

## Real $2 \times 2$ Matrices with Complex Eigenvalues

It turns out that a  $2 \times 2$  matrix with complex eigenvalues, in general, represents a “rotation and dilation (rescaling)” in a new coordinate system.

The text handles much of its discussion in this section without any proof. I'll try to spell out some of the details here.

**Lemma** Suppose  $A$  is a  $2 \times 2$  real matrix that has a (complex) eigenvalue  $\lambda = a - bi$  ( $b \neq 0$ ) and an associated eigenvector  $\mathbf{v}$ . Then

$$\begin{aligned} A(\operatorname{Re} \mathbf{v}) &= a(\operatorname{Re} \mathbf{v}) + b(\operatorname{Im} \mathbf{v}) \\ A(\operatorname{Im} \mathbf{v}) &= -b(\operatorname{Re} \mathbf{v}) + a(\operatorname{Im} \mathbf{v}) \end{aligned}$$

**Proof** Write  $\mathbf{v} = \operatorname{Re} \mathbf{v} + i \operatorname{Im} \mathbf{v}$ , so that

$$A\mathbf{v} = \underbrace{A}_{\swarrow} \underbrace{\operatorname{Re} \mathbf{v}}_{\nearrow} + i \underbrace{A}_{\swarrow} \underbrace{\operatorname{Im} \mathbf{v}}_{\nearrow} \quad (**)$$

these are real since  $A$  is a real matrix

But on the other hand,

$$\begin{aligned} A\mathbf{v} &= \lambda\mathbf{v} = (a - bi)\mathbf{v} = (a - bi)(\operatorname{Re} \mathbf{v} + i \operatorname{Im} \mathbf{v}) \\ &= (a \operatorname{Re} \mathbf{v} + b \operatorname{Im} \mathbf{v}) + (-b \operatorname{Re} \mathbf{v} + a \operatorname{Im} \mathbf{v}) i \quad (***) \end{aligned}$$

Equating real and imaginary parts of (\*\*) and (\*\*\*) gives

$$\begin{aligned} A \operatorname{Re} \mathbf{v} &= (a \operatorname{Re} \mathbf{v} + b \operatorname{Im} \mathbf{v}) \\ A \operatorname{Im} \mathbf{v} &= (-b \operatorname{Re} \mathbf{v} + a \operatorname{Im} \mathbf{v}), \text{ so we're done.} \end{aligned}$$

**Lemma** Suppose  $A$  is a  $2 \times 2$  matrix with a complex eigenvalue  $a - bi$  and corresponding complex eigenvector  $\mathbf{v}$ . Then  $\operatorname{Re} \mathbf{v}$  and  $\operatorname{Im} \mathbf{v}$  are linearly independent vectors in  $\mathbb{R}^2$ .

**Proof** We are assuming the eigenvalue and eigenvector are complex, so  $b \neq 0$  and  $\operatorname{Im} \mathbf{v} \neq \mathbf{0}$ .

Suppose  $c_1 \operatorname{Re} \mathbf{v} + c_2 \operatorname{Im} \mathbf{v} = \mathbf{0}$ , and assume  $c_1 \neq 0$ , so that  $\operatorname{Re} \mathbf{v} = -\frac{c_2}{c_1} \operatorname{Im} \mathbf{v}$ .

Then

$$\begin{aligned}
 \mathbf{0} &= A(c_1 \operatorname{Re} \mathbf{v} + c_2 \operatorname{Im} \mathbf{v}) \\
 &= c_1 A \operatorname{Re} \mathbf{v} + c_2 A \operatorname{Im} \mathbf{v} \\
 &= c_1(a \operatorname{Re} \mathbf{v} + b \operatorname{Im} \mathbf{v}) + c_2(-b \operatorname{Re} \mathbf{v} + a \operatorname{Im} \mathbf{v}) \quad (\text{by the first Lemma}) \\
 &= c_1(a(-\frac{c_2}{c_1} \operatorname{Im} \mathbf{v}) + b \operatorname{Im} \mathbf{v}) + c_2(-b(-\frac{c_2}{c_1} \operatorname{Im} \mathbf{v}) + a \operatorname{Im} \mathbf{v}) \\
 &= -c_2 a \operatorname{Im} \mathbf{v} + bc_1 \operatorname{Im} \mathbf{v} + \frac{bc_2^2}{c_1} \operatorname{Im} \mathbf{v} + ac_2 \operatorname{Im} \mathbf{v} \\
 &= bc_1 \operatorname{Im} \mathbf{v} + \frac{bc_2^2}{c_1} \operatorname{Im} \mathbf{v} \\
 &= b(c_1 \operatorname{Im} \mathbf{v} + \frac{c_2^2}{c_1} \operatorname{Im} \mathbf{v})
 \end{aligned}$$

