$ for some $B \in \mathcal{B}\}$.

Assuming, as we have stated, that filters will provide an equivalent theory of convergence in a space X, we see that there is no longer a set-theoretic issue. The set of all filters in X makes perfectly good sense: each filter $\mathcal{F} \in \mathcal{P}(\mathcal{P}(X))$, so the set of all filters in X is just a subset of $\mathcal{P}(\mathcal{P}(X))$.

In a topological space X, there is a completely familiar example of a filter : $\mathcal{N}_x$, the neighborhood system at x. A base for this filter is what we have been calling a neighborhood base, $\mathcal{B}_x$. In fact, $\mathcal{N}_x$ and $\mathcal{B}_x$ are what inspired the general definitions of filter and filter base in the first place.

If we start with a filter $\mathcal{F}$, then a base $\mathcal{B}$ for a filter $\mathcal{F}$ must have certain properties: $\mathcal{B}$ is nonempty (or else $\mathcal{F} = \emptyset$), and each $B \in \mathcal{B}$ is nonempty (or else $\emptyset \in \mathcal{F}$). Moreover, if $B_1, B_2 \in \mathcal{B} \subseteq \mathcal{F}$, then $B_1 \cap B_2 \in \mathcal{F}$ so (by definition of a base) $B_1 \cap B_2 \supseteq B_3$ for some $B_3 \in \mathcal{B}$.

If, on the other hand, we start with any nonempty collection $\mathcal{B}$ of nonempty sets in X such that whenever $B_1, B_2 \in \mathcal{B}$, there is a $B_3 \in \mathcal{B}$ such that $B_1 \cap B_2 \supseteq B_3$ (*)

and define $\mathcal{F} = \{F : F \supseteq B$ for some $B \in \mathcal{B}\}$, then $\mathcal{F}$ is a filter and $\mathcal{B}$ is a base for $\mathcal{F}$ (*check!*). $\mathcal{F}$ is called is called the filter generated by $\mathcal{B}$; it is the smallest filter containing $\mathcal{B}$.

If we begin with a nonempty collection $\mathcal{S}$ of nonempty sets in X with the finite intersection property, then

$\mathcal{B} = \{B : B$ is a finite intersection of sets from $\mathcal{S}\}$ has property (*)

so $\mathcal{B}$ is a base for a filter $\mathcal{F}$ called the filter generated by $\mathcal{S}$; it is smallest filter containing $\mathcal{S}$.

There can be many neighborhood bases $\mathcal{B}_x$ for a neighborhood system $\mathcal{N}_x$ in a space X. Similarly, a filter $\mathcal{F}$ can have many different filter bases $\mathcal{B}$. In particular, notice that $\mathcal{F}$ itself is a filter base – but usually we want to choose the simplest base possible.

Definition 3.2 A filter base $\mathcal{B}$ converges to $x \in X$ if for every $N \in \mathcal{N}_x$, there is a $B \in \mathcal{B}$ such that $B \subseteq N$. In this case we write $\mathcal{B} \to x$. (Since a filter $\mathcal{F}$ is also a filter base, we have also just defined the meaning of $\mathcal{F} \to x$.)

If $\mathcal{F}$ is the filter generated by $\mathcal{B}$, then clearly $\mathcal{B} \to x$ iff $\mathcal{F} \supseteq \mathcal{N}_x$. (**)

The following theorem is very simple. It is stated explicitly just to make sure that there are no confusions.

Theorem 3.3 Let $\mathcal{F}$ be a filter in the space X. For any $x \in X$, the following are equivalent:

i) $\mathcal{F} \to x$
ii) $\mathcal{F} \supseteq \mathcal{N}_x$
iii) $\mathcal{F}$ has a base $\mathcal{B}$ where $\mathcal{B} \to x$.
iv) Every base $\mathcal{B}'$ for $\mathcal{F}$ satisfies $\mathcal{B}' \to x$.

Proof From Definition 3.2 and the observation (**) shows that i) ⇔ ii) ⇔ iii).

iii) ⇒ iv) Suppose $\mathcal{B}$ is a base for $\mathcal{F}$ and that $\mathcal{B} \to x$. Then for each $N \in \mathcal{N}_x$, there is a $B \in \mathcal{B}$ for which $N \supseteq B$. If $\mathcal{B}'$ is another base for $\mathcal{F}$ then, since $B \in \mathcal{F}$, B contains some set $B' \in \mathcal{B}'$. Therefore $N \supseteq B \supseteq B'$, so $\mathcal{B}' \to x$. •

iv) ⇒ i) because $\mathcal{F}$ is a base for $\mathcal{F}$. •

Example 3.4

1) Let $x \in (X, \mathcal{T})$. If $\mathcal{B}_x$ is a neighborhood base at x, then $\mathcal{B}_x \to x$. In particular, $\mathcal{N}_x \to x$.

2) Suppose $\mathcal{F}$ is a filter in X. If $A \subseteq X$ and $A \cap F \neq \emptyset$ for all $F \in \mathcal{F}$, then the collection $\mathcal{F} \cup \{A\} = \mathcal{S}$ is a nonempty collection with the finite intersection property. Therefore $\mathcal{S}$ generates a filter $\mathcal{F}' \supseteq \mathcal{F}$. So: if $A \notin \mathcal{F}$ and $A \cap F \neq \emptyset$ for all $F \in \mathcal{F}$, we can enlarge $\mathcal{F}$ to a filter $\mathcal{F}'$ that also contains the set A.

3) If $x_0 \in X$, then $\mathcal{F} = \{A \subseteq X : x_0 \in A\}$ is a filter. Since $\mathcal{F} \supseteq \mathcal{N}_{x_0}$, we have $\mathcal{F} \to x_0$. The simplest base for $\mathcal{F}$ is $\mathcal{B} = \{\{x_0\}\}$ and $\mathcal{B} \to x_0$.

Suppose $B \notin \mathcal{F}$. Then $x_0 \notin B$ so $\mathcal{F}$ cannot be enlarged to a filter $\mathcal{F}'$ that also contains the set B – such an $\mathcal{F}'$ would contain both $\{x_0\}$ and B, and therefore also contain $\emptyset = \{x_0\} \cap B$, which is impossible. (*This also follows from Theorem 3.5, below.*)

Therefore, $\mathcal{F}$ is a maximal filter in X. A maximal filter is called an ultrafilter.

For each $x_0 \in X$, $\mathcal{F} = \{A \subseteq X : x_0 \in A\}$ is called a trivial ultrafilter. In general, it takes some more work (and AC) to decide whether a set X also contains "nontrivial" ultrafilters – that is, ultrafilters not of this form. (*See Theorem 5.4 and the examples that follow*).

The next theorem gives us a simple characterization of ultrafilters. It is completely set-theoretic, not topological.

Theorem 3.5 A filter $\mathcal{F}$ in X is an ultrafilter iff for every $A \subseteq X$, either $A \in \mathcal{F}$ or $X - A \in \mathcal{F}$.

Proof ($\Rightarrow$) If $A \subseteq X$ and $A \notin \mathcal{F}$, then $A \supseteq F$ is false for every $F \in \mathcal{F}$, so $F \cap (X - A) \neq \emptyset$ for all $F \in \mathcal{F}$. Therefore the collection $\mathcal{S} = \mathcal{F} \cup \{X - A\}$ has the finite intersection property, so $\mathcal{S}$ generates a filter $\mathcal{F}' \supseteq \mathcal{F}$. So if $\mathcal{F}$ is an ultrafilter, we must have $\mathcal{F}' = \mathcal{F}$ and therefore $X - A \in \mathcal{F}$.

($\Leftarrow$) Suppose that for every $A \subseteq X$, either A or $X - A$ is in $\mathcal{F}$. If $\mathcal{F}'$ is a filter and $\mathcal{F}' \supseteq \mathcal{F}$, then $\mathcal{F}' = \mathcal{F}$: for if $A \in \mathcal{F}' - \mathcal{F}$, then $X - A \in \mathcal{F}$ which would mean $A \cap (X - A) = \emptyset \in \mathcal{F}'$. ●

Note: The proof ($\Rightarrow$) *shows that if a filter $\mathcal{F}$ contains neither A nor $X - A$, then $\mathcal{F}$ can be enlarged to a new filter $\mathcal{F}'$ containing either A or $X - A$ – whichever of the two you wish.*

Definition 3.6 A point $x \in X$ is a cluster point for a filter base $\mathcal{B}$ if $N \cap B \neq \emptyset$ for every $N \in \mathcal{N}_x$ and every $B \in \mathcal{B}$. (Since a filter $\mathcal{F}$ is also a filter base, we have also just defined a cluster point for a filter $\mathcal{F}$.)

It is immediate from the definition that x is a cluster point of a filter base $\mathcal{B}$ iff x is in the closure of each set in $\mathcal{B}$, that is, if and only if $x \in \bigcap\{\text{cl}\, B : B \in \mathcal{B}\}$.

Clearly, if $\mathcal{B} \to x$, then x is a cluster point of $\mathcal{B}$ (*explain!*).

Theorem 3.7 Suppose $\mathcal{F}$ is a filter in a space X. For $x \in X$, the following are equivalent:

i) x is a cluster point of $\mathcal{F}$
ii) $x \in \bigcap\{\text{cl}\, F : F \in \mathcal{F}\}$
iii) there is a filter base $\mathcal{B}$ for $\mathcal{F}$ such that x is a cluster point of $\mathcal{B}$
iv) for every filter base $\mathcal{B}$ for $\mathcal{F}$, x is a cluster point of $\mathcal{B}$.

