Chapter VII
Separation Properties

1. Introduction

"Separation" refers here to whether objects such as points or disjoint closed sets can be enclosed in disjoint open sets. In spite of the similarity of terminology, "separation properties" have no direct connection to the idea of "separated sets" that appeared in Chapter 5 in the context of connected spaces.

We have already met some simple separation properties of spaces: the T_0, T_1 and T_2 (Hausdorff) properties. In this chapter, we look at these and others in more depth. As hypotheses for "more separation" are added, spaces generally become nicer and nicer – especially when "separation" is combined with other properties. For example, we will see that "enough separation" and "a nice base" guarantees that a space is metrizable.

"Separation axioms" translates the German term *Trennungsaxiome* used in the original literature. Therefore the standard separation axioms were historically named T_0, T_1,T_2,T_3, and T_4, each one stronger than its predecessor in the list. Once these had become common terminology, another separation axiom was discovered to be useful and "interpolated" into the list: $T_{3\frac{1}{2}}$. It turns out that the $T_{3\frac{1}{2}}$ spaces (also called Tychonoff spaces) are an extremely well-behaved class of spaces with some very nice properties.

2. The Basics

Definition 2.1 A topological space X is called a

1) T_0-space if, whenever $x \neq y \in X$, there either exists an open set U with $x \in U$, $y \notin U$ or there exists an open set V with $y \in V$, $x \notin V$

2) T_1-space if, whenever $x \neq y \in X$, there exists an open set U with $x \in U, y \notin V$ and there exists an open set V with $x \notin U, y \in V$

3) T_2-space (or, Hausdorff space) if, whenever $x \neq y \in X$, there exist disjoint open sets U and V in X such that $x \in U$ and $y \in V$.

It is immediately clear from the definitions that $T_2 \Rightarrow T_1 \Rightarrow T_0$.

Example 2.2

1) X is a T_0-space if and only if: whenever $x \neq y$, then $\mathcal{N}_x \neq \mathcal{N}_y$ – that is, different points in X have different neighborhood systems.

2) If X has the trivial topology and $|X| > 1$, then X is not a T_0-space.

3) A pseudometric space (X, d) is a metric space in and only if (X, d) is a T_0-space.

Clearly, a metric space is T_0. On the other hand, suppose (X, d) is T_0 and that $x \neq y$. Then for some $\epsilon > 0$ either $x \notin B_\epsilon(y)$ or $y \notin B_\epsilon(x)$. Either way, $d(x, y) \geq \epsilon$, so d is a metric.

4) In any topological space X we can define an equivalence relation $x \sim y$ iff $\mathcal{N}_x = \mathcal{N}_y$. Let $g : X \to X/\sim\ = Y$ by $g(x) = [x]$. Give Y the quotient topology. Then g is continuous, onto, open (not automatic for a quotient map!) and the quotient is a T_0 space:

If O is open in X, we want to show that $g[O]$ is open in Y, and because Y has the quotient topology this is true iff $g^{-1}[g[O]]$ is open in X. But $g^{-1}[g[O]]$
$= \{x \in X : g(x) \in g[O]\} = \{x \in X :$ for some $y \in O, g(x) = g(y)\}$
$= \{x \in X :\ x$ is equivalent to some point y in $O\} = O$.

If $[x] \neq [y] \in Y$, then x is not equivalent to y, so there is an open set $O \subseteq X$ with (say) $x \in O$ and $y \notin O$. Since g is open, $g[O]$ is open in Y and $[x] \in g[O]$. Moreover, $[y] \notin g[O]$ or else y would be equivalent to some point of O – implying $y \in O$.

Y is called the T_0-identification of X. This identification turns any space into a T_0-space by identifying points that have identical neighborhoods. If X is a T_0-space to begin with, then g is one-to-one and g is a homeomorphism. Applied to a T_0 space, the T_0-identification accomplishes nothing. If (X, d) is a pseudometric space, the T_0-identification is the same as the metric identification discussed in Example VI.5.6 because, in that case, $\mathcal{N}_x = \mathcal{N}_y$ if and only if $d(x, y) = 0$.

5) For $i = 0, 1, 2$: if $(X, \mathcal{T})$ is a T_i space and $\mathcal{T}' \supseteq \mathcal{T}$ is a new topology on X, then $(X, \mathcal{T}')$ is also a T_i space.

Example 2.3

1) (Exercise) It is easy to check that a space X is a T_1 space

iff for each $x \in X$, $\{x\}$ is closed
iff for each $x \in X$, $\{x\} = \bigcap\{O :\ O$ is open and $x \in O\}$

2) A finite T_1 space is discrete.

3) Sierpinski space $X = \{0, 1\}$ with topology $\mathcal{T} = \{\emptyset, \{1\}, \{0, 1\}\})$ is T_0 but not T_1: $\{1\}$ is an open set that contains 1 and not 0; but there is no open set containing 0 and not 1.

4) $\mathbb{R}$, with the right-ray topology, is T_0 but not T_1: if $x < y \in \mathbb{R}$, then $O = (x, \infty)$ is an open set that contains y and not x; but there is no open set that contains x and not y.

5) With the cofinite topology, $\mathbb{N}$ is T_1 but not T_2 because, in an infinite cofinite space, any two nonempty open sets have nonempty intersection.

These separation properties are very well-behaved with respect to subspaces and products.

Theorem 2.4 For $i = 0, 1, 2$:

a) A subspace of a T_i-space is a T_i-space
b) If $X = \prod_{\alpha \in A} X_\alpha \neq \emptyset$, then X is a T_i-space iff each X_α is a T_i-space.

Proof All of the proofs are easy. We consider here only the case $i = 1$, leaving the other cases as an exercise.

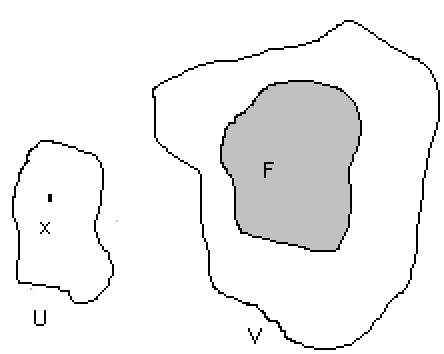
a) Suppose $a \neq b \in A \subseteq X$, where X is a T_1 space. If U' is an open set in X containing x but not y, then $U = U' \cap A$ is an open set in A containing x but not y. Similarly we can find an open set V in A containing y but not x. Therefore A is a T_1-space.

b) Suppose $X = \prod_{\alpha \in A} X_\alpha$ is a nonempty T_1-space. Each X_α is homeomorphic to a subspace of X, so, by part a), each X_α is T_1. Conversely, suppose each X_α is T_1 and that $x \neq y \in X$. Then $x_\alpha \neq y_\alpha$ for some α. Pick an open set U_α in X_α containing x_α but not y_α. Then $U = \, < U_\alpha >$ is an open set in X containing x but not y. Similarly, we find an open set V in X containing y but not x. Therefore X is a T_1-space. •

Exercise 2.5 Is a continuous image of a T_i-space necessarily a T_i-space? How about a quotient? A continuous open image?

We now consider a slightly different kind of separation axiom for a space X : formally, the definition is "just like" the definition of T_2, but with a closed set replacing one of the points.