Since  $b \neq 0$ ,  $c_1 \operatorname{Im} \mathbf{v} + \frac{c_2^2}{c_1} \operatorname{Im} \mathbf{v} = (c_1 + \frac{c_2^2}{c_1}) \operatorname{Im} \mathbf{v} = \mathbf{0}$ , and since  $\operatorname{Im} \mathbf{v} \neq \mathbf{0}$

$$(c_1 + \frac{c_2^2}{c_1}) = 0, \text{ so}$$

$$c_1 = -\frac{c_2^2}{c_1}$$

$$c_1^2 = -c_2^2.$$

But we are assuming  $c_1 \neq 0$ , so  $c_1^2 > 0$ , so this equation is impossible.

Assuming  $c_1 \neq 0$  leads to an impossibility, so it must be that  $c_1 = 0$ .

Since  $c_1 \operatorname{Re} \mathbf{v} + c_2 \operatorname{Im} \mathbf{v} = \mathbf{0}$ , we then have  $c_2 \operatorname{Im} \mathbf{v} = \mathbf{0}$ . But  $\operatorname{Im} \mathbf{v} \neq \mathbf{0}$ , so  $c_2 = 0$  also. Therefore  $c_1, c_2$  are both 0, so  $\operatorname{Re} \mathbf{v}$  and  $\operatorname{Im} \mathbf{v}$  are linearly independent.

**Theorem** Suppose  $A$  is a  $2 \times 2$  real matrix with a complex eigenvalue  $a - bi$  and a corresponding (complex) eigenvector  $\mathbf{v}$ .

Then  $A = PCP^{-1}$ , where  $\begin{cases} \text{the columns of } P \text{ are the vectors } \operatorname{Re} \mathbf{v} \text{ and } \operatorname{Im} \mathbf{v} \text{ and \\ } C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \end{cases}$

**Proof** From the Lemma, we know that the columns of  $P$  are linearly independent, so  $P$  is invertible. To check that  $A = PCP^{-1}$ , we only need to verify that  $AP = PC$ .

$$AP = A[\operatorname{Re} \mathbf{v} \ \operatorname{Im} \mathbf{v}] = [A \operatorname{Re} \mathbf{v} \ A \operatorname{Im} \mathbf{v}] = [a \operatorname{Re} \mathbf{v} + b \operatorname{Im} \mathbf{v} \quad -b \operatorname{Re} \mathbf{v} + a \operatorname{Im} \mathbf{v}]$$

$$\begin{aligned} PC &= [\operatorname{Re} \mathbf{v} \ \operatorname{Im} \mathbf{v}] \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = [ [\operatorname{Re} \mathbf{v} \ \operatorname{Im} \mathbf{v}] \begin{bmatrix} a \\ b \end{bmatrix} \quad [\operatorname{Re} \mathbf{v} \ \operatorname{Im} \mathbf{v}] \begin{bmatrix} -b \\ a \end{bmatrix} ] \\ &= [ a \operatorname{Re} \mathbf{v} + b \operatorname{Im} \mathbf{v} \quad -b \operatorname{Re} \mathbf{v} + a \operatorname{Im} \mathbf{v} ] = AP. \end{aligned}$$

Suppose  $A$  is a real  $2 \times 2$  matrix with a complex eigenvalue  $a - bi$  and a corresponding eigenvector  $\mathbf{v}$ . Let  $P = [\operatorname{Re} \mathbf{v} \ \operatorname{Im} \mathbf{v}]$ .

By the theorem,  $A = [\operatorname{Re} \mathbf{v} \ \operatorname{Im} \mathbf{v}] \begin{bmatrix} a & -b \\ b & a \end{bmatrix} [\operatorname{Re} \mathbf{v} \ \operatorname{Im} \mathbf{v}]^{-1}$ .