Proof It is clear from the definition and the following remarks that i) $\Leftrightarrow$ ii) $\Leftrightarrow$ iii).

iii) $\Rightarrow$ iv) Suppose x is a cluster point of $\mathcal{B}$ and $\mathcal{B}'$ is another base for $\mathcal{F}$. If $B' \in \mathcal{B}' \subseteq \mathcal{F}$ then $B' \supseteq B$ for some set $B \in \mathcal{B}$ – because $\mathcal{B}$ is a base for $\mathcal{F}$. But every neighborhood of x intersects B, so every neighborhood of x also intersects B'. Therefore x is a cluster point of $\mathcal{B}'$.

iv) $\Rightarrow$ i) This follows since $\mathcal{F}$ is a base for $\mathcal{F}$. •

Example 3.8

1) Let $\mathcal{F} = \{F \subseteq \mathbb{R} : F \supseteq [0,1]\}$. Each real number r has a neighborhood N for which $N \not\supseteq [0,1]$ and therefore $N \notin \mathcal{F}$. So $\mathcal{F} \not\supseteq \mathcal{N}_r$ so $\mathcal{F} \not\to r$.

If $r \in [0,1]$, then $r \in A \subseteq \text{cl}\, A$ for each $A \in \mathcal{F}$, so r is a cluster point of $\mathcal{F}$.

If $r \notin [0,1]$, then $N = \mathbb{R} - [0,1]$ is a neighborhood of r disjoint from $[0,1] \in \mathcal{F}$. Therefore r is not a cluster point of $\mathcal{F}$. So the set of cluster points of $\mathcal{F}$ is precisely the interval $[0,1]$.

2) Let $\mathbb{N}$ have the usual topology. $\mathcal{F} = \{A \subseteq \mathbb{N} : \mathbb{N} - A \text{ is finite}\}$ is a filter (*check!*). For each $n \in \mathbb{N}$, we have $F = \{n+1, n+2, ...\} \in \mathcal{F}$ and $n \notin F = \text{cl}\, F$. Therefore n is not a cluster point of $\mathcal{F}$. And since $\mathcal{F}$ has no cluster points, certainly $\mathcal{F}$ does not converge.

3) Let $\mathcal{F}$ be a filter in a space X. If $\mathcal{F} \to x$ and $\mathcal{F} \to y$, then $\mathcal{F} \supseteq \mathcal{N}_x$ and $\mathcal{F} \supseteq \mathcal{N}_y$. Since $\emptyset \notin \mathcal{F}$, every neighborhood of x must intersect every neighborhood of y. Therefore, if X is Hausdorff, we must have $x = y$, that is, a filter in a Hausdorff space can have at most one limit. (*In fact, X is Hausdorff iff every filter in $\mathcal{F}$ in X has at most one limit. We could prove this directly,here and now (try it!). But this fact follows later "for free" from Theorem 2.6 via the duality between nets and filters: see Corollary 4.5 below.*)

4. The Relationship Between Nets and Filters

Nets and filters are dual to each other in a natural way – it is possible to move back-and-forth between them. Although this back-and-forth process is not perfectly symmetric, it is still very useful because the process preserves limits and cluster points.

Definition 4.1 Let $(\Lambda, \leq)$ be a directed set. For a net (x_λ) in X, we define its associated filter $\mathcal{F}$ (or, the filter generated by (x_λ)) : :

Let $T_\lambda = \{x_\mu : \mu \in \Lambda, \mu \geq \lambda\} =$ the λ^{th} tail of the net, and let $\mathcal{B} = \{T_\lambda : \lambda \in \Lambda\}$. $\mathcal{B}$ is nonempty since $\Lambda \neq \emptyset$, and each $T_\lambda \neq \emptyset$ since $x_\lambda \in T_\lambda$. Moreover, if $\lambda_3 \geq \lambda_1$ and $\lambda_3 \geq \lambda_2$, then $T_{\lambda_1} \cap T_{\lambda_2} \supseteq T_{\lambda_3}$. Therefore the collection of tails $\mathcal{B} = \{T_\lambda : \lambda \in \Lambda\}$ is a filter base and the filter it generates is the associated filter of (x_λ).

Definition 4.2 For a filter $\mathcal{F}$ in X, we define its associated net (or, the net generated by $\mathcal{F}$) :

> Let $\Lambda = \{(x, F) : x \in F \in \mathcal{F}\}$ and define $(x, F) \leq (x', F')$ if $F \supseteq F'$. Then $(\Lambda, \leq)$ is directed. (*Why?*)) The net $f : \Lambda \to X$ defined by $f((x, F)) = x$ is the associated net of $\mathcal{F}$.
>
> *Notice that* $(\Lambda, \leq)$ *is not a poset: for example if* $x \neq x' \in F \in \mathcal{F}$*, then* $(x, F) \geq (x', F)$ *and* $(x', F) \geq (x, F)$ *but* $(x, F) \neq (x', F)$.

If we begin with a filter $\mathcal{F}$, form its associated net, and then form its associated filter $\mathcal{F}'$, we are back to where we started:

$$\mathcal{F} \quad \rightsquigarrow \quad \text{associated net } (x_\lambda) \quad \rightsquigarrow \quad \text{associated filter } \mathcal{F}' = \mathcal{F}$$

> To see this: a base for $\mathcal{F}'$ is the collection of tails $\{T_{(x,F)} : (x, F) \in \Lambda\}$. But $T_{(x,F)} = \{f(x', F') : (x', F') \geq (x, F) = \{x' : (x', F') \geq (x, F)\}$ $= \{x' : x' \in F'$ and $F' \subseteq F$, where $F, F' \in \mathcal{F}\} = \bigcup\{F' : F' \subseteq F \in \mathcal{F}\} = \mathcal{F}$. So the collection of tails is $\mathcal{F}$ itself !

On the other hand, if we begin with a net $(x_\lambda : \lambda \in \Lambda\}$, form its associated filter $\mathcal{F}$, and then form the net associated with $\mathcal{F}$, we do not return to the original net (x_λ) :

$$(x_\lambda) \quad \rightsquigarrow \quad \text{associated filter } \mathcal{F} \quad \rightsquigarrow \quad \text{net associated to } \mathcal{F} \neq (x_\lambda)$$

> First of all, the net associated with $\mathcal{F}$ has for its directed set $\Lambda' = \{(x, F) : x \in F \in \mathcal{F}\} \neq \Lambda$. But the problem runs even deeper than that:
>
> Consider the net (x_n) in $\mathbb{R}$ with directed set $\mathbb{N}$, where $x_n = 0$ for all n. The collection of tails is $\mathcal{B} = \{\{0\}\}$, which generates the associated $\mathcal{F} = \{A \subseteq \mathbb{R} : 0 \in A\}$ (a trivial ultrafilter). In turn, $\mathcal{F}$ generates a net whose directed set $\Lambda = \{(x, A) : x \in A \in \mathcal{F}\}$ and in Λ, $(0, \{0\})$ is a maximal element. The directed set for the new net is not even order isomorphic to $\mathbb{N}$!

Nevertheless, the following theorem shows that this back-and-forth process between associated nets and associated filters is good enough to be very useful for topological purposes.

Theorem 4.3 In any space X,

> 1) x is a cluster point of a net (x_λ) iff x is a cluster point of the associated filter $\mathcal{F}$.
> 2) x is a cluster point of a filter $\mathcal{F}$ iff x is a cluster point of the associated net (x_λ)

Proof 1) x is a cluster point of (x_λ) iff (x_λ) is frequently in every neighborhood of x
iff $T_\lambda \cap N \neq \emptyset$ for every tail T_λ and every $N \in \mathcal{N}_x$
iff x is a cluster point of the filter base $\mathcal{B} = \{T_\lambda : \lambda \in \Lambda\}$
iff x is a cluster point of $\mathcal{F}$.

2) We use 1). Given a filter $\mathcal{F}$, consider its associated net (x_λ). By 1), x is a cluster point of (x_λ) iff x is a cluster point of its associated filter $\mathcal{F}'$. But $\mathcal{F}' = \mathcal{F}$. ∙

> *Notice: if we had somehow proved part 2) first, and then tried to use 2) to prove part 1), we would run into trouble − we would end up looking at a net different from the one we started*

with. The "asymmetry" in the back-and-forth process between nets and filters shows up here. .

Theorem 4.4 In any space X,

1) a net $(x_\lambda) \to x$ iff its associated filter $\mathcal{F} \to x$.
2) a filter $\mathcal{F} \to x$ iff its associated net $(x_\lambda) \to x$.

Proof The proof is left as an exercise. As in Theorem 4.3, part 2) follows "for free" from 1) using the duality between nets and filters. •

Corollary 4.5 A space X is Hausdorff iff every filter $\mathcal{F}$ has at most one limit in X.

Proof The result follows immediately by duality: use Theorem 4.4 and Theorem 2.6. •

The following theorem shows how subnets and larger filters are related: <u>subnets generate larger filters and vice-versa</u>.

Theorem 4.6 Suppose $\mathcal{F}$ is a filter in X generated by some net $f : \Lambda \to X$. (*This is not a restriction on $\mathcal{F}$ because every filter is generated by a net: for example, by its associated net.*)

1) Each subnet of f generates a filter $\mathcal{G} \supseteq \mathcal{F}$
2) Each filter $\mathcal{G} \supseteq \mathcal{F}$ is generated by a subnet of f.

Proof 1) A base for $\mathcal{F}$ is $\{B_\lambda : \lambda \in \Lambda\}$, where B_λ is the λ^{th} tail of f. Suppose $\phi : M \to \Lambda$ and that $f \circ \phi : M \to X$ is a subnet of f. The filter base $\{C_\mu : \mu \in M\}$, where C_μ is the μ^{th} tail of $f \circ \phi$, generates a filter $\mathcal{G}$.