Definition 2.6 A topological space X is called regular if whenever F is a closed set and $x \notin F$, there exist disjoint open sets U and V such that $x \in U$ and $F \subseteq V$.



There are some easy equivalents of the definition of "regular" that are useful to recognize.

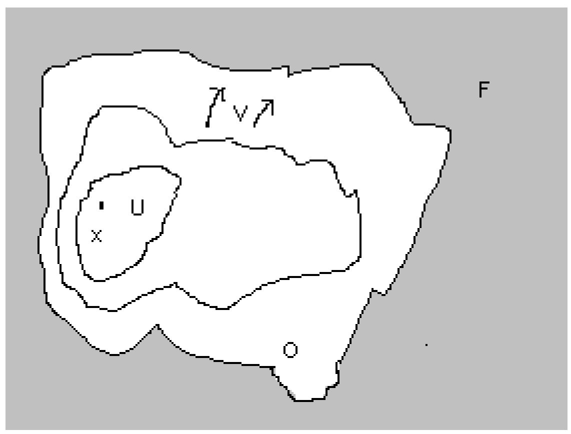
Theorem 2.7 The following are equivalent for any space X:

i) X is regular

ii) if O is an open set containing x, then there exists an open set $U \subseteq X$ such that $x \in U \subseteq \text{cl}\, U \subseteq O$

iii) at each point $x \in X$ there exists a neighborhood base consisting of closed neighborhoods.

Proof i) $\Rightarrow$ ii) Suppose X is regular and O is an open set with $x \in O$. Letting $F = X - O$, we use regularity to get disjoint open sets U, V with $x \in U$ and $F \subseteq V$ as illustrated below:



Then $x \in U \subseteq \text{cl}\, U \subseteq O$ (since $\text{cl}\, U \subseteq X - V$).

ii) $\Rightarrow$ iii) If $N \in \mathcal{N}_x$, then $x \in O = \text{int}\, N$. By ii), we can find an open set U so that $x \in U \subseteq \text{cl}\, U \subseteq O \subseteq N$. Since $\text{cl}\, U$ is a neighborhood of x, the closed neighborhoods of x form a neighborhood base at x.

iii) $\Rightarrow$ i) Suppose F is closed and $x \notin F$. By ii), there is a closed neighborhood K of x such that $x \in K \subseteq X - F$. We can choose $U = \text{int}\, K$ and $V = X - K$ to complete the proof that X is regular. $\bullet$

Example 2.8 Every pseudometric space (X, d) is regular. Suppose $a \notin F$ and F is closed. We have a continuous function $f(x) = d(x, F)$ for which $f(a) = c > 0$ and $f|F = 0$. This gives us disjoint open sets with $a \in U = f^{-1}[(\frac{c}{2}, \infty)]$ and $F \subseteq V = f^{-1}[(-\infty, \frac{c}{2})]$. Therefore X is regular.

At first glance, one might think that regularity is a stronger condition than T_2. But this is false: if (X, d) is a pseudometric space but not a metric space, then X is regular but not even T_0.

To bring things into line, we make the following definition.

Definition A topological space X is called a T_3-space if X is regular and T_1.

It is easy to show that $T_3 \Rightarrow T_2$ ($\Rightarrow T_1 \Rightarrow T_0$): suppose X is T_3 and $x \neq y \in X$. Then $F = \{y\}$ is closed so, by regularity, there are disjoint open sets U, V with $x \in U$ and $y \in \{y\} \subseteq V$.

Caution *Terminology varies from book to book. For some authors, the definition of "regular" includes T_1 : for them, "regular" means what we have called "T_3." Check the definitions when reading other books.*

Exercise 2.10 Show that a regular T_0 space must be T_3 (*so it would have been equivalent to use "T_0" instead of "T_1" in the definition of "T_3"*).

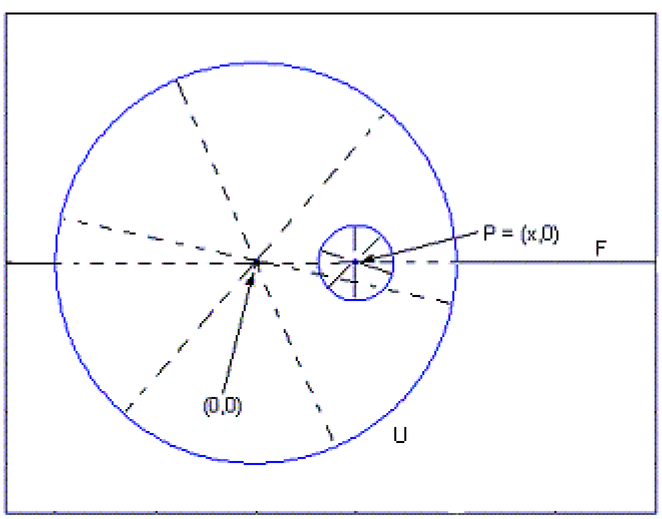
Example 2.11 $T_2 \not\Rightarrow T_3$. We will put a new topology on the set $X = \mathbb{R}^2$. At each point $p \in X$, let a neighborhood base $\mathcal{B}_p$ consist of all sets N of the form

$$N = B_\epsilon(p) - (\text{a finite number of straight lines through } p) \cup \{p\} \text{ for some } \epsilon > 0.$$

(*Check that the conditions in the Neighborhood Base Theorem III.5.2 are satisfied.*) With the resulting topology, X is called the slotted plane. Note that $B_\epsilon(p) \in \mathcal{B}_p$ (*because "0" is a finite number*), so each $B_\epsilon(p)$ is among the basic neighborhoods in $\mathcal{B}_p$ – so the slotted plane topology on $\mathbb{R}^2$ contains the usual Euclidean topology. It follows that X is T_2.

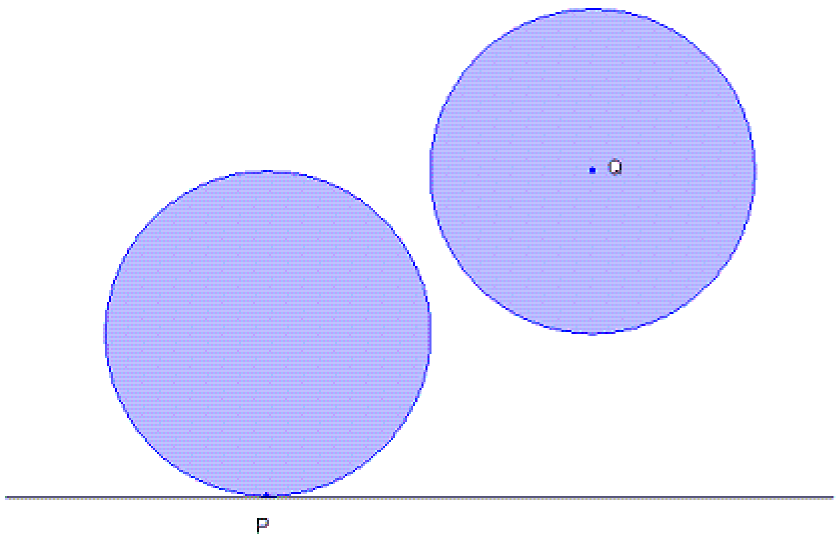
The set $F = \{(x, 0) : x \neq 0\} =$ "the x-axis with the origin deleted" is a closed set in X (*why?*).