$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$  can be written as  $r \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$ , where  $r = \sqrt{a^2 + b^2}$ .

Thus  $C$  represents a counterclockwise rotation (if  $\theta$  is chosen  $> 0$ ) around the origin through the angle  $\theta$ , followed by a rescaling factor of  $r$ .

If we use  $\mathcal{B} = \{\operatorname{Re} \mathbf{v}, \operatorname{Im} \mathbf{v}\}$  as a new basis for  $\mathbb{R}^2$ , then the change of coordinate matrix  $P_{\mathcal{B}} = P = [\operatorname{Re} \mathbf{v} \ \operatorname{Im} \mathbf{v}]$ . The effect of  $A$ , broken into several steps, is:

$$\begin{array}{lll} \mathbf{x} & \mapsto & P_{\mathcal{B}}^{-1} \mathbf{x} = [\mathbf{x}]_{\mathcal{B}} & \mapsto & C[\mathbf{x}]_{\mathcal{B}} & \mapsto & P_{\mathcal{B}} C[\mathbf{x}]_{\mathcal{B}} = A\mathbf{x} \\ \text{switch} & & & & \text{rotate and} & & \text{switch back} \\ \text{to} & & & & \text{dilate in new} & & \text{so standard} \\ \mathcal{B} \text{ coordinates} & & & & \text{coordinate} & & \text{coordinates} \\ & & & & \text{system} & & \end{array}$$

1) If  $r = 1$ ,  $C$  represents a “pure“ rotation (in the new coordinates)

- 2) If  $r > 1$ , then the successive images  $\mathbf{x}_0, \mathbf{x}_1 = A\mathbf{x}_0, \dots, \mathbf{x}_{n+1} = A\mathbf{x}_n, \dots$  move further and further away from the origin (assuming  $\mathbf{x}_0 \neq \mathbf{0}$ )
- 3) If  $r < 1$ , then the successive images  $\mathbf{x}_0, \mathbf{x}_1 = A\mathbf{x}_0, \dots, \mathbf{x}_{n+1} = A\mathbf{x}_n, \dots$  approach the origin.

**Example** Let  $A = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{3}{4} & \frac{5}{4} \end{bmatrix}$ . The characteristic equation is

$$\det \begin{bmatrix} \frac{1}{2} - \lambda & -\frac{1}{2} \\ \frac{3}{4} & \frac{5}{4} - \lambda \end{bmatrix} = (\frac{1}{2} - \lambda)(\frac{5}{4} - \lambda) + \frac{3}{8} = \frac{5}{8} - \frac{7}{4}\lambda + \lambda^2 + \frac{3}{8}$$

$$= \lambda^2 - \frac{7}{4}\lambda + 1 = 0, \text{ which has the same solutions as } 4\lambda^2 - 7\lambda + 4 = 0$$

$$\text{The eigenvalues are } \lambda = \frac{7 \pm \sqrt{49 - 64}}{8} = \frac{7}{8} \pm \frac{\sqrt{-15}}{8} = \frac{7}{8} \pm \frac{\sqrt{15}}{8} i.$$

For no particular reason, choose the eigenvalue  $\lambda = \frac{7}{8} - \frac{\sqrt{15}}{8} i$ .

To find the corresponding eigenspace: solve  $(A - \lambda I)\mathbf{x} = \mathbf{0}$

$$\begin{bmatrix} \frac{1}{2} - (\frac{7}{8} - \frac{\sqrt{15}}{8} i) & -\frac{1}{2} & 0 \\ \frac{3}{4} & \frac{5}{4} - (\frac{7}{8} - \frac{\sqrt{15}}{8} i) & 0 \end{bmatrix} = \begin{bmatrix} -\frac{3}{8} + \frac{\sqrt{15}}{8} i & -\frac{1}{2} & 0 \\ \frac{3}{4} & \frac{3}{8} + \frac{\sqrt{15}}{8} i & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} -3 + \sqrt{15} i & -4 & 0 \\ 6 & 3 + \sqrt{15} i & 0 \end{bmatrix}.$$

We could continue row reducing but it's easier, here, to note that the first equation says

$$(-3 + \sqrt{15} i)x_1 - 4x_2 = 0, \quad \text{that is,} \quad x_2 = \frac{-3 + \sqrt{15} i}{4} x_1.$$