Let $F \in \mathcal{F}$. Then $F \supseteq B_{\lambda_0}$ for some $\lambda_0 \in \Lambda$. Pick $\mu_0 \in M$ so that $\mu \geq \mu_0 \Rightarrow \phi(\mu) \geq \lambda_0$. Then the subnet tail $C_{\mu_0} = \{f \circ \phi(\mu) : \mu \geq \mu_0\} \subseteq B_{\lambda_0} \subseteq F$, so F is in the filter $\mathcal{G}$ generated by $f \circ \phi$. Therefore $\mathcal{F} \subseteq \mathcal{G}$.

2) Conversely, let $\mathcal{G}$ be a filter containing $\mathcal{F}$. We claim $\mathcal{G}$ is generated by a subnet of f. Let B_λ denote the λ^{th} tail of f. We claim that $\mathcal{B} = \{G \cap B_\lambda : G \in \mathcal{G}, \lambda \in \Lambda\}$ is a base for $\mathcal{G}$:

> $\mathcal{B}$ is clearly base for some filter $\mathcal{F}'$ and $\mathcal{G} \subseteq \mathcal{F}'$ (since $G \supseteq G \cap B_\lambda$). On the other hand, each $B_\lambda \in \mathcal{F} \subseteq \mathcal{G}$, so each $G \cap B_\lambda \in \mathcal{G}$ so $\mathcal{F}' \subseteq \mathcal{G}$

Let $M = \{(\xi, G \cap B_\lambda) : G \in \mathcal{G}, \lambda, \xi \in \Lambda, \xi \geq \lambda \text{ and } f(\xi) = x_\xi \in G \cap B_\lambda\}$. (For each $G \cap B_\lambda$ there is at least one such ξ because $G \cap B_\lambda \neq \emptyset$). Order M by defining $(\xi, G \cap B_\lambda) \leq (\xi', G' \cap B_{\lambda'})$ if $\lambda' \geq \lambda$ and $G' \cap B_{\lambda'} \subseteq G \cap B_\lambda$. It is easy to check that $\leq$ is transitive and reflexive. In fact, $(M, \leq)$ is a directed set:

> If $\lambda'' \geq \lambda, \lambda'$, we have $B_{\lambda''} \subseteq B_{\lambda'} \cap B_\lambda$. Since $G'' = G \cap G' \in \mathcal{G}$, we have $G'' \cap B_{\lambda''} \neq \emptyset$, so there exists a $\xi'' \geq \lambda''$ such that $f(\xi'') = x_{\xi''} \in G'' \cap B_{\lambda''}$. Thus $(\xi'', G'' \cap B_{\lambda''}) \in M$, and we have $(\xi'', G'' \cap B_{\lambda''}) \geq (\xi, G \cap B_\lambda)$ and $(\xi'', G'' \cap B_{\lambda''}) \geq (\xi', G' \cap B_{\lambda'})$.

Define $\phi : M \to \Lambda$ by $\phi((\xi, G \cap B_\lambda)) = \xi$. Then $f \circ \phi$ is a subnet of f :

Suppose $\lambda_0 \in \Lambda$. Pick any $G_0 \in \mathcal{G}$ and any $\xi_0 \geq \lambda_0$ such that $f(\xi_0) = x_{\xi_0} \in G_0 \cap B_{\lambda_0}$. Let $\mu_0 = (\xi_0, G_0 \cap B_{\lambda_0}) \in M$. If $\mu = (\xi, G \cap B_\lambda) \geq \mu_0$ in M, then $\phi(\mu) = \xi \geq \xi_0 \geq \lambda_0$.

The filter generated by the subnet $f \circ \phi$ has the set of tails C_{μ_0} a base. If $\mu_0 = (\xi_0, G_0 \cap B_{\lambda_0}) \in M$, we claim that the tail $C_{\mu_0} = G_0 \cap B_{\lambda_0}$. <u>If this is true, we are done</u>, since the sets of this form, as noted above, are a base for the filter $\mathcal{G}$.

$C_{\mu_0} = \{f \circ \phi(\mu) : \mu \geq \mu_0\}$. If $\mu \in M$ and $\mu = (\xi, G \cap B_\lambda) \geq \mu_0 = (\xi_0, G_0 \cap B_{\lambda_0})$, then $G \cap B_\lambda \subseteq G_0 \cap B_{\lambda_0}$, so $f \circ \phi(\mu) = f(\xi) \in G \cap B_\lambda \subseteq G_0 \cap B_{\lambda_0}$.
Therefore $C_{\mu_0} \subseteq G_0 \cap B_{\lambda_0}$.

Conversely, if $y \in G_0 \cap B_{\lambda_0}$, then $y = f(\xi) = x_\xi$ for some $\xi \in \Lambda$, $\xi \geq \lambda_0$.
Then $\mu = (\xi, G_0 \cap B_\xi) \in M$ and $\mu \geq \mu_0$ so $y = f(\xi) = f \circ \phi(\mu) \in C_{\mu_0}$.
Therefore $C_{\mu_0} \supseteq G_0 \cap B_{\lambda_0}$. •

The remark following Definition 2.10 is relevant here. To prove part 2) of Theorem 4.6: we need the more generous definition of subnets to be sure that we have "enough" subnets to generate all possible filters containing $\mathcal{F}$. In particular, the subnet defined in proving part 2) is <u>not</u> a "subnet" using the more restrictive definition for subnet.

Theorem 4.7 A point $x \in X$ is a cluster point of the filter $\mathcal{F}$ iff there exists a filter $\mathcal{G} \supseteq \mathcal{F}$ such that $\mathcal{G} \to x$.

Proof ($\Rightarrow$) If such a filter $\mathcal{G}$ exists, then $\mathcal{N}_x \subseteq \mathcal{G}$. Therefore each neighborhood of x intersects each set in $\mathcal{G}$, and therefore, in particular, intersects each set in $\mathcal{F}$. Therefore x is a cluster point of $\mathcal{F}$.

($\Leftarrow$) If x is a cluster point of $\mathcal{F}$, then the set $\mathcal{B} = \{N \cap F : N \in \mathcal{N}_x \text{ and } F \in \mathcal{F}\}$ is a filter base that generates a filter $\mathcal{G}$. For $F \in \mathcal{F}$, we have $F = X \cap F \in \mathcal{B}$, so $\mathcal{F} \subseteq \mathcal{G}$; and for each $N \in \mathcal{N}_x$, we have $N = N \cap X \in \mathcal{B}$, so $\mathcal{N}_x \subseteq \mathcal{G}$ and therefore $\mathcal{G} \to x$. •

Corollary 4.8 x is a cluster point of the net (x_λ) in X iff there exists a subnet $(x_{\lambda_\mu}) \to x$.
(*This result was stated earlier, without proof, as Theorem 2.11.*)

Proof x is a cluster point of (x_λ) iff x is a cluster point of the associated filter $\mathcal{F}$
iff there exists a filter $\mathcal{G} \supseteq \mathcal{F}$ with $\mathcal{G} \to x$
iff (x_λ) has a subnet converging to x. •

Example 4.9 Think about each of the following parallel statements about nets and filters. Which ones follow "by duality" from the others?

1) If $x_\lambda = a$ for all $\lambda \geq \lambda_0$, then $(x_\lambda) \to a$	1′) If $\mathcal{F}$ consists of all sets containing a, then $\mathcal{F} \to a$

2) $(x_\lambda) \to a$ iff every subnet converges to a	2$'$) If $\mathcal{F} \to a$ iff $\mathcal{G} \to a$ for every filter $\mathcal{G} \supseteq \mathcal{F}$
3) If a subnet of (x_λ) has cluster point a, then a is a cluster point of (x_λ)	3$'$) If a is a cluster point of $\mathcal{G}$ and $\mathcal{G} \supseteq \mathcal{F}$, then a is a cluster point of $\mathcal{F}$.

5. Ultrafilters and Universal Nets

Nets and filters are objects that can be defined in any set X. The same is true for ultrafilters and universal nets. No topology is needed unless we want to talk about convergence, cluster points and other ideas that involve "nearness." So many results in this section are purely set-theoretic.

Definition 5.1 An ultrafilter $\mathcal{U}$ in X is called fixed (or trivial) if $\bigcap \mathcal{U} \neq \emptyset$. $\mathcal{U}$ is called free (or nontrivial) if $\bigcap \mathcal{U} = \emptyset$.

For example (*see Example 3.4*)) the ultrafilter $\mathcal{F} = \{A \subseteq X : p \in A\}$ is fixed (trivial).

Theorem 5.2 For an ultrafilter $\mathcal{U}$ in X, then the following are equivalent.

i) for some $p \in X$, $\mathcal{U} = \{A \subseteq X : p \in A\}$

ii) $\bigcap \mathcal{U} = \{p\}$ for some $p \in X$

iii) $\mathcal{U}$ is fixed – that is, $\bigcap \mathcal{U} \neq \emptyset$.

Proof It is clear that i) $\Rightarrow$ ii) $\Rightarrow$ iii)

iii) $\Rightarrow$ i) Suppose i) is false. Then $\{x\} \notin \mathcal{U}$ for every $x \in X$ (*why?*) so by Theorem 3.5, $X - \{x\} \in \mathcal{U}$ for every x. Therefore $\bigcap \mathcal{U} \subseteq \bigcap \{X - \{x\} : x \in X\} = \emptyset$. •

Example 5.3 If $\mathcal{F}$ is a filter and $\bigcap \mathcal{F} \neq \emptyset$, $\mathcal{F}$ might not be an ultrafilter.

Let $\mathcal{F}$ be the filter in $\mathbb{N}$ generated by $\{B_n : n \in \mathbb{N}\}$, where $B_n = \{1\} \cup \{k : k \geq n\}$. Then $\bigcap \mathcal{F} = \bigcap_{n=1}^{\infty} B_n = \{1\}$. Since neither $\mathbb{E} = \{2, 4, 6, ...\} \subseteq \mathbb{N}$ nor $\mathbb{N} - \mathbb{E}$ contains one of the sets B_n, neither $\mathbb{E}$ nor $\mathbb{N} - \mathbb{E}$ is in $\mathcal{F}$. Therefore $\mathcal{F}$ is not an ultrafilter.