If U is any open set containing the origin $(0, 0)$, then there is a basic neighborhood N with $(0, 0) \in N \subseteq U$. Using the ϵ in the definition of N, we can choose a point $p = (x, 0) \in F$ with $0 < x < \epsilon$. Every basic neighborhood set of p must intersect N (*why?) and therefore must intersect* U. It follows that $(0, 0)$ and F cannot be separated by disjoint open sets, so the slotted plane is not regular (and therefore not T_3).



Note: The usual topology in $\mathbb{R}^2$ is regular. This example shows that an "enlargement" of a regular (or T_3) topology may not be regular (or T_3). Although the enlarged topology has more open sets to work with, there are also more "point/closed set pairs x, F" that need to be separated. By contrast, it is easy to see that an "enlargement" of a T_i topology $(i = 0, 1, 2)$ is still T_i.

Example 2.12 The Moore plane Γ (Example III.5.6) is clearly T_2. In fact, at each point, there is a neighborhood base of closed neighborhoods. The figure illustrates this for a point P on the x-axis and a point Q above the x-axis. Therefore Γ is T_3.



Theorem 2.13 a) A subspace of a regular (T_3) space is regular (T_3).
b) Suppose $X = \prod_{\alpha \in A} X_\alpha \neq \emptyset$. X is regular (T_3) iff each X_α is regular (T_3).

Proof a) Let $A \subseteq X$ where X is regular. Suppose $a \in A$ and that F is a closed set in A that does not contain a. There exists a closed set F' in X such that $F' \cap A = F$. Choose disjoint open sets U' and V' in X with $a \in U'$ and $F' \subseteq V'$. Then $U = U' \cap A$ and $V = V' \cap A$ are open in A, disjoint, $a \in U$, and $F \subseteq V$. Therefore A is regular.

b) If $X = \prod_{\alpha \in A} X_\alpha \neq \emptyset$ is regular, then part a) implies that each X_α is regular, because each X_α is homeomorphic to a subspace of X. Conversely, suppose each X_α is regular and that $U = \, < U_{\alpha_1}, ..., U_{\alpha_n} >$ is a basic open set containing x. For each α_i, we can pick an open set V_{α_i} in X_{α_i} such that $x_{\alpha_i} \in V_{\alpha_i} \subseteq \text{cl}\, V_{\alpha_i} \subseteq U_{\alpha_i}$. Then $x \in V = \, < V_{\alpha_1}, ..., V_{\alpha_n} > \, \subseteq \text{cl}\, V \subseteq \, < \text{cl}\, V_{\alpha_1}, ..., \text{cl}\, V_{\alpha_n} > \, \subseteq U$. (*Why is the last inclusion true?*) Therefore X is regular.

Since the T_1 property is hereditary and productive, a) and b) also hold for T_3-spaces •

The obvious "next step up" in separation is the following:

Definition 2.14 A topological space X is called normal if, whenever A, B are disjoint closed sets in X, there exist disjoint open sets U, V in X with $A \subseteq U$ and $B \subseteq V$. X is called a T_4-space if X is normal and T_1.

Example 2.15 a) Every pseudometric space (X, d) is normal (so every metric space is T_4). In fact, if A and B are disjoint closed sets, we can define $f(x) = \frac{d(x,A)}{d(x,A)+d(x,B)}$. Since the denominator cannot be 0, f is continuous and $f|A = 0$, $f|B = 1$. The open sets $U = \{x : f(x) < \frac{1}{2}\}$ and $V = \{x : f(x) > \frac{1}{2}\}$ are disjoint that contain A and B respectively. Therefore X is normal. *Note: the argument given is slick and clean. Can you show (X, d) is normal by directly constructing a pair of disjoint open sets that contain A and B ?*

b) Let $\mathbb{R}$ have the right ray topology $\mathcal{T} = \{(x, \infty); x \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$. $(\mathbb{R}, \mathcal{T})$ is normal because the only possible pair of disjoint closed sets is $\emptyset$ and X and we can separate these using the disjoint open sets $U = \emptyset$ and $V = X$. Also, $(\mathbb{R}, \mathcal{T})$ is not regular: for example 1 is not in the closed set $F = (-\infty, 0]$, but every open set that contains F also contains 1. So normal $\not\Rightarrow$ regular. But $(\mathbb{R}, \mathcal{T})$ is not T_1 and therefore not T_4.

When we combine "normal + T_1" into T_4, we have a property that fits perfectly into the separation hierarchy.

Theorem 2.16 $T_4 \Rightarrow T_3$ ($\Rightarrow T_2 \Rightarrow T_1 \Rightarrow T_0$)

Proof Suppose X is T_4. If F is a closed set not containing x, then $\{x\}$ and F are disjoint closed sets. By normality, we can find disjoint open sets separating $\{x\}$ and F. It follows that X is regular and therefore T_3. •

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Exercises

E1. X is called a door space if every subset is either open or closed. Prove that if a T_2-space X contains two points that are not isolated, then X is not a door space, and that otherwise X is a door space.

E2. A base for the closed sets in a space X is a collection of $\mathcal{F}$ of closed subsets such that every closed set F is an intersection of sets from $\mathcal{F}$. Clearly, $\mathcal{F}$ is a base for the closed sets in X iff $\mathcal{B} = \{O : O = X - F,\ F \in \mathcal{F}\}$ is a base for the open sets in X.

For a polynomial P in n real variables, define the zero set of P as

$$Z(P) = \{(x_1, x_2, ..., x_n) \in \mathbb{R}^n : P\ (x_1, x_2, ..., x_n) = 0\}$$

a) Prove that $\{Z(P) : P$ a polynomial in n real variables$\}$ is the base for the closed sets of a topology (called the Zariski topology) on $\mathbb{R}^n$.

b) Prove that the Zariski topology on $\mathbb{R}^n$ is T_1 but not T_2.

c) Prove that the Zariski topology on $\mathbb{R}$ is the cofinite topology, but that if $n > 1$, the Zariski topology on $\mathbb{R}^n$ is not the cofinite topology.

Note: The Zariski topology arises in studying algebraic geometry. After all, the sets $Z(P)$ are rather special geometric objects—those "surfaces" in $\mathbb{R}^n$ which can be described by polynomial equations $P(x_1, x_2, ..., x_n) = 0$.

E3. A space X is a $T_{5/2}$ space if, whenever $x \neq y \in X$, there exist open sets U and V such that $x \in U$, $y \in V$ and $\text{cl}\, U \cap \text{cl}\, V = \emptyset$. (*Clearly,* $T_3 \Rightarrow T_{5/2} \Rightarrow T_2$.)

a) Prove that a subspace of a $T_{5/2}$ space is a $T_{5/2}$ space.

b) Suppose $X = \prod X_\alpha \neq \emptyset$. Prove that X is $T_{5/2}$ iff each X_α is $T_{5/2}$.

c) Let $S = \{(x, y) \in \mathbb{R}^2 : y \geq 0\}$ and $L = \{(x, y) \in S : y = 0\}$. Define a topology on S with the following neighborhood bases:

if $p \in S - L$: $\quad \mathcal{B}_p = \{B_\epsilon(p) \cap S : \epsilon > 0\}$
if $p \in L$: $\quad \mathcal{B}_p = \{B_\epsilon(p) \cap (S - L) \cup \{p\} : \epsilon > 0\}$

You may assume that these $\mathcal{B}_p$'s satisfy the axioms for a neighborhood base.