*Note: for a  $2 \times 2$  homogeneous system in general, the first equation might solve as  $x_2 = k_1 x_1$ ; the second equation might solve as  $x_2 = k_2 x_1$ . If the system had only the trivial solution, then  $k_1$  and  $k_2$  would be different, and just looking at the first equation might mislead us into thinking there were nontrivial solutions. However, in this example, we know in advance that the system has nontrivial solutions (since  $\lambda = \frac{7}{8} - \frac{\sqrt{15}}{8} i$  is an eigenvalue). Therefore the second equation will give the same relationship between  $x_2$  and  $x_1$ . (Check it!)*

Choosing  $x_1 = 1$ , we can write down an eigenvector  $\begin{bmatrix} 1 \\ \frac{-3 + \sqrt{15} i}{4} \end{bmatrix}$ . A neater

eigenvector would be 4 times this one:  $\mathbf{v} = \begin{bmatrix} 4 \\ -3 + \sqrt{15} i \end{bmatrix}$ .

In the notation of the Theorem, we have:

- $A = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{3}{4} & \frac{5}{4} \end{bmatrix}$

- An eigenvalue  $\lambda = \frac{7}{8} - \frac{\sqrt{15}}{8}i$  (so  $a = \frac{7}{8}$ ,  $b = \frac{\sqrt{15}}{8}$ ), and

- A corresponding eigenvector  $\mathbf{v} = \begin{bmatrix} 4 \\ -3 + \sqrt{15}i \end{bmatrix} = \begin{bmatrix} 4 + 0 \cdot i \\ -3 + \sqrt{15}i \end{bmatrix}$ , for  
which  $\text{Re } \mathbf{v} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$  and  $\text{Im } \mathbf{v} = \begin{bmatrix} 0 \\ \sqrt{15} \end{bmatrix}$ .

So  $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} \frac{7}{8} & -\frac{\sqrt{15}}{8} \\ \frac{\sqrt{15}}{8} & \frac{7}{8} \end{bmatrix}$  and  $P = [\text{Re } \mathbf{v} \quad \text{Im } \mathbf{v}] = \begin{bmatrix} 4 & 0 \\ -3 & \sqrt{15} \end{bmatrix}$ .

Theorem 9 states that  $A$  factors as  $PCP^{-1}$  :

$$\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{3}{4} & \frac{5}{4} \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ -3 & \sqrt{15} \end{bmatrix} \begin{bmatrix} \frac{7}{8} & -\frac{\sqrt{15}}{8} \\ \frac{\sqrt{15}}{8} & \frac{7}{8} \end{bmatrix} \begin{bmatrix} 4 & 0 \\ -3 & \sqrt{15} \end{bmatrix}^{-1}.$$

*I checked this for errors using Matlab: rounded to 4 places, Matlab gives*

$$AP = \begin{bmatrix} 3.5 & -1.9365 \\ -.75 & 4.8412 \end{bmatrix} = PC$$

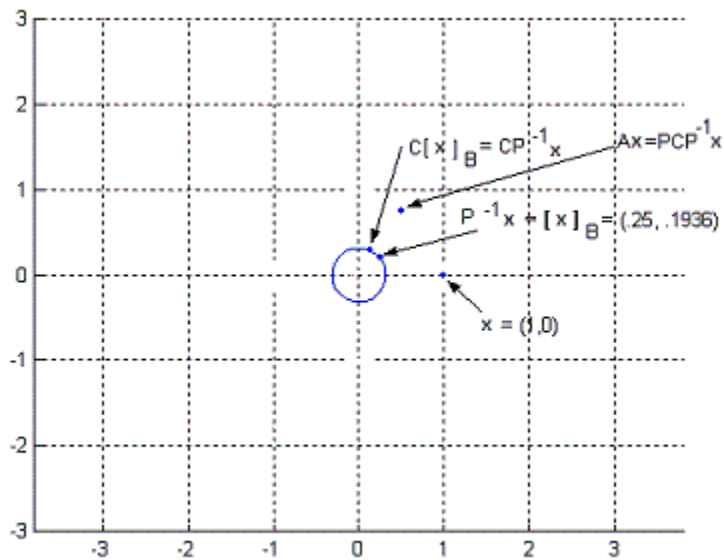
What does this mean geometrically? As in our first example, we can write

$$C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = r \begin{bmatrix} \frac{a}{r} & -\frac{b}{r} \\ \frac{b}{r} & \frac{a}{r} \end{bmatrix}.$$