$\mathcal{F} \cup \{\mathbb{E}\}$ is a filter base that generates a filter $\mathcal{F}'$ strictly larger than $\mathcal{F}$. Similarly, $\mathcal{F} \cup \{\mathbb{N} - \mathbb{E}\}$ generates a filter $\mathcal{F}''$ strictly larger than $\mathcal{F}$. Then $\mathcal{F}' \neq \mathcal{F}''$ (because $\mathbb{E}$ and $\mathbb{N} - \mathbb{E}$ cannot be in the same filter). So $\mathcal{F}$ can be "enlarged" in at least two different ways.

Theorem 5.4 If $\mathcal{F}$ is a filter in X, then $\mathcal{F} \subseteq \mathcal{U}$ for some ultrafilter $\mathcal{U}$.

Proof Let $\mathcal{P} = \{\mathcal{G} : \mathcal{G}$ is a filter and $\mathcal{G} \supseteq \mathcal{F}\}$, ordered by inclusion. $\mathcal{P}$ is a nonempty poset since $\mathcal{F} \in \mathcal{P}$.. Let $\{\mathcal{G}_\alpha : \alpha \in I\}$ be a chain in $\mathcal{P}$. We claim that $\bigcup \mathcal{G}_\alpha \in \mathcal{P}$.

Clearly, $\bigcup \mathcal{G}_\alpha \supseteq \mathcal{F}$; and $\bigcup \mathcal{G}_\alpha$ is a filter:

$\bigcup \mathcal{G}_\alpha \neq \emptyset$ since each $\mathcal{G}_\alpha \neq \emptyset$, and each set in $\bigcup \mathcal{G}_\alpha$ is nonempty.
If $A, B \in \bigcup \mathcal{G}_\alpha$, then both A, B are in a single filter $\mathcal{G}_{\alpha_0}$. Therefore $A \cap B \in \mathcal{G}_{\alpha_0}$, so $A \cap B \in \bigcup \mathcal{G}_\alpha$. In addition, if $C \supseteq A$, then $C \in \mathcal{G}_{\alpha_0}$, so $C \in \bigcup \mathcal{G}_\alpha$.

Therefore the chain $\{\mathcal{G}_\alpha : \alpha \in I\}$ has an upper bound $\bigcup \mathcal{G}_\alpha$ in $\mathcal{P}$. By Zorn's Lemma, $\mathcal{P}$ contains a maximal element $\mathcal{U}$. ●

Example 5.5 For each $n \in \mathbb{N}$, there is a fixed (trivial) ultrafilter $\mathcal{U}_n = \{A \subseteq \mathbb{N} : n \in A\}$. It is the ultrafilter for which $\bigcap \mathcal{U}_n = \{n\}$. By Theorem 5.2, the $\mathcal{U}_n$'s are <u>the only</u> fixed ultrafilters in $\mathbb{N}$. There are also nontrivial (free) ultrafilters in $\mathbb{N}$:

Let $B_n = \{k \in \mathbb{N} : k \geq n\}$. The collection $\mathcal{B} = \{B_n : n \in \mathbb{N}\}$ is a filter base and $\mathcal{B}$ generates a filter $\mathcal{F}$. By Theorem 5.4 there is an ultrafilter $\mathcal{U} \supseteq \mathcal{F}$, and $\bigcap \mathcal{U} \subseteq \bigcap_{n=1}^{\infty} B_n = \emptyset$.

$\mathcal{U}$ is <u>not the only</u> ultrafilter containing $\mathcal{F}$. If $\mathbb{E}$ and $\mathbb{O}$ are the sets of even and odd natural numbers, then $\mathcal{B} \cup \{\mathbb{E}\}$ and $\mathcal{B} \cup \{\mathbb{O}\}$ are filter bases that generate filters $\mathcal{F}'$ and $\mathcal{F}''$. Then there are ultrafilters $\mathcal{U}' \supseteq \mathcal{F}'$ and $\mathcal{U}'' \supseteq \mathcal{F}''$. Since $\mathbb{E} \in \mathcal{U}'$, $\mathbb{O} \in \mathcal{U}''$ and $\mathbb{E} \cap \mathbb{O} = \emptyset$, we have $\mathcal{U}' \neq \mathcal{U}''$. Since $\mathcal{B} \subseteq \mathcal{U}'$ and $\mathcal{B} \subseteq \mathcal{U}''$, we know that $\bigcap \mathcal{U}' = \emptyset$ and $\bigcap \mathcal{U}'' = \emptyset$, so $\mathcal{U}'$ and $\mathcal{U}'$ are free ultrafilters.

It is a nice exercise to prove that <u>if there is only one ultrafilter $\mathcal{U}$ containing</u> $\mathcal{F}$, then $\mathcal{F} = \mathcal{U}$: that is, if $\mathcal{F}$ is <u>not</u> an ultrafilter, then $\mathcal{F}$ can always be enlarged to an ultrafilter in more than one way.

If $\mathcal{F}$ is a filter in $\mathbb{N}$, then $\mathcal{F} \in \mathcal{P}(\mathcal{P}(\mathbb{N}))$, so there are <u>at most</u> $|\mathcal{P}(\mathcal{P}(\mathbb{N}))| = 2^{2^{\aleph_0}} = 2^c$ filters in $\mathbb{N}$.

<u>Comment without proof</u>: There are <u>exactly</u> 2^c different filters in $\mathbb{N}$. In fact, there are 2^c different ultrafilters in $\mathbb{N}$, and since $\mathbb{N}$ contains only countably many fixed ultrafilters $\mathcal{U}_n$, there are in fact 2^c free ultrafilters !

Theorem 5.6 An ultrafilter $\mathcal{U}$ in a space X converges to each of its cluster points.

Proof If x is a cluster point of $\mathcal{U}$, then there is a filter $\mathcal{G} \supseteq \mathcal{U}$ such that $\mathcal{G} \to x$. But $\mathcal{G} = \mathcal{U}$ since $\mathcal{U}$ is an ultrafilter. ●

Corollary 5.7 An ultrafilter $\mathcal{U}$ in a T_2 space X has at most one cluster point.

We now define the analogue of ultrafilters for nets.

Definition 5.8 A net (x_λ) in X is called <u>universal</u> <u>net</u> (or <u>ultranet</u>) if for every $A \subseteq X$, (x_λ) is either eventually in A or eventually in $X - A$.

For example, a net (x_λ) which is eventually constant, say $x_\lambda = p$ for all $\lambda \geq \lambda_0$, is a universal net. It is referred to as a trivial universal net because its associated filter is the trivial ultrafilter $\{A \subseteq X : p \in A\}$.

Theorem 5.9 In any set X,

1) a net (x_λ) is universal iff its associated filter is an ultrafilter, and
2) a filter $\mathcal{F}$ is an ultrafilter iff its associated net is universal.

Proof 1) (x_λ) is universal iff for every $A \subseteq X$, (x_λ) is eventually in A or $X - A$
iff for every $A \subseteq X$, A or $X - A$ contains a tail of (x_λ)
iff for every $A \subseteq X$, A or $X - A$ is in the associated filter
iff the associated filter is an ultrafilter.

2) We use duality. For a given a filter $\mathcal{F}$, consider its associated net (x_λ).
By a), (x_λ) is universal iff its associated filter $\mathcal{F}'$ is an ultrafilter. But $\mathcal{F}' = \mathcal{F}$. •

Corollary 5.10 In any space X,

1) a subnet of a universal net is universal
2) a universal net converges to each of its cluster points
3) every net has a universal subnet.

Proof 1) If (x_λ) is universal, then (x_λ) generates an ultrafilter $\mathcal{U}$. By Theorem 4.6, each subnet (x_{λ_μ}) generates a filter $\mathcal{G} \supseteq \mathcal{U}$. So $\mathcal{G} = \mathcal{U}$. Because the filter associated to (x_{λ_μ}) is an ultrafilter, (x_{λ_μ}) is universal.

2) If x is a cluster point of the universal net (x_λ), then x is a cluster point of the associated ultrafilter $\mathcal{U}$. By Theorem 5.6 $\mathcal{U} \to x$ and therefore $(x_\lambda) \to x$.

3) Let $\mathcal{F}$ be the associated filter for the net (x_λ) and let $\mathcal{U}$ be an ultrafilter containing $\mathcal{F}$. By Theorem 4.6(2), $\mathcal{U}$ is generated by a subnet (x_{λ_μ}) of (x_λ). Since the filter associated with (x_{λ_μ}) is an ultrafilter, (x_{λ_μ}) is a universal net. •

Corollary 5.11 A universal net in a T_2 space has at most one cluster point.

Both nets and filters are sufficient to describe the topology in any space, so we should be able to use them to describe continuous functions.

Theorem 5.12 Suppose X and Y are topological spaces, $f : X \to Y$ and $a \in X$. The following are equivalent:

1) f is continuous at a
2) whenever a net $(x_\lambda) \to a$ in X, then $(f(x_\lambda)) \to f(a)$ in Y
3) whenever a universal net $(x_\lambda) \to a$ in X, then $(f(x_\lambda)) \to f(a)$ in Y
4) whenever a is a cluster point of a net (x_λ) in X, then $f(a)$ is a cluster point of $(f(x_\lambda))$

in Y

5) whenever a filter base $\mathcal{B} \to a$ in X, then the filter base $f[\mathcal{B}] = \{f[B] : B \in \mathcal{B}\} \to f(a)$ in Y
6) whenever an ultrafilter $\mathcal{U} \to a$ in X, then the filter base $f[\mathcal{U}] \to f(a)$ in Y.
7) whenever a is a cluster point of a filter base $\mathcal{B}$ in X, then $f(a)$ is a cluster point of the filter base $f[\mathcal{B}]$ in Y.