Prove that S is $T_{5/2}$ but not T_3.

E4. Suppose $A \subseteq X$. Define a topology on X by

$$\mathcal{T} = \{O \subseteq X : O \supseteq A\} \cup \{\emptyset\}$$

Decide whether or not $(X, \mathcal{T})$ is normal.

E5. A function $f : X \to Y$ is called perfect if f is continuous, closed, onto, and, for each $y \in Y$, $f^{-1}(y)$ is compact. Prove that if X is regular and f is perfect, then Y is regular; and that if X is T_3, the Y is also T_3.

E6. a) Suppose X is finite. Prove that $(X, \mathcal{T})$ is regular iff there is a partition $\mathcal{B}$ of X that is a base for the topology.

b) Give an example to show that a compact subset K of a regular space X need not be closed. However, show that if X is regular then $\operatorname{cl} K$ is compact.

c) Suppose F is closed in a T_3-space X. Prove that

i) Prove that $F = \bigcap\{O : O \text{ is open and } F \subseteq O\}$.

ii) Define $x \sim y$ iff $x = y$ or $x, y \in F$. Prove that the quotient space $X/\sim$ is Hausdorff.

d) Suppose B is an infinite subset of a T_3-space X. Prove that there exists a sequence of open sets U_n such that each $U_n \cap B \neq \emptyset$ and that $\operatorname{cl} U_n \cap \operatorname{cl} U_m = \emptyset$ whenever $n \neq m$.

e) Suppose each point y in a space Y has a neighborhood V such that $\operatorname{cl} V$ is regular. Prove that Y is regular.

3. Completely Regular Spaces and Tychonoff Spaces

The T_3 property is well-behaved. For example, we saw in Theorem 2.13 that the T_3 property is hereditary and productive. However, the T_3 property is not sufficiently strong to give us really nice theorems.

For example, it's very useful if a space has many (nonconstant) continuous real-valued functions available to use. Remember how many times we have used the fact that continuous real-valued functions f can be defined on a metric space (X, d) using formulas like $f(x) = d(x, a)$ or $f(x) = d(x, F)$; when $|X| > 1$, we get many nonconstant real functions defined on (X, d). But a T_3-space can sometimes be very deficient in continuous real-valued functions – in 1946, Hewett gave an example of a infinite T_3-space H on which the only continuous real-valued functions are the constant functions.

In contrast, we will see that the T_4 property is strong enough to guarantee the existence of lots of continuous real-valued functions and, therefore, to prove some really nice theorems (for example, see Theorems 5.2 and 5.6 later in this chapter). The downside is that T_4-spaces turn out also to exhibit some very bad behavior: the T_4 property is not hereditary (*explain why a proof analogous to the one given for Theorem 2.13b) doesn't work*) and it is not even finitely productive. Examples of such bad behavior are a little hard to find right now, but later they will appear rather naturally.

These observations lead us to look first at a class of spaces with separation somewhere "between T_3 and T_4." We want a group of spaces that is well-behaved, but also with enough separation to give us some very nice theorems. We begin with some notation and a lemma.

Recall that $C(X) = \{f \in \mathbb{R}^X : f \text{ is continuous}\} =$ the collection of continuous real-valued functions on X

$C^*(X) = \{f \in C(X) : f \text{ is bounded}\} =$ the collection of continuous bounded real-valued functions on X

Lemma 3.1 Suppose $f, g \in C(X)$. Define real-valued functions $f \vee g$ and $f \wedge g$ by

$$(f \vee g)(x) = \max\{f(x), g(x)\}$$
$$(f \wedge g)(x) = \min\{f(x), g(x)\}$$

Then $f \vee g$ and $f \wedge g$ are in $C(X)$.

Proof We want to prove that the max or min of two continuous real-valued functions is continuous. But this follows immediately from the formulas

$$(f \vee g)(x) = \frac{f(x) + g(x)}{2} + \frac{|f(x) - g(x)|}{2}$$

$$(f \wedge g)(x) = \frac{f(x) + g(x)}{2} - \frac{|f(x) - g(x)|}{2} \quad \bullet$$

Definition 3.2 A space X is called completely regular if whenever F is a closed set and $a \notin F$, there exists a function $f \in C(X)$ such that $f(a) = 0$ and $f|F = 1$.

Informally, "completely regular" means that "a and F can be separated by a continuous real-valued function."

Note i) The definition requires that $f|F = 1$, in other words, that $F \subseteq f^{-1}[\{1\}]$. However, these two sets might not be equal.

ii) If there is such a function f, there is also a continuous $g : X \to [0, 1]$ such that $g(a) = 0$ and $g|F = 1$. For example, we could use $g = (f \vee 0) \wedge 1$ which, by Lemma 3.1, is continuous.

iii) Suppose $g : X \to [0, 1]$ is continuous and $g(a) = 0$, $g|F = 1$. The particular values $0, 1$ in the definition are not important.: they could be any real numbers $r < s$. (*If we choose a homeomorphism $\phi : [0, 1] \to [r, s]$, then it must be true that either $\phi(0) = r$, $\phi(1) = s$ or $\phi(0) = s$, $\phi(1) = r$ – why?. Then $h = \phi \circ g : X \to [r, s]$ and, depending on how you chose ϕ, $h(a) = r$ and $h|F = b$ or vice-versa.*)

Putting these observations together, we see Definition 3.2 is equivalent to:

Definition 3.2′ A space X is called completely regular if whenever F is a closed set, $a \notin F$, and r, s are real numbers with $r < s$, then there exists a continuous function $f : X \to [r, s]$ for which $f(a) = r$ and $f|F = s$.

In one way, the definition of "completely regular space" is very different the definitions for the other separation properties: the definition isn't "internal"because an "external" space, $\mathbb{R}$ is an integral part of the definition. While it is possible to contrive a purely internal definition for "completely regular," the definition is complicated and seems completely unnatural: it simply imposes some very unintuitive conditions to force the existence of enough functions in $C(X)$.

Example 3.3 Suppose (X, d) is a pseudometric space with a closed subset F and $a \notin F$. Then $f(x) = \frac{d(x,F)}{d(a,F)}$ is continuous, $f(a) = 1$ and $f|F = 0$. So (X, d) is completely regular, but if d is not a metric, then this space is not even T_0.

Definition 3.4 A completely regular T_1-space X is called a Tychonoff space (or $T_{3\frac{1}{2}}$-space).

Theorem 3.5 $T_{3\frac{1}{2}} \Rightarrow T_3$ ($\Rightarrow T_2 \Rightarrow T_1 \Rightarrow T_0$)
Proof Suppose F is a closed set in X not containing a. If X is $T_{3\frac{1}{2}}$, we can choose $f \in C(X)$ with $f(a) = 0$ and $f|F = 1$. Then $U = f^{-1}[(-\infty, \frac{1}{2})]$ and $V = f^{-1}[(\frac{1}{2}, \infty)]$ are disjoint open sets with $a \in U$, $F \subseteq V$. Therefore X is regular. Since X is T_1, X is T_3. ∙

Hewitt's example of a T_3 space on which every continuous real-valued function is constant is more than enough to show that a T_3 space may not be $T_{3\frac{1}{2}}$ (the example, in *Ann. Math., 47(1946) 503-509*, is rather complicated.). For that purpose, it is a little easier – but still nontrivial – to find a T_3 space X containing two points p, q such that for all $f \in C(X)$, $f(p) = f(q)$. Then p and $\{q\}$ cannot be separated by a function from $C(X)$ so X is not $T_{3\frac{1}{2}}$. (See D.J. Thomas, *A regular space, not completely regular,* American Mathematical Monthly, 76(1969), 181-182). The space X can then be used to construct an infinite T_3 space H (simpler than Hewitt's example) on which every continuous real-valued function is constant (see Gantner, *A regular space on which every continuous real-valued function is constant,* American Mathematical Monthly, 78(1971), 52.) Although we will not present these constructions here, we will occasionally refer to H in comments later in this section.