Here,  $r = \sqrt{a^2 + b^2} = \sqrt{\frac{49}{64} + \frac{15}{64}} = \sqrt{\frac{64}{64}} = 1$ , so

$$C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = 1 \cdot \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} = \begin{bmatrix} \frac{7}{8} & -\frac{\sqrt{15}}{8} \\ \frac{\sqrt{15}}{8} & \frac{7}{8} \end{bmatrix}$$

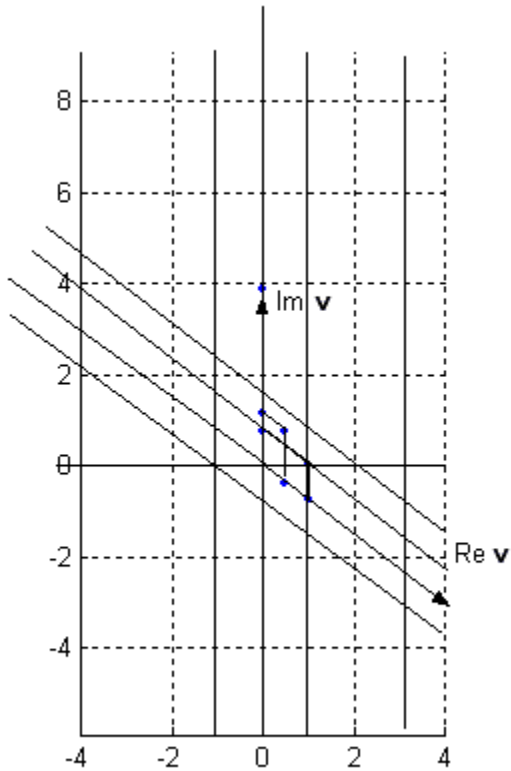
so  $C$  represent a “pure rotation” (the rescaling factor  $r = 1$ ), and since  $\cos \theta = \frac{7}{8}$ , we can choose  $\theta \approx 0.5054$  (radians)  $\approx 28.96^\circ$



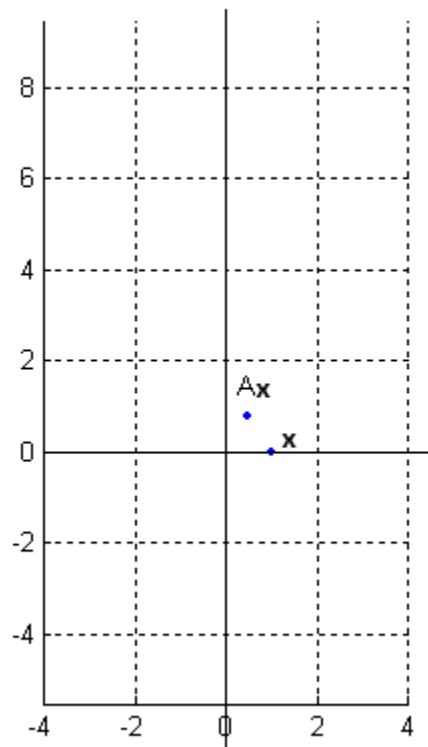
where  $\cos \theta = \frac{7}{8}$ ,  $\sin \theta = \frac{\sqrt{15}}{8}$ . Using a calculator or Matlab we find that  $\approx 0.5054$  radians (rounded to 4 places), or  $\theta \approx 28.96^\circ$ .

Multiplication by  $C$  rotates a vector counterclockwise by an angle  $\theta$  around the origin (and then rescales it by a factor of  $r$ , but in this example  $r = 1$ ).