Proof 1) $\Rightarrow$ 2) Suppose V is a neighborhood of $f(a)$. Since f is continuous at a, there is a neighborhood U of a such that $f[U] \subseteq V$. If $(x_\lambda) \to a$, then (x_λ) is eventually in U so $(f(x_\lambda))$ is eventually in V. Therefore $(f(x_\lambda)) \to f(a)$.

2) $\Rightarrow$ 3) This is immediate.

3) $\Rightarrow$ 4) If a is a cluster point of (x_λ), then there is a subnet $(x_{\lambda_\mu}) \to a$. Let $(x_{\lambda_{\mu_\nu}})$ be a universal subnet of (x_{λ_μ}). Then $(x_{\lambda_{\mu_\nu}}) \to a$ and by iii), $(f(x_{\lambda_{\mu_\nu}})) \to f(a)$. Then $(f(x_\lambda))$ has a subnet converging to $f(a)$, so $f(a)$ is a cluster point of $(f(x_\lambda))$.

4) $\Rightarrow$ 1) Suppose f is not continuous at a. Then there is a neighborhood V of $f(a)$ such that $f[N] \not\subseteq V$ for every $N \in \mathcal{N}_a$. Let $\Lambda = \mathcal{N}_a$, ordered by reverse inclusion, and define a net $g : \Lambda \to X$ by $g(N) = x_N =$ a point in N for which $f(x_N) \notin V$. Since $(x_N) \to a$, (x_N) has a cluster point at a. But the net $(f(x_N))$ does not have a cluster point at $f(a)$ because $(f(x_N))$ is <u>never</u> in V. Therefore 4) fails.

Knowing that 1)-4) are equivalent, we could show that each of 2)-4) is equivalent to its filter counterpart – e.g., that 2) $\Leftrightarrow$ 5), etc. This involves a little more than simply saying "by duality" because, in each case, the function f also comes into the argument. Instead, for practice, we will show directly that 1) $\Rightarrow$ 5) $\Rightarrow$ 6) $\Rightarrow$ 7) $\Rightarrow$ 1).

It is easy to check that if $\mathcal{B}$ is a filter base in X, then $f[\mathcal{B}] = \{f[B] : B \in \mathcal{B}\}$ is a filter base in Y.

1) $\Rightarrow$ 5) Suppose $\mathcal{B} \to a$ in X and let $f[B] \in f[\mathcal{B}]$. If N is any neighborhood of $f(a)$ in Y, then by continuity $f^{-1}[N]$ is a neighborhood of a in X. Therefore $f^{-1}[N] \supseteq B$ for some $B \in \mathcal{B}$, so $N \supseteq f[f^{-1}[N]] \supseteq f[B]$. Therefore $f(\mathcal{B}) \to f(a)$.

5) $\Rightarrow$ 6) This is immediate.

6) $\Rightarrow$ 7) Suppose a is a cluster point of the filter base $\mathcal{B}$. We can choose an ultrafilter $\mathcal{U} \supseteq \mathcal{B}$ with $\mathcal{U} \to a$. By 6), $f[\mathcal{U}] \to f(a)$. Therefore the filter $\mathcal{U}'$ generated by $f[\mathcal{U}]$ converges to $f(a)$, so $\mathcal{U}' \supseteq \mathcal{N}_{f(a)}$. Since $f[\mathcal{B}] \subseteq \mathcal{U}'$, each set in $f[\mathcal{B}]$ intersects every set in $\mathcal{N}_{f(a)}$. Therefore $f(a)$ is a cluster point of $f[\mathcal{B}]$.

7) $\Rightarrow$ 1) Suppose f is not continuous at a. Then there is a neighborhood V of $f(a)$ such that $f[N] \not\subseteq V$ for all $N \in \mathcal{N}_a$, that is, $N - f^{-1}[V] \neq \emptyset$ for all $N \in \mathcal{N}_a$. The collection $\mathcal{B} = \{N - f^{-1}[V] : N \in \mathcal{N}_a\}$ is a filter base (*why?*) that has a cluster point at a. However $f[\mathcal{B}]$ does not cluster at $f(a)$ since no set in $f[\mathcal{B}]$ intersects V. $\bullet$

Corollary 5.13 Let (x_λ) be a net in the product $X = \prod\{X_\alpha : \alpha \in A\}$. Then $(x_\lambda) \to x$ in X iff $(\pi_\alpha(x_\lambda)) \to \pi_\alpha(x)$ in X_α for each $\alpha \in A$.

Proof If $(x_\lambda) \to x \in X$, then by continuity $(\pi_\alpha(x_\lambda)) \to \pi_\alpha(x) \in X_\alpha$ for every $\alpha \in A$.

Conversely, suppose $(\pi_\alpha(x_\lambda)) \to \pi_\alpha(x)$ for every α and let $U = < U_{\alpha_1}, U_{\alpha_2}, ..., U_{\alpha_n} >$ be a basic open set containing x in X. For each $i = 1, ..., n$ we have $(\pi_{\alpha_i}(x_\lambda)) \to \pi_{\alpha_i}(x) \in U_{\alpha_i}$. Therefore we can choose a $\lambda_i \in \Lambda$ so that $\pi_{\alpha_i}(x_\lambda) \in U_{\alpha_i}$ when $\lambda \geq \lambda_i$. Pick $\lambda^* \geq \lambda_1, ..., \lambda_n$. If $\lambda \geq \lambda^*$, we have $\pi_{\alpha_i}(x_\lambda) \in U_{\alpha_i}$ for each $i = 1, ..., n$. Therefore $x_\lambda \in U$ for $\lambda \geq \lambda^*$ so $(x_\lambda) \to x$. •

6. Compactness Revisited and The Tychonoff Product Theorem

With nets and filters available, we can give a nice characterization of compact spaces in terms of convergence.

Theorem 6.1 For any space X, the following are equivalent:

1) X is compact
2) if $\mathcal{F}$ is a family of closed sets with the finite intersection property, then $\bigcap \mathcal{F} \neq \emptyset$
3) every filter $\mathcal{F}$ has a cluster point
4) every filter can be enlarged to a filter that converges
5) every net has a cluster point
6) every net has a convergent subnet
7) every universal net converges
8) every ultrafilter converges.

Proof We proved earlier that 1) and 2) are equivalent (*see Theorem IV.8.4*).

2) $\Rightarrow$ 3) If $\mathcal{F}$ is a filter in X, then $\mathcal{F}$ has the finite intersection property, so $\{\text{cl}\, F : F \in \mathcal{F}\}$ is a family of closed sets also with the finite intersection property. By ii), $\exists\, x \in \bigcap\{\text{cl}\, F : F \in \mathcal{F}\}$ so x is a cluster point of $\mathcal{F}$.

3) $\Rightarrow$ 4) If $\mathcal{F}$ is a filter, then $\mathcal{F}$ has a cluster point x so, by Theorem 4.7, there is a filter $\mathcal{G} \supseteq \mathcal{F}$ such that $\mathcal{G} \to x$.

4) $\Rightarrow$ 5) If (x_λ) is a net, consider the associated filter $\mathcal{F}$. By 4) there is a filter $\mathcal{G} \supseteq \mathcal{F}$ where $\mathcal{G} \to x \in X$. $\mathcal{G}$ is generated by a subnet (x_{λ_μ}) and by duality, $(x_{\lambda_\mu}) \to x$.

5) $\Rightarrow$ 6) If (x_λ) has a cluster point x, then by Corollary 4.8, there is a subnet $(x_{\lambda_\mu}) \to x$.

6) $\Rightarrow$ 7) If (x_λ) is a universal net, then 6) gives that (x_λ) has a subnet that converges to a point x. Then x is a cluster point of (x_λ). Since (x_λ) is universal, $(x_\lambda) \to x$ by Corollary 5.10.

7) $\Rightarrow$ 8) This is immediate from the duality between universal nets and ultrafilters (Theorems 5.9 and 4.4)

8) $\Rightarrow$ 1) Suppose X is not compact and let $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ be an open cover with no finite subcover. Then for any $\alpha_1, \alpha_2, ..., \alpha_n \in A$, $X \neq U_{\alpha_1} \cup U_{\alpha_2} \cup ... \cup U_{\alpha_n}$; so, by complements, $\emptyset \neq (X - U_{\alpha_1}) \cap ... \cap (X - U_{\alpha_n})$. Therefore $\mathcal{S} = \{X - U_\alpha : \alpha \in A\}$ is a collection of closed sets with the finite intersection property. The set of all finite intersections of sets from $\mathcal{S}$ is a filter base which generates a filter $\mathcal{F}$, and we can find an ultrafilter $\mathcal{V} \supseteq \mathcal{F}$.

Every point x is in U_α for some α, so $x \notin X - U_\alpha = \text{cl}\,(X - U_\alpha)$. Therefore x is not a cluster point of $\mathcal{V}$. In particular, this implies $\mathcal{V} \not\to x$. Since x was arbitrary, this means $\mathcal{V}$ does not converge. ●

Theorem 6.1 gives us a fresh look at the relationship between some of the "compactness-like" properties that we defined in Chapter IV:

X is compact iff every <u>net</u> has a convergent <u>subnet</u> (Theorem 6.1)	X is sequentially compact iff every <u>sequence</u> has a convergent <u>subsequence</u> (Definition IV.8.7)
$\searrow$	$\swarrow$

X is countably compact iff
every sequence has a cluster point (Theorem IV.8.10) iff
every <u>sequence</u> has a convergent <u>subnet</u> (Corollary 4.8)

It is now easy to prove that every product of compact spaces is compact.