Note: We have not yet shown that $T_4 \Rightarrow T_{3\frac{1}{2}}$: this is true (as the notation suggests), but it is not at all easy to prove: try it! This result is in Corollary 5.3.

Tychonoff spaces continue the pattern of good behavior that we saw in preceding separation axioms, and they will also turn out to be a rich class of spaces to study.

Theorem 3.7 a) A subspace of a completely regular ($T_{3\frac{1}{2}}$) space is completely regular ($T_{3\frac{1}{2}}$).
b) Suppose $X = \prod_{\alpha \in A} X_\alpha \neq \emptyset$. X is completely regular ($T_{3\frac{1}{2}}$) iff each X_α is completely regular ($T_{3\frac{1}{2}}$) .

Proof Suppose $a \notin F \subseteq A \subseteq X$, where X is completely regular and F is a closed set in A. Pick a closed set K in X such that $K \cap A = F$ and an $f \in C(X)$ such that $f(a) = 0$ and $f|K = 1$. Then $g = f|A \in C(A)$, $g(a) = 0$ and $g|F = 1$. Therefore A is completely regular.

If $\emptyset \neq X = \prod_{\alpha \in A} X_\alpha$ is completely regular, then each X_α is homeomorphic to a subspace of X so each X_α is completely regular. Conversely, suppose each X_α is completely regular and that F is a closed set in X not containing a. There is a basic open set U such that

$$a \in U = \; < U_{\alpha_1}, ... U_{\alpha_n} > \; \subseteq X - F$$

For each $i = 1, ..., n$ we can pick a continuous function $f_{\alpha_i} : X_{\alpha_i} \to [0, 1]$ with $f_{\alpha_i}(a_{\alpha_i}) = 0$ and $f_{\alpha_i}|(X_{\alpha_i} - U_{\alpha_i}) = 1$. Define $f : X \to [0, 1]$ by

$$f(x) = \max\{(f_{\alpha_i} \circ \pi_{\alpha_i})(x) : i = 1, ..., n\} = \max\{f_{\alpha_i}(x_{\alpha_i}) : i = 1, ..., n\}$$

Then f is continuous and $f(a) = \max\{f_{\alpha_i}(a_{\alpha_i}) : i = 1, ..., n\} = 0$. If $x \in F$, then for some i, $x_{\alpha_i} \notin U_{\alpha_i}$ and $f_{\alpha_i}(x_{\alpha_i}) = 1$, so $f(x) = 1$. Therefore $f|F = 1$ and X is completely regular.
.

Since the T_1 property is both hereditary and productive, the statements in a) and b) also hold for $T_{3\frac{1}{2}}$. •

Corollary 3.8 For any cardinal m, the "cube" $[0, 1]^m$ and all its subspaces are $T_{3\frac{1}{2}}$.

Since a Tychonoff space X is defined using functions in $C(X)$, we expect that these functions will have a close relationship to the topology on X. We want to explore that connection.

Definition 3.9 Suppose $f \in C(X)$. Then $Z(f) = f^{-1}[\{0\}] = \{x \in X : f(x) = 0\}$ is called the zero set of f. If $A = Z(f)$ for some $f \in C(X)$, we call A a zero set in X. The complement of a zero set in X is called a cozero set: $\text{coz}(f) = X - Z(f) = \{x \in X : f(x) \neq 0\}$.

A zero set $Z(f)$ in X is closed because f is continuous. In addition, $Z(f) = \bigcap_{n=1}^{\infty} O_n$, where $O_n = \{x \in X : |f(x)| < \frac{1}{n}\}$. Each O_n is open. Therefore a zero set is always a closed G_δ-set. Taking complements shows that $\text{coz}\,(f)$ is always an open F_σ-set in X.

For $f \in C(X)$, let $g = (-1 \vee f) \wedge 1 \in C^*(X)$. Then $Z(f) = Z(g)$. Therefore $C(X)$ and $C^*(X)$ produce the same zero sets in X (and therefore also the same cozero sets).

Example 3.10

1) A closed set F in a pseudometric space (X, d) is a zero set: $F = Z(f)$, where $f(x) = d(x, F)$.

2) In general, a closed set in X might not be a zero set – in fact, a closed set in X might not even be a G_δ set.

Suppose X is uncountable and $p \in X$. Define a topology on X by letting $\mathcal{B}_x = \{\{x\}\}$ be a neighborhood at each point $x \neq p$ and letting $\mathcal{B}_p = \{B : p \in B \text{ and } X - B \text{ is countable}\}$ be the neighborhood base at p. (*Check that the conditions of the Neighborhood Base Theorem III.5.2 are satisfied.*)

All points in $X - \{p\}$ are isolated and X is clearly T_1. In fact, X is T_4.

If A and B are disjoint closed sets in X, then one of them (say A) satisfies $A \subseteq X - \{p\}$, so A is clopen. We then have open sets $U = A$ and $V = X - A \supseteq B$, so X is normal.

We do not know yet that $T_4 \Rightarrow T_{3\frac{1}{2}}$ in general, but it's easy to see that this space X is also $T_{3\frac{1}{2}}$.

If F is a closed set not containing x, then either $F \subseteq X - \{p\}$ or $\{x\} \subseteq X - \{p\}$.
So one of the sets F or $\{x\}$ is clopen and he characteristic function of that clopen set is continuous and works to show that X is completely regular.

The set $\{p\}$ is closed but $\{p\}$ is not a G_δ set in X, so $\{p\}$ <u>is not a zero set</u> in X.

Suppose $p \in \bigcap_{n=1}^{\infty} O_n$ where O_n is open. For each n, there is a basic neighborhood B_n of p such that $p \in B_n \subseteq O_n$, so $X - O_n \subseteq X - B_n$ is countable. Therefore $X - \bigcap_{n=1}^{\infty} O_n = \bigcup_{n=1}^{\infty}(X - O_n)$ is countable. Since X is uncountable, we conclude that $\{p\} \neq \bigcap_{n=1}^{\infty} O_n$.

Even when F is both closed <u>and</u> a G_δ set, F might not be a zero set. We will see examples later.