An alternate way of looking at this is to think of the first and last steps as simply renaming points in the new coordinates, rather than moving points around in  $\mathbb{R}^2$ :



New coordinate system  
and new graph paper



Standard coordinate system  
and graph paper

$$\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

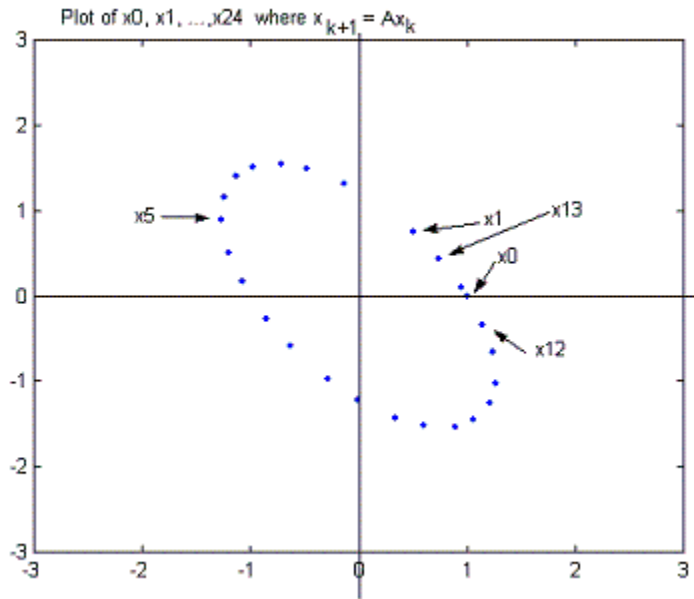
$$P^{-1}\mathbf{x} = \begin{bmatrix} .25 \\ .1936 \end{bmatrix}$$

$$CP^{-1}\mathbf{x} = \begin{bmatrix} .125 \\ .2905 \end{bmatrix}$$

$$PCP^{-1}\mathbf{x} = \begin{bmatrix} .50 \\ .75 \end{bmatrix}$$



The figure below shows an example where we take  $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$  and plot  $\mathbf{x}_0, \mathbf{x}_1 = A\mathbf{x}_0, \mathbf{x}_2 = A\mathbf{x}_1, \dots, \mathbf{x}_{24} = A\mathbf{x}_{23}$ .



The images run along an elliptical “orbit” – even though  $C$  is a “pure” rotation ( $r = 1$ ), the new coordinate basis vectors  $\begin{bmatrix} 4 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ \sqrt{15} \end{bmatrix}$  are not perpendicular and have different lengths, so the orbit isn't a circle.

To briefly illustrate the case where  $r > 1$ , let  $A = \begin{bmatrix} 2 & -2 \\ 3 & 5 \end{bmatrix} = 4 \cdot (\text{matrix } A \text{ in the preceding example})$ .

Multiplying a square matrix by 4 multiplies the eigenvalues by 4 but doesn't change the eigenvectors (*why? -- check this from the definitions of eigenvalue and eigenvector.*)

So  $A = \begin{bmatrix} 2 & -2 \\ 3 & 5 \end{bmatrix}$  has a complex eigenvalue  $\lambda = \frac{7}{2} - \frac{\sqrt{15}}{2}i$  with a corresponding eigenvector  $\begin{bmatrix} 4 \\ -3 + \sqrt{15}i \end{bmatrix}$ . According to the Theorem, we can write

$$\begin{bmatrix} 2 & -2 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ -3 & \sqrt{15} \end{bmatrix} \begin{bmatrix} \frac{7}{2} & -\frac{\sqrt{15}}{2} \\ \frac{\sqrt{15}}{2} & \frac{7}{2} \end{bmatrix} \begin{bmatrix} 4 & 0 \\ -3 & \sqrt{15} \end{bmatrix}^{-1}.$$

*I checked this for errors using Matlab: rounded to 4 places, Matlab gives*

$$AP = \begin{bmatrix} 14 & -7.7460 \\ -3 & 19.3649 \end{bmatrix} = PC$$

$$\begin{aligned} \text{Here, } C &= \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = r \begin{bmatrix} \frac{a}{r} & -\frac{b}{r} \\ \frac{b}{r} & \frac{a}{r} \end{bmatrix} = 4 \begin{bmatrix} \frac{7}{8} & -\frac{\sqrt{15}}{8} \\ \frac{\sqrt{15}}{8} & \frac{7}{8} \end{bmatrix} \\ &= 4 \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}, \end{aligned}$$

where  $\cos \theta = \frac{7}{8}$ ,  $\sin \theta = \frac{\sqrt{15}}{8}$ . Again, we can choose  $\theta \approx 0.5054$  or  $\theta \approx 50.1338^\circ$ . The matrix  $C$  rotates the new coordinate vector and then rescales it by a factor of 4.

The following figure illustrates the first few iterations, starting with  $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$  :