Theorem 6.2 (Tychonoff Product Theorem) Suppose $X = \prod\{X_\alpha : \alpha \in A\} \neq \emptyset$. X is compact iff each X_α is compact.

Proof For each α, $X_\alpha = \pi_\alpha[X]$ so if X is compact, then each X_α is compact.

Conversely, suppose each X_α is compact and let (x_λ) be a universal net in $X = \prod\{X_\alpha : \alpha \in A\}$.
For each α, $(\pi_\alpha(x_\lambda))$ is a universal net in X_α (*Check: if $A \subseteq X_\alpha$, then (x_λ) is eventually in $\pi_\alpha^{-1}[A]$ or $\pi_\alpha^{-1}[X_\alpha - A]$, so $(\pi_\alpha(x_\lambda))$ eventually in A or $X - A$.*) But X_α is compact, so by Theorem 6.1 $(\pi_\alpha(x_\lambda)) \to$ some point $z_\alpha \in X_\alpha$. Let $z = (z_\alpha) \in X$. By Corollary 5.13, $(x_\lambda) \to z$. Since every universal net in X converges, X is compact by Theorem 6.1. ●

Remark: A quite different approach to the Tychonoff Product Theorem is to show first that a space X is compact iff every open cover by <u>sub</u>basic open sets has a finite subcover. This is called the <u>Alexander Subbase Theorem</u> and the proof is nontrivial: it involves an argument using Zorn's Lemma or one of its equivalents.

After that, it is fairly straightforward to show that any cover of $X = \prod\{X_\alpha : \alpha \in A\}$ by sets of the form $\pi_\alpha^{-1}[U_\alpha]$ has a finite subcover. See Exercise E10.

At this point we restate a result which we stated earlier but without a complete proof (Corollary VII.3.16).

Corollary 6.3 A space X is Tychonoff iff it is homeomorphic to a subspace of a compact Hausdorff space. (*In other words, the Tychonoff spaces are exactly the subspaces of compact Hausdorff spaces.*)

Proof A compact Hausdorff space is T_4, and therefore Tychonoff. Since the Tychonoff property is hereditary, every subspace of a compact T_2 space is Tychonoff.

Conversely, every Tychonoff space X is homeomorphic to a subspace of some cube $[0,1]^m$. This cube is Hausdorff and it is compact by the Tychonoff Product Theorem. •

Remark Suppose X is embedded in some cube $[0,1]^m$. To simplify notation, assume $X \subseteq [0,1]^m$. Then $X \subseteq \text{cl}\, X = K \subseteq [0,1]^m$. K is a compact T_2 space containing X as a dense subspace and K is called a compactification of X. Since every Tychonoff space can be embedded in a cube, we have therefore shown that every Tychonoff space X has a compactification.

Conversely, if K is a compactification of X, then K is Tychonoff and its subspace X is also Tychonoff. Therefore X has a compactification iff X is Tychonoff.

Our proof of the Tychonoff Product Theorem used the Axiom of Choice (AC) in the form of Zorn's Lemma (to get the necessary universal nets or ultrafilters). The following theorem shows that, in fact, the Tychonoff Product Theorem and AC are equivalent. This is perhaps somewhat surprising since AC is a purely set theoretical statement while Tychonoff's Theorem is topological. On the other hand, if "all mathematics can be embedded in set theory" then every mathematical statement is purely set theoretical.

Theorem 6.5 (Kelley, 1950) The Tychonoff Product Theorem implies the Axiom of Choice (so the two are equivalent).

Proof Suppose $\{X_\alpha : \alpha \in A\}$ is a collection of nonempty sets. The Axiom of Choice is equivalent to the statement that $\prod_{\alpha \in A} X_\alpha \neq \emptyset$ (*see Theorem 6.2.2*).

Let $Y_\alpha = X_\alpha \cup \{p\}$ where $p \notin \bigcup_{\alpha \in A} X_\alpha$ and give Y_α the very simple topology $\mathcal{T}_\alpha = \{Y_\alpha, \{p\}, \emptyset\}$. Then Y_α is compact, so $Y = \prod_{\alpha \in A} Y_\alpha$ is compact by the Tychonoff Product Theorem.

$\{p\}$ is open in Y_α, so X_α is closed in Y_α. Therefore $\mathcal{F} = \{\pi_\alpha^{-1}[X_\alpha] : \alpha \in A\}$ is a family of closed sets in Y. We claim that $\mathcal{F}$ has the finite intersection property.

> Suppose $\alpha_1, \alpha_2, ..., \alpha_n \in A$. Since the X_α's are nonempty, there exist points $x_{\alpha_1} \in X_{\alpha_1}, ...,$ $x_{\alpha_n} \in X_{\alpha_n}$.
>
> Define $f : A \to \bigcup_{\alpha \in A} Y_\alpha$ by:
>
> $$\text{for } \alpha \in A,\ f(\alpha) = \begin{cases} x_{\alpha_i} & \text{if } \alpha = \alpha_i \\ p & \text{if } \alpha \neq \alpha_1, \alpha_2, ..., \alpha_n \end{cases}$$
>
> To be more formal $-$ since this is the crucial set-theoretic issue in the argument $-$ we can formally and precisely define f in ZF by:

$$f = \{(\alpha, y) \in A \times \bigcup_{\alpha \in A} Y_\alpha : (\alpha = \alpha_1 \wedge y = x_{\alpha_1}) \vee ... \vee (\alpha = \alpha_n \wedge y = x_{\alpha_n}) \\ \vee\ ((\alpha \neq \alpha_1) \wedge ... \wedge (\alpha \neq \alpha_n)) \wedge y = p)\}$$

Then $f \in \pi_{\alpha_1}^{-1}[X_{\alpha_1}] \cap ... \cap \pi_{\alpha_n}^{-1}[X_{\alpha_n}]$.

Since $\mathcal{F}$ has the finite intersection property and Y is compact, $\bigcap \mathcal{F} \neq \emptyset$. If $g \in \bigcap \mathcal{F}$, then $g \in \prod_{\alpha \in A} X_\alpha$ and therefore $\prod_{\alpha \in A} X_\alpha \neq \emptyset$. •

If $\prod X_\alpha \neq \emptyset$ and any X_α's are noncompact, then $\prod X_\alpha$ is noncompact. And we note that if infinitely many X_α's are noncompact, then $\prod X_\alpha$ is "dramatically" noncompact as the following theorem indicates.

Theorem 6.6 Let $X = \prod\{X_\alpha : \alpha \in A\} \neq \emptyset$. If infinitely many of the X_α's are not compact, then every compact closed subset of X is nowhere dense. (*Thus, all closed compact subsets of X are "very skinny" and "far from" being all of X.*)

Proof Suppose B is a compact closed set in X and that B is not nowhere dense. Then there is a point x and indices $\alpha_1, \alpha_2, ..., \alpha_n$ such that $x \in\ < U_{\alpha_1}, U_{\alpha_2}, ... U_{\alpha_n} >\ \subseteq \text{int}\,(\text{cl}\, B) = \text{int}\, B \subseteq B$. Then $\pi_\alpha[B] = X_\alpha$ for $\alpha \neq \alpha_1, \alpha_2, ..., \alpha_n$, so X_α is compact if $\alpha \neq \alpha_1, \alpha_2, ..., \alpha_n$. •

7. Applications of the Tychonoff Theorem

We have already used the Tychonoff Theorem in several ways (*see, for example, Corollary 6.3 and the remarks following.*) It's a result that is useful in nearly all parts of analysis and topology, although its full generality is not always necessary. In this section we sketch how it can be used in more "unexpected" settings. The following examples also provide additional insight into the significance of compactness.

The Compactness Theorem for Propositional Calculus

Propositional calculus is a part of mathematical logic that deals with expressions such as $p \wedge q$, $p \vee q$, $p \Rightarrow q$, $\sim p$, $(p \Rightarrow q) \vee r$, etc. Letters such as $p, q, r, ...$ are often used to represent "propositions" that can have "truth values" T (true) or F (false). These letters are the "alphabet" for propositional calculus. For example, we could think of p as representing the (false) proposition "$2 + 2 = 5$" or the (true) proposition "$\forall x \in \mathbb{R}\,(x \geq 0 \Rightarrow \exists y \in \mathbb{R}\ (y^2 = x))$". However, p could not represent an expression like "$x = 5$", because this expression has no truth value: it contains a "free variable" x.

In propositional calculus, propositions $p, q, ...$ are thought of as "atoms" − that is, the internal structure of the propositions $p, q, ...$ (such as variables and quantifiers) is ignored. Propositional calculus deals with "basic" or "atomic" propositions such as $p, q, ...$, with compounds built up from them such as $(p \vee q)$ and $\sim(\sim p \vee q)$, and with the relations between their truth values. We want to allow the possibility of infinitely many propositions, so we will use A_1, ... , A_n, ... as our alphabet instead of the letters $p, q, \ldots$ that one usually sees in beginning treatments of propositional calculus.

Here is a slightly more formal description of propositional calculus.

Propositional calculus has an alphabet $\mathcal{A} = \{A_1, A_2, \ldots, A_n, \ldots\}$. We will assume $\mathcal{A}$ is countable, although that restriction is not really necessary for anything we do. Propositional calculus also has connective symbols: (,) , $\vee$, and $\sim$.

Inductively we define a collection $\mathcal{W}$ of well-formed formulas (called wffs, for short) that are the "legal expressions" in propositional calculus:

1) for each n, A_n is a wff
2) if ϕ and ψ are wffs, so are $(\phi \vee \psi)$ and $\sim \phi$.