For purely technical purposes, it is convenient to notice that zero sets and cozero sets can be described in a many different forms. For example, if $f \in C(X)$, then we can see that <u>each set in the left column is a zero set</u> by choosing a suitable $g \in C(X)$:

$Z = \{x : f(x) = r\} \quad = Z(g)$, where $g(x) = f(x) - r$
$Z = \{x : f(x) \geq 0\} \quad = Z(g)$, where $g(x) = f(x) - |f(x)|$
$Z = \{x : f(x) \leq 0\} \quad = Z(g)$, where $g(x) = f(x) + |f(x)|$
$Z = \{x : f(x) \geq r\} \quad = Z(g)$, where $g(x) = (f(x) - r) - |f(x) - r|$
$Z = \{x : f(x) \leq r\} \quad = Z(g)$, where $g(x) = (f(x) - r) + |f(x) - r|$

On the other hand, if $g \in C(X)$, we can write $Z(g)$ in any of the forms listed above by choosing an appropriate function $f \in C(X)$:

$$Z(g) = \{x : f(x) = r\} \quad \text{where} \quad f(x) = g(x) + r$$
$$Z(g) = \{x : f(x) \geq 0\} \quad \text{where} \quad f(x) = -|g(x)|$$
$$Z(g) = \{x : f(x) \leq 0\} \quad \text{where} \quad f(x) = |g(x)|$$
$$Z(g) = \{x : f(x) \geq r\} \quad \text{where} \quad f(x) = r - |g(x)|$$
$$Z(g) = \{x : f(x) \leq r\} \quad \text{where} \quad f(x) = r + |g(x)|$$

Taking complements, we get the corresponding results for cozero sets: if $f \in C(X)$

i) the sets $\{x : f(x) \neq r\}$, $\{x : f(x) < 0\}$, $\{x : f(x) > 0\}$, $\{x : f(x) < r\}$, $\{x : f(x) > r\}$ are cozero sets, and

ii) any given cozero set can be written in any one of these forms.

Using the terminology of cozero sets, we can see a nice comparison/contrast between regularity and complete regularity. Suppose $x \notin F$, where F is closed in X. If X is regular, we can find disjoint open sets U and V with $x \in U$ and $F \subseteq V$. But if X is completely regular, we can separate x and F with "special" open sets U and V – cozero sets! Just choose $f \in C(X)$ with $f(x) = 0$ and $f|F = 1$, then

$$x \in U = \{x : f(x) < \tfrac{1}{2}\} \quad \text{and} \quad F \subseteq V = \{x : f(x) > \tfrac{1}{2}\}$$

In fact this observation characterizes completely regular spaces – that is, if a regular space fails to be completely regular, it is because there is a "shortage" of cozero sets – because there is a "shortage" of functions in $C(X)$ (*see Theorem 3.12, below)*. For the extreme case of a T_3 space H on which the only continuous real valued functions are constant, the only cozero sets are $\emptyset$ and H !

The next theorem reveals the connections between cozero sets, $C(X)$ and the weak topology on X.

Theorem 3.11 For any space $(X, \mathcal{T})$, $C(X)$ and $C^*(X)$ induce the same weak topology $\mathcal{T}_w$ on X, and a base for $\mathcal{T}_w$ is the collection of all cozero sets in X.

Proof A subbase for $\mathcal{T}_w$ consists of all sets of the form $f^{-1}[U]$, where U is open in $\mathbb{R}$ and $f \in C(X)$. Without loss of generality, we can assume the sets U are subbasic open sets of the form (a, ∞) and $(-\infty, b)$, so that the sets $f^{-1}[U]$ have form $\{x \in X : f(x) > a\}$ or $\{x \in X : f(x) < b\}$. But these are cozero sets of X, and every cozero set in X has this form. So the cozero sets are a subbase for $\mathcal{T}_w$ In fact, the cozero sets are actually a base because $\text{coz}(f) \cap \text{coz}(g) = \text{coz}(fg)$: the intersection of two cozero sets is a cozero set.

The same argument, with $C^*(X)$ replacing $C(X)$, shows that the cozero sets of $C^*(X)$ are a base for the weak topology on X generated by $C^*(X)$. But $C(X)$ and $C^*(X)$ produce the same cozero sets in X, and therefore generate the same weak topology $\mathcal{T}_w$ on X. •

Now we can now see the close connection between X and $C(X)$ in completely regular spaces. For any space $(X, \mathcal{T})$ the functions in $C(X)$ certainly are continuous with respect to $\mathcal{T}$ (by definition of $C(X)$). But is $\mathcal{T}$ the smallest topology making this collection of functions continuous? In other words, is $\mathcal{T}$ the weak topology on X generated by $C(X)$? The next theorem says that is true precisely when X is completely regular.

Theorem 3.12 For any space $(X, \mathcal{T})$, the following are equivalent:

a) X is completely regular
b) The cozero sets of X are a base for the topology on X (equivalently, the zero sets of X are a base for the closed sets—meaning that every closed set is an intersection of zero sets)
c) X has the weak topology from $C(X)$ (equivalently, from $C^*(X)$)
d) $C(X)$ (equivalently, $C^*(X)$) separates points from closed sets.

Proof The preceding theorem shows that b) and c) are equivalent.

a) $\Rightarrow$ b) Suppose $a \in O$ where O is open. Let $F = X - O$. Then we can choose $f \in C(X)$ with $f(a) = 0$ and $f|F = 1$. Then $U = \{x : f(x) < \frac{1}{2}\}$ is a cozero set for which $a \in U \subseteq O$. Therefore the cozero sets are a base for X.

b) $\Rightarrow$ d) Suppose F is a closed set not containing a. By b), we can choose $f \in C(X)$ so that $a \in \text{coz}\,(f) \subseteq X - F$. Then $f(a) = r \neq 0$, so $f(a) \notin \text{cl}\, f[F] = \{0\}$. Therefore $C(X)$ separates points and closed sets.

d) $\Rightarrow$ a) Suppose F is a closed set not containing a. There is some $f \in C(X)$ for which $f(a) \notin \text{cl}\, f[F]$. Without loss of generality (*why?*), we can assume $f(a) = 0$. Then, for some $\epsilon > 0$, $(-\epsilon, \epsilon) \cap f[F] = \emptyset$, so that for $x \in F$, $|f(x)| \geq \epsilon$. Define $g \in C^*(X)$ by $g(x) = \min\{|f(x)|, \epsilon\}$. Then $g(a) = 0$ and $g|F = \epsilon$, so X is completely regular.

At each step of the proof, $C(X)$ can be replaced by $C^*(X)$ (*check!*) •

The following corollary is curious and the proof is a good test of whether one understands the idea of "weak topology."

Corollary 3.13 Suppose X is a set and let $\mathcal{T}_{\mathcal{F}}$ be the weak topology on X generated by <u>any</u> family of functions $\mathcal{F} \subseteq \mathbb{R}^X$. Then $(X, \mathcal{T}_{\mathcal{F}})$ is completely regular. $(X, \mathcal{T}_{\mathcal{F}})$ is Tychonoff is $\mathcal{F}$ separates points.

Proof Give X the topology the weak topology $\mathcal{T}_{\mathcal{F}}$ generated by $\mathcal{F}$. Now X has a topology, so the collection $C(X)$ makes sense. Let T_w be the weak topology on X generated by $C(X)$.

The topology $\mathcal{T}_{\mathcal{F}}$ <u>does</u> make all the functions in $C(X)$ continuous, so $\mathcal{T}_w \subseteq \mathcal{T}_{\mathcal{F}}$.

On the other hand: $\mathcal{F} \subseteq C(X)$ by definition of $\mathcal{T}_{\mathcal{F}}$, and the larger collection of functions $C(X)$ generates a (potentially) larger weak topology. Therefore $\mathcal{T}_{\mathcal{F}} \subseteq \mathcal{T}_w$.

