The contents of the document include a preface, an alphabetical list of propositions referred to by names, an introduction to the second edition, an introduction, and chapters on preliminary explanations of ideas and notations, the theory of logical types, and incomplete symbols. There is also a part on mathematical logic with sections on the theory of deduction, the theory of apparent variables, classes and relations, and the universal class, the null class, and the existence of relations.
#52. THE CARDINAL NUMBER 1

Summary of #52.

In this number, we introduce the cardinal number 1, defined as the class of all unit classes. The fact that 1 so defined is a cardinal number is not relevant at present, and cannot of course be proved until "cardinal number" has been defined. For the present, therefore, 1 is to be regarded simply as the class of all unit classes, unit classes being such classes as are of the form \( t' s \) for some \( s \).

Like \( \Lambda \) and \( V \), 1 is ambiguous as to type: it means "all unit classes of the type in question." The symbol \( 1 \alpha \), where \( \alpha \) is a type, will mean "all unit classes whose sole members belong to the type \( \alpha \)" (cf. #63). Thus e.g. "\( \xi \in 1 \text{ (Indiv)} \)" will mean "\( \xi \) is a class consisting of one individual," if "Indiv" stands for the class of individuals.

The properties of 1 to be proved in the present number are what we may call logical as opposed to arithmetical properties, i.e., they are not concerned with the arithmetical operations (addition, etc.) which can be performed with 1, but with the relations of 1 to unit classes. The arithmetical properties of 1 will be considered later, in Part III.

The propositions of the present number which are most used are the following:

#5216. \( \vdash a \in 1 \iff \exists \alpha : x, y \in \alpha \cdot \exists \alpha, y \cdot x = y \)

I.e. \( a \) is a unit class if, and only if, it is not null, and all its members are identical.

#5222. \( \vdash \sim \alpha \in 1 \)

#5224. \( \vdash a \in 1 \lor x = 1 \iff x = 1 \land y \in a \cdot \exists \alpha, y \cdot x = y \)

We shall define 0 as \( \sim 1 \). Thus the above proposition states that a class has one member or none when, and only when, all its members are identical.

#5241. \( \vdash \forall \alpha, \beta : a \sim 1 \iff \exists \alpha, x \cdot x = \alpha \cdot x = \beta \cdot x = y \)

This proposition is obtainable from #52-4 by transposition, i.e., by negating each side of the equivalence.

#5246. \( \vdash a, \beta \in 1 \Rightarrow a \subseteq \beta \equiv a = \beta \equiv \exists \alpha : a \cap \beta \equiv 1 \cdot (a \cap \beta) \)

I.e. two unit classes are identical when, and only when, one is contained in the other, and when and only when they have a common part.

#5201. \( \vdash \exists \alpha \in \alpha \cdot \alpha = \sim \alpha \)

#521. \( \vdash a \in 1 \iff \exists \alpha \in (\forall \alpha) \cdot a = \sim \alpha \) \([\#20-3, \#52-01] \)


\[\text{PROLEGOMENA TO CARDINAL ARITHMETIC} \quad \text{[PART II]}\]

\#5211. \(\vdash \alpha \in 1 \Rightarrow (\forall \alpha) \forall \beta, \gamma, \delta. y = \varepsilon \) \[\text{[\#521. \#511.]}\]

\#5212. \(\vdash \exists (\forall \alpha) \in 1. \exists ! (\forall \beta) \in (\forall \gamma), \gamma, \delta. y = \varepsilon \)

\[\text{Dem.}\]

\(\vdash \exists 1 \Rightarrow (\forall \alpha) \in (\forall \beta) \Rightarrow (\forall \gamma), \gamma, \delta. y = \varepsilon \) 

\[\text{[\#521. \#511.]}\]

\#5213. \(\vdash 1 = D'\)

\[\text{Dem.}\]

\(\vdash \exists 1 \Rightarrow (\forall \alpha) = (\forall \beta) \Rightarrow (\forall \gamma), \gamma, \delta. y = \varepsilon \) 

\[\text{[\#521. \#511.]}\]

\#5214. \(\vdash 1 \in \mathcal{C} \)

\[\text{[\#521. \#511.]}\]

\#5215. \(\vdash \alpha \in 1 \Rightarrow (\forall \beta) \in \alpha \) 

\[\text{[\#521. \#511.]}\]

\#5216. \(\vdash \alpha \in 1 \Rightarrow (\exists x) \vdash x \in \alpha \) 

\[\text{[\#521. \#511.]}\]

\#5217. \(\vdash \alpha \in 1 \Rightarrow (\exists x) \vdash x \in \alpha \) 

\[\text{[\#521. \#511.]}\]

\#5218. \(\vdash \alpha \in 1 \Rightarrow (\forall \beta) \in \alpha \) 

\[\text{[\#521. \#511.]}\]

\#5219. \(\vdash \alpha \in 1 \Rightarrow (\exists x) \vdash x \in \alpha \) 

\[\text{[\#521. \#511.]}\]

\#5220. \(\vdash \alpha \in 1 \Rightarrow (\exists x) \vdash x \in \alpha \) 

\[\text{[\#521. \#511.]}\]

\#5221. \(\vdash \alpha \in 1 \Rightarrow (\exists x) \vdash x \in \alpha \) 

\[\text{[\#521. \#511.]}\]

\#5222. \(\vdash \alpha \in 1 \Rightarrow (\exists x) \vdash x \in \alpha \) 

\[\text{[\#521. \#511.]}\]