For example, $((A_{13} \vee A_2) \vee \sim A_4)$ and $\sim (A_1 \vee A_4)$ are wffs but the string of symbols ($\vee$ A_3 $\vee$ $\vee$) is not a wff. If we like, we can add additional connectives $\wedge$ and $\Rightarrow$ to our propositional calculus, defining them as follows:

Given wffs ϕ and ψ,

$$\phi \wedge \psi = \sim (\sim \phi \vee \sim \psi) \quad \text{and}$$
$$\phi \Rightarrow \psi = \psi \vee \sim \phi.$$

Since $\wedge$ and $\Rightarrow$ can be defined in terms of $\vee$ and $\sim$, it is simpler and involves no loss to develop the theory using just the smaller set of connectives.

A truth assignment is a function s that assigns a truth value to each A_n. More formally, $s : \mathcal{A} \to \{T, F\}$, so $s \in \{T, F\}^{\aleph_0} = C$. We give C the product topology. By the Tychonoff Theorem, C is compact (in fact, by Theorem VI.2.19, C is homeomorphic to the Cantor set).

A truth assignment s can be used to assign a unique truth value to every wff in $\mathcal{W}$, that is, we can extend s to a function $\overline{s} : \mathcal{W} \to \{T, F\}$ as follows:

For any wff σ we define:

if $\sigma = A_n$, then $\overline{s}\,(\sigma) = s(\sigma)$

if $\sigma = \phi \vee \psi$, then $\overline{s}\,(\sigma) = \begin{cases} T & \text{if } \overline{s}\,(\phi) = T \text{ or } \overline{s}\,(\psi) = T \\ F & \text{otherwise} \end{cases}$

if $\sigma = \sim \phi$, then $\overline{s}\,(\sigma) = \begin{cases} T & \text{if } \overline{s}\,(\phi) = F \\ F & \text{if } \overline{s}\,(\phi) = T \end{cases}$

We say that a truth assignment s satisfies a wff σ if $\overline{s}\,(\sigma) = T$. A set of wffs Σ is called satisfiable if there exists a truth assignment $s \in \{T, F\}^{\aleph_0}$ such that s satisfies σ for every $\sigma \in \Sigma$.

For example,

$\Sigma = \{A_1, A_1 \vee A_2, A_2\}$ is satisfiable (*We can use any s for which* $s(A_1) = s(A_2) = T$)

$\Sigma = \{\ A_1,\ \sim (A_1 \vee\ \sim A_2)\}$ is not satisfiable   (*If* $s(A_1) = T$*, then*
$\overline{s}\ (\ \sim (A_1 \vee\ \sim A_2)) = F$)

Theorem 7.1 (Compactness Theorem for Propositional Calculus) Let Σ be a set of wffs. If every finite subfamily of Σ is satisfiable, then Σ is satisfiable.

Proof Let $C = \{T, F\}^{\aleph_0}$ and suppose that every finite subset of Σ is satisfiable. Then for each $\sigma \in \Sigma$, $A_\sigma = \{s \in C : s$ satisfies $\sigma\} \neq \emptyset$. We claim that A_σ is closed in C:

Suppose $s \notin A_\sigma$. We need to produce an open set U containing s for which $U \cap A_\sigma = \emptyset$.

For a sufficiently large n, the list $A_1, ..., A_n$ will contain all the letters that occur in σ. Let $U = \{t \in C\colon\ t_i = s_i, i = 1, ..., n\} = \{s_1\} \times ... \times \{s_n\} \times \{T, F\}^{\aleph_0}$. In other words, U is the set of truth assignments that agree with s for all the letters $A_1, ..., A_n$ that may occur in σ. Since s fails to satisfy σ, each $t \in U$ also fails to satisfy σ, so $\sigma \in U \subseteq C - A_\sigma$.

The A_σ's have the finite intersection property $-$ in fact, this is precisely equivalent to saying that every finite subset of Σ is satisfiable. Since C is compact, $\bigcap\{A_\sigma : \sigma \in \Sigma\} \neq \emptyset$, i.e., Σ is satisfiable. •

If we assume the Alexander Subbase Theorem (*see the remarks following Theorem 6.2, as well as Exercise E10*), then we can also prove that the Compactness Theorem 7.1 is equivalent to the statement that $C = \{T, F\}^{\aleph_0}$ is compact.

Suppose the Compactness Theorem 7.1 is true. To show that C is compact it is sufficient, by the Alexander Subbase Theorem, to show that every cover of C by subbasic open sets has a finite subcover. Each subbasic set has the form $\pi_n^{-1}[\{T\}]$ or $\pi_n^{-1}[\{F\}]$.

Taking complements, we see that it is sufficient to show that:

if $\mathcal{F}$ is any family of closed sets with the finite intersection property and each set in $\mathcal{F}$ has form $C - \pi_n^{-1}[\{T\}] = \pi_n^{-1}[\{F\}]$ or $C - \pi_n^{-1}[\{F\}] = \pi_n^{-1}[\{T\}]$, then $\bigcap \mathcal{F} \neq \emptyset$.

Let $\mathcal{F} = \{F_\alpha : \alpha \in A\}$ be any such family. For each $\alpha \in A$, define a wff

$$\sigma_\alpha = \begin{cases} A_n & \text{if } F_\alpha = \pi_n^{-1}[\{T\}] \\ \sim A_n & \text{if } F_\alpha = \pi_n^{-1}[\{F\}] \end{cases}$$

Clearly, σ_α is satisfied precisely by the truth assignments in F_α.

Let $\Sigma = \{\sigma_\alpha : \alpha \in A\}$. Since the F_α's have the finite intersection property, any finite subset of Σ is satisfiable. By the Compactness Theorem, Σ is satisfiable, so $\bigcap \mathcal{F} \neq \emptyset$. •

Note: If the propositional calculus is allowed to have an uncountable alphabet of cardinality m, then the compactness theorem is equivalent to the statement that $\{T, F\}^m$ is compact; the proof requires only minor notational changes.

A "map-coloring" theorem

Imagine that M is a (geographical) map containing infinitely many countries $C_1, C_2, ..., C_n, ...$.
A valid coloring c of M with 4 colors (red, white, blue, and green, say) is a function

$$c : \{C_1, C_2, ..., C_n, ...\} \rightarrow \{R, W, B, G\}$$

such that no two adjacent countries are assigned the same color.

Intuitively, if M doesn't have a valid covering, it must be because some "finite piece" of the map M has a configuration of countries for which a valid coloring can't be done. That is the content of the following theorem.

Theorem 7.2 Suppose M is a (geographical) map with infinitely countries $C_1, ..., C_n, ...$. If every finite submap of M has a valid coloring, then M has a valid coloring. (*Any reasonable definition of "adjacent" and "submap" will work in the proof.*)

Proof Consider set of all colorings $C = \{R, W, B, G\}^{\aleph_0}$ with the product topology. For each finite submap F, let $V_F = \{c \in C : c \text{ is a valid coloring of } F\} \neq \emptyset$. We claim that V_F is closed in C.

Suppose $c \notin V_F$. We need to produce an open set U with $c \in U \subseteq C - V_F$. If n is large enough, the list $C_1, ...C_n$ will include all the countries in the submap F. Then the open set

$$U = \{c(C_1)\} \times ... \times \{c(C_n)\} \times \{R, W, B, G\}^{\aleph_0}$$

works − any coloring in U is invalid because it colors the countries in F the same way c does.

If F_1 and F_2 are finite submaps of M, then so is $F_1 \cup F_2$. Since $F_1 \cup F_2$ has a valid coloring by hypothesis, $V_{F_1 \cup F_2} \subseteq V_{F_1} \cap V_{F_2} \neq \emptyset$. Therefore $\mathcal{F} = \{V_F : F \text{ a finite submap of } M\}$ has the finite intersection property. Since C is compact, $\bigcap \mathcal{F} \neq \emptyset$, and any $c \in \bigcap \mathcal{F}$ is a valid covering of the whole map M. •

(*It is clear that a nearly identical proof would work for any finite number of colors and for maps with uncountably many countries.*)

Exercises

E1. Let X be a topological space and $a \in X$. For each $N \in \mathcal{N}_a$, pick a point $x_N \in N$. If we order $\mathcal{N}_a$ by reverse inclusion, then (x_N) is a net in X. Prove that $(x_N) \to a$.

E2. a) Let $(C, \leq)$ be an uncountable chain in which each element has only countably many predecessors. Suppose $f : C \to \mathbb{R}$ and that $f(\lambda) > 0$ for each $\lambda \in C$. Show that the net f does not converge to 0 in $\mathbb{R}$.

b) Give an example to show that part a) is false if $(C, \leq)$ is an uncountable poset in which each element has only countably many predecessors.

b) Is it possible to have a net $f : [0, \omega_2) \to \mathbb{R}$ and that $f(\lambda) > 0$ for each λ? Is it possible that f converges to 0 in $\mathbb{R}$? (*Recall that ω_α denotes the first ordinal with $\aleph_\alpha$ predecessors*.)

E3. Suppose X is a compact Hausdorff space and that $(\Lambda, \leq)$ is a directed set. For each $\lambda \in \Lambda$, let A_λ be a nonempty closed subset in X such that $A_{\lambda_2} \subseteq A_{\lambda_1}$ iff $\lambda_1 \leq \lambda_2$. Prove that $\bigcap\{A_\lambda : \lambda \in \Lambda\} \neq \emptyset$.

E4. a) Let $f : \Lambda \to X$ be a net in a space X and write $f(\lambda) = x_\lambda$. For each $\alpha \in \Lambda$, let $T_\alpha = \{x_\lambda : \lambda \geq \alpha\}$ = "the α^{th} tail of the net." Show that a point $x \in X$ is a cluster point of (x_λ) iff $x \in \bigcap\{\operatorname{cl} T_\alpha : \alpha \in \Lambda\}$.

b) Suppose x is a cluster point of the net (x_λ) a product $\prod\{X_\alpha : \alpha \in A\}$. Show that for each α, $\pi_\alpha(x) \in X_\alpha$ is a cluster point of the net $(\pi_\alpha(x_\lambda))$.

c) Give an example to show that the converse to part b) is false.

d) Let (X, d) be a metric space and $f : [0, \omega_1) \to X$ a function given by $f(\alpha) = x_\alpha$. Show that the net (x_α) converges iff (x_α) is eventually constant.