Therefore $\mathcal{T}_{\mathcal{F}} = \mathcal{T}_w$. By Theorem 3.12, $(X, \mathcal{T}_{\mathcal{F}})$ is completely regular. •

Example 3.14

1) If $\mathcal{F} = \{f \in \mathbb{R}^{\mathbb{R}} : f$ is nowhere differentiable$\}$, then the weak topology $\mathcal{T}_{\mathcal{F}}$ on $\mathbb{R}$ generated by $\mathcal{F}$ is completely regular.

2) If H is an infinite T_3 space on which every continuous real-valued function is constant (*see the comments at the beginning of this Section 3*), then the weak topology generated by $C(X)$ has for a base the collection of cozero sets $\{\emptyset, H\}$. So the weak topology generated by $C(X)$ is the trivial topology, not the original topology on X.

Theorem 3.12 leads to a lovely characterization of Tychonoff spaces.

Corollary 3.15 Suppose X is a Tychonoff space. For each $f \in C^*(X)$, we have $\text{ran}(f) \subseteq [a_f, b_f] = I_f$ for some $a_f < b_f \in \mathbb{R}$. The evaluation map $e : X \to \prod \{I_f : f \in C^*(X)\}$ is an embedding.

Proof X is T_1, the f's are continuous and the collection of f's $(= C^*(X))$ separates points and closed sets. By Corollary VI.4.11, e is an embedding. •

Since each I_f is homeomorphic to $[0,1]$, $\prod\{I_f : f \in C^*(X)\}$ is homeomorphic to $[0,1]^m$, where $m = |C^*(X)|$. Therefore any Tychonoff space can be embedded in a "cube." On the other hand (Corollary 3.8) $[0,1]^m$ and all its subspaces are Tychonoff. So we have:

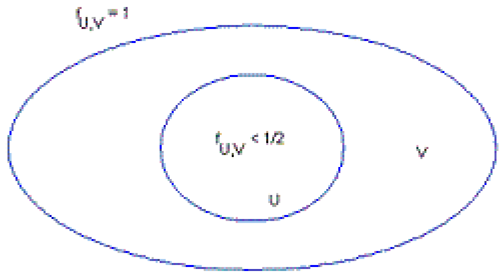
Corollary 3.16 A space X is Tychonoff iff X is homeomorphic to a subspace of a "cube" $[0,1]^m$ for some cardinal number m.

The exponent $m = |C^*(X)|$ in the corollary may not be the smallest exponent possible. As an extreme case, for example, we have $c = |C^*(\mathbb{R})|$, even though we can embed $\mathbb{R}$ in $[0,1] = [0,1]^1$. The following theorem improves the value for m in certain cases (*and we proved a similar result for metric spaces* (X,d)*: see Example VI.4.5.*)

Theorem 3.17 Suppose X is Tychonoff with a base $\mathcal{B}$ of cardinality m. Then X can be embedded in $[0,1]^m$. In particular, X can be embedded in $[0,1]^{w(X)}$.

Proof Suppose m is finite. Since X is T_1, $\{x\} = \bigcap\{B : B$ is a basic open set containing $x\}$. Only finitely many such intersections are possible, so X is finite and therefore discrete. Hence $X \overset{\text{top}}{\subseteq} [0,1] \overset{\text{top}}{\subseteq} [0,1]^m$.

Suppose $\mathcal{B}$ is a base of cardinal m where m is infinite. Call a pair $(U,V) \in \mathcal{B} \times \mathcal{B}$ distinguished if there exists a continuous $f_{U,V} : X \to [0,1]$ with $f_{U,V}(x) < \frac{1}{2}$ for all $x \in U$ and $f_{U,V}(x) = 1$ for all $x \in X - V$. Clearly, $U \subseteq V$ for a distinguished pair (U,V). For each distinguished pair, pick such a function $f_{U,V}$ and let $\mathcal{F} = \{f_{U,V} : (U,V) \in \mathcal{B} \times \mathcal{B}$ is distinguished$\}$.



We note that if $a \in V \in \mathcal{B}$, then there must exist $U \in \mathcal{B}$ such that $a \in U$ and (U, V) is distinguished. To see this, pick an $f : X \to [0, 1]$ so that $f(a) = 0$ and $f|X - V = 1$. Then choose $U \in \mathcal{B}$ so that $a \in U \subseteq f^{-1}[[0, \frac{1}{2})] \subseteq V$.

We claim that $\mathcal{F}$ separates points and closed sets:

> Suppose F is a closed set not containing a. Choose a basic set $V \in \mathcal{B}$ with $a \in V \subseteq X - F$. There is a distinguished pair (U, V) with $a \in U \subseteq V \subseteq X - F$. Then $f_{U,V}(a) = r < \frac{1}{2}$ and $f_{U,V}|F = 1$, so $f_{U,V}(a) \notin \text{cl}\, f_{U,V}[F] = \{1\}$.

By Corollary VI.4.11, $e : X \to [0, 1]^{|\mathcal{F}|}$ is an embedding. Since m is infinite, $|\mathcal{F}| \leq |\mathcal{B} \times \mathcal{B}| = m^2 = m$. •

A theorem that states that certain topological properties of a space X imply that X is metrizable is called a "metrization theorem." Typically the hypotheses of a metrization theorem involve that
1) X has "enough separation" and 2) X has a "sufficiently nice base." The following theorem is a simple example.

Corollary 3.18 ("Baby Metrization Theorem") A second countable Tychonoff space X is metrizable.

Proof By Theorem 3.17, $X \overset{\text{top}}{\subseteq} [0, 1]^{\aleph_0}$. Since $[0, 1]^{\aleph_0}$ is metrizable, so is X. •

In Corollary 3.18, X turns out to be metrizable and separable (since X is second countable). On the other hand $[0, 1]^{\aleph_0}$ and all its subspaces are separable metrizable spaces. Thus, the corollary tells us that the separable metrizable spaces (topologically) are precisely the second countable Tychonoff spaces.

Exercises

E7. Prove that if X is a countable Tychonoff space, then there is a neighborhood base of clopen sets at each point. (Such a space X is sometimes called zero-dimensional.)

E8. Prove that in any space X, a countable union of cozero sets is a cozero set – or, equivalently, that a countable intersection of zero sets is a zero set.

E9. Prove that the following are equivalent in any Tychonoff space X:

a) every zero set is open
b) every G_δ set is open
c) for each $f \in C(X)$: if $f(p) = 0$ then there is a neighborhood N of p such that $f \mid N \equiv 0$

E10. Let $i : \mathbb{R} \to \mathbb{R}$ be the identity map and let

$$(i) = \{f \in C(\mathbb{R}) : f = gi \text{ for some } g \in C(\mathbb{R})\}.$$

(i) is called the ideal in $C(\mathbb{R})$ generated by the element i.