E5. a) Suppose X is infinite set with the cofinite topology. Let $\mathcal{F}$ be the filter generated by the filter base consisting of all cofinite sets. To what points does $\mathcal{F}$ converge?

b) Translate the work in part a) into statements about nets.

E6. Show that if a filter $\mathcal{F}$ is contained in a unique ultrafilter $\mathcal{U}$, then $\mathcal{F} = \mathcal{U}$.
(*Thus, if $\mathcal{F}$ is not an ultrafilter, it can enlarged to an ultrafilter in more than one way.*)

E7. a) State and prove a theorem of the form: Suppose x is a point in a space X. Then $\mathcal{N}_x$ is an ultrafilter $\Leftrightarrow$ ....

b) Prove or disprove: Suppose $X \neq \emptyset$ and that $\mathcal{F}$ is a maximal family of subsets of X with the finite intersection property. Then $\mathcal{F}$ is an ultrafilter.

E8. a) Let $\mathcal{F}$ be a filter in a set X. Prove that $\mathcal{F}$ is the intersection of all ultrafilters containing $\mathcal{F}$.

b) Let $\mathcal{U}$ be an ultrafilter in X and suppose that $A_1 \cup ... \cup A_n \in \mathcal{U}$. Prove that at least one A_i must be in $\mathcal{U}$. (*This is the filter analogue for a fact in ring theory: in a commutative ring with a unit, every maximal ideal is a prime ideal.*)

c) Give an example to show that part b) is not true for infinite unions.

d) "By duality," there is a result similar to b) about universal nets. State the result and prove it directly.

E9. Let $\mathcal{U}$ be a free ultrafilter in $\mathbb{N}$ and let $\Sigma = \mathbb{N} \cup \{\sigma\}$, where $\sigma \notin \mathbb{N}$. Define a topology $\mathcal{T}$ on Σ by $\mathcal{T} = \{O : O \subseteq \mathbb{N} \text{ or } O = U \cup \{\sigma\} \text{ where } U \in \mathcal{U}\}$

a) Prove that Σ is T_4 and that $\mathbb{N}$ is dense in Σ.

b) Prove that a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$ cannot have a countable base.
Hint: Since $\mathcal{U}$ is free, each set in $\mathcal{U}$ must be infinite. Why?

c) Prove that no sequence in $\mathbb{N}$ can converge to σ (and therefore there can be no countable neighborhood base at σ in Σ)
(*Hint: Work you did in b) might help.)*

Thus, Σ is another example of a countable space where only one point is not isolated and which is not first countable, See the space L in Example III.9.8.

d) How would the space Σ be different if $\mathcal{U}$ were a fixed ultrafilter?

Note: if $\mathcal{U}'$ is a free ultrafilter on $\mathbb{N}$ and $\mathcal{U}' \neq \mathcal{U}$, then the corresponding spaces Σ' and Σ may not be homeomorphic: the neighborhood systems of σ may look quite different. In this sense, free ultrafilters in $\mathbb{N}$ do not all "look alike."

E10. Suppose $(X, \mathcal{T})$ has a some property P. $\mathcal{T}$ is called a maximal-P topology (or minimal-P topology) if any larger (smaller) topology on X fails to have property P. Prove that if $(X, \mathcal{T})$ is a compact Hausdorff space, then $\mathcal{T}$ is maximal-compact and minimal-Hausdorff. (*Compare Exercise IV E23.*)

In one sense, this "justifies" the choice of the product topology over the box topology: for a product of compact Hausdorff spaces, a larger topology would not be compact and a smaller one would not be Hausdorff. The product topology is "just right" to ensure that the property "compact Hausdorff" is productive.

E11. Show that the map coloring Theorem 7.2 is equivalent to the statement that $\{R, W, B, G\}^{\aleph_0}$ is compact.

E12. A family $\mathcal{B}$ of subsets of X is called inadequate if it does not cover X, and $\mathcal{B}$ is called finitely inadequate if no finite subfamily covers X.

a) Use Zorn's Lemma to prove that any finitely inadequate family $\mathcal{B}$ is contained in a maximal finitely inadequate family.

b) Prove (by contradiction) that a maximal finitely inadequate family $\mathcal{B}$ has the following property: if $C_1, ..., C_n$ are subsets of X and $C_1 \cap ... \cap C_n \in \mathcal{B}$, then at least one set $C_i \in \mathcal{B}$.

c) The following are equivalent (Alexander's Subbase Theorem)

i) X has a subbase $\mathcal{S}$ such that each cover of X by members of $\mathcal{S}$ has a finite subcover

ii) X has a subbase $\mathcal{S}$ such that each finitely inadequate collection from $\mathcal{S}$ is inadequate

iii) every finitely inadequate family of open sets in X is inadequate

iv) X is compact

d) Use c) to prove the Tychonoff Product Theorem.

E13. This exercise gives still another proof of the Tychonoff Product Theorem. Suppose X_α is compact for all $\alpha \in A$. We want to prove that $\prod X_\alpha$ is compact. We proceed assuming X is not compact.

a) Show that there is a maximal open cover $\mathcal{U}$ of X having no finite subcover.

b) Show that if O is open in X and $O \notin \mathcal{U}$, then there are sets $U_1, ..., U_n \in \mathcal{U}$ such that $\{U_1, ..., U_n, O\}$ covers X.

c) Show that for each α, $\{V_\alpha \subseteq X_\alpha : V_\alpha$ is open and $< V_\alpha > \in \mathcal{U}\}$ cannot cover X_α. Conclude that for each α we can choose $x_\alpha \in X_\alpha$ so that $x_\alpha \notin$ any open set V_α for which $< V_\alpha > \in \mathcal{U}$.

d) Let $x = (x_\alpha) \in X$ and suppose $x \in U \in \mathcal{U}$. Pick open sets $V_{\alpha_i} \subseteq X_{\alpha_i}$ so that $x \in < V_{\alpha_1}, ..., V_{\alpha_k} > \subseteq U$. Explain why each $< V_{\alpha_i} > \notin \mathcal{U}$.

e) Show that for each $i = 1, ..., k$, there is a finite family $\mathcal{U}_i \subseteq \mathcal{U}$ such that $\mathcal{U}_i \cup < V_{\alpha_i} >$ covers X.

f) Show that $\bigcup_{i=1}^k \mathcal{U}_i \cup \{ < V_1, ..., V_{\alpha_k} > \}$ covers X, and then arrive at the contradiction that $\bigcup_{i=1}^k \mathcal{U} \cup \{U\}$ covers X.

Chapter IX Review

Explain why each statement is true, or provide a counterexample.

1. Order $C(X)$ by $f \leq g$ iff $f(x) \leq g(x)$ (in $\mathbb{R}$) for every $x \in X$. Then $(C(X), \leq)$ is a directed set.

2. x is a limit point of A in the space X iff there exists a filter $\mathcal{F}$ such that $A - \{x\} \in \mathcal{F}$ and $\mathcal{F} \rightarrow x$.

3. A set U is open in X iff U belongs to every filter $\mathcal{F}$ which converges to a point of U.

4. In a space X, let $\Phi_x = \{\mathcal{F} : \mathcal{F}$ is a filter converging to $x\}$. Then $\bigcap\{\mathcal{F} : \mathcal{F} \in \Phi_x\} = \mathcal{N}_x$.

5. Suppose $\mathcal{A}$ is a family of subsets of a space X such that if $A, B \in \mathcal{A}$ then $A \cap B \supseteq C$ for some $C \in \mathcal{A}$. Suppose (x_λ) is a net which is frequently in each set of $\mathcal{A}$. Then (x_λ) has a subnet which is eventually in each set in $\mathcal{A}$.

6. If a net $(x_\lambda) \rightarrow x$ in X and $|X| = \aleph_0$, then (x_λ) has a subsequence (i.e., a subnet whose directed set is $\mathbb{N}$) which converges to x.

7. If X is a nonempty finite set and $|X| = n$, then there are exactly $2^n - 1$ different filters and exactly n different ultrafilters on X.

8. A universal <u>sequence</u> must be eventually constant.

9. Suppose X is infinite. The collection $\mathcal{S} = \{A \subseteq X : |X - A| < |X|\}$ is an ultrafilter.

10. In $\mathbb{R}$, a filter $\mathcal{F} \rightarrow x$ iff $\forall\, \epsilon > 0\;\; \exists F \in \mathcal{F}$ such that $x \in F$ and diam $(F) < \epsilon$.

11. If X is compact, then every net in X has a convergent subsequence. (*Note: a "subsequence of a net" is a subnet whose directed set is* $\mathbb{N}$.)

12. If $x \neq y$ in a space X and if $\mathcal{N}_x \cup \mathcal{N}_y$ generates a filter, then X is not T_1 .

13. If $\mathcal{F}$ is a filter in $[0, \omega_1]$, then there must be a filter $\mathcal{G} \supseteq \mathcal{F}$ such that $\mathcal{G}$ has a cluster point.

14. Call a set in $[0,1]$ an endset if it has the form $[0,\epsilon)$ or $(1-\epsilon,1]$ for some $\epsilon > 0$. To show $[0,1]$ is compact, it is sufficient to show that any cover by endsets has a finite subcover.

15. Every separable metric space (X,d) has an equivalent totally bounded metric.

16. Suppose Λ is the collection of finite subsets of $[0,1]$, directed by $\subseteq$, and $f(F) = 1$ for all $F \in \Lambda$. The net f converges to 1 in $[0,1]$.