For those who know a bit of algebra: if we definite addition and multiplication of functions pointwise, then $C(\mathbb{R})$ (or, more generally, $C(X)$) is a commutative ring. The constant function $\mathbf{0}$ is the zero element in the ring; there is also a unit element, namely the constant function "1."

a) Prove that $(i) = \{f \in C(\mathbb{R}) : f(0) = 0 \text{ and the derivative } f'(0) \text{ exists}\}$.

b) Exhibit two functions f, g in $C(\mathbb{R})$ for which $fg \in (i)$ yet $f \notin (i)$ and $g \notin (i)$.

c) Let X be a Tychonoff space with more than one point. Prove that there are two functions $f, g \in C(X)$ such that $fg = \mathbf{0}$ on X yet neither f nor g is identically 0 on X.
Thus, there are functions $f, g \in C(X)$ for which $fg = \mathbf{0}$ although $f \neq \mathbf{0}$ and $g \neq \mathbf{0}$. In an algebra course, such elements f and g in the ring $C(X)$ are called "zero divisors."

d) Prove that there are exactly two functions $f \in C(\mathbb{R})$ for which $f^2 = f$. (*In* $C(X)$, the notation $f^2(x)$ *means* $f(x) \cdot f(x)$, *not* $f(f(x))$.)

e) Prove that there are exactly *c* functions f in $C(\mathbb{Q})$ for which $f^2 = f$.

An element in $C(X)$ that equals its own square is called an idempotent. Part d) shows that $C(\mathbb{R})$ and $C(\mathbb{Q})$ are not isomorphic rings since they have different numbers of idempotents. Is either $C(\mathbb{R})$ or $C(\mathbb{Q})$ isomorphic to $C(\mathbb{N})$?

One classic part of general topology is to explore the relationship between the space X and the rings $C(X)$ and $C^*(X)$. For example, if X is homeomorphic to Y, then $C(X)$ is isomorphic to $C(Y)$. This necessarily implies (why?) that $C^*(X)$ is isomorphic to $C^*(Y)$. The question "when does *isomorphism imply homeomorphism?" is more difficult. Another important area of study is how the maximal ideals of the ring $C(X)$ are related to the topology of X. The best introduction to this material is the classic book Rings of Continuous Functions (Gillman-Jerison).*

f) Let $D(\mathbb{R})$ be the set of differentiable functions $f : \mathbb{R} \to \mathbb{R}$. Are the rings $C(\mathbb{R})$ and $D(\mathbb{R})$ isomorphic? *Hint: An isomorphism between $C(\mathbb{R})$ and $D(\mathbb{R})$ preserves cube roots.*

E11. Suppose X is a connected Tychonoff space with more than one point. Prove $|X| \geq c$..

E12. Let X be a topological space. Suppose $f, g \in C(X)$ and that $Z(f)$ is a neighborhood of $Z(g)$ (that is, $Z(g) \subseteq \operatorname{int} Z(f)$.)

a) Prove that f is a multiple of g in $C(X)$, that is, prove there is a function $h \in C(X)$ such that $f(x) = g(x)h(x)$ for all $x \in X$.

b) Give an example where $Z(f) \supseteq Z(g)$ but f is not a multiple of g in $C(X)$.

E13. Let X be a Tychonoff space with subspaces F and A, where F is closed and A is countable. Prove that if $F \cap A = \emptyset$, then A is disjoint from some zero set that contains F.

E14. A space X is called pseudocompact if every continuous $f : X \to \mathbb{R}$ is bounded, that is, if $C(X) = C^*(X)$ (s*ee Definition IV.8.7*). Consider the following condition (*) on a space X:

(*) Whenever $V_1 \supseteq V_2 \supseteq ... \supseteq V_n \supseteq ...$ is a decreasing sequence of nonempty open sets,

then $\bigcap_{n=1}^{\infty} \operatorname{cl} V_n \neq \emptyset$.

a) Prove that if X satisfies (*), then X is pseudocompact.

b) Prove that if X is Tychonoff and pseudocompact, then X satisfies (*).

Note: For Tychonoff spaces, part b) gives an "internal" characterization of pseudocompactness – that is, a characterization that makes no explicit reference to $\mathbb{R}$.

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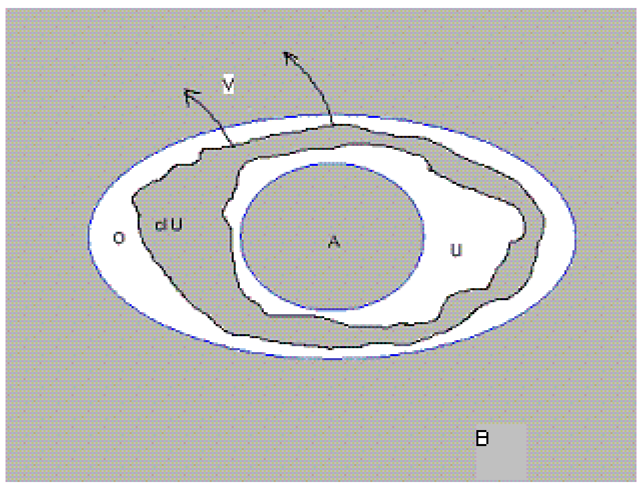
4. Normal and T_4-Spaces

We now return to a topic in progress: normal spaces (and T_4-spaces). Even though normal spaces are badly behaved in some ways, there are still some very important (and nontrivial) theorems that we can prove. One of these will give "$T_4 \Rightarrow T_{3\frac{1}{2}}$" as an immediate corollary.

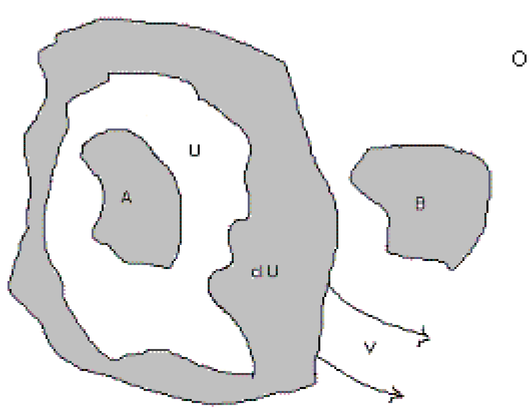
To begin, the following technical variation on the definition of normality is very useful.

Lemma 4.1 A space X is normal iff whenever A is closed, O is open and $A \subseteq O$, there exists an open set U with $A \subseteq U \subseteq \text{cl}\, U \subseteq O$.

Proof Suppose X is normal and that O is an open set containing the closed set A. Then A and $B = X - O$ are disjoint closed sets. By normality, there are disjoint open sets U and V with $A \subseteq U$ and $B \subseteq V$. Then $A \subseteq U \subseteq \text{cl}\, U \subseteq X - V \subseteq O$.



Conversely, suppose X satisfies the stated condition and that A, B are disjoint closed sets.



Then $A \subseteq O = X - B$, so there is an open set U with $A \subseteq U \subseteq \text{cl}\, U \subseteq X - B$. Let $V = X - \text{cl}\, U$. U and V are disjoint closed sets containing A and B respectively, so X is normal. $\bullet$

Theorem 4.2 a) A closed subspace of a normal (T_4) space is normal (T_4).
b) A continuous closed image of a normal (T_4) space normal (T_4).

Proof a) Suppose F is a closed subspace of a normal space X and let A and B be disjoint closed sets in F. Then A, B are also closed in X so we can find disjoint open sets U' and V' in X containing A and B respectively. Then $U = U' \cap F$ and $V = V' \cap F$ are disjoint open sets in F that contain A and B, so F is normal.

b) Suppose X is normal and that $f : X \to Y$ is continuous, closed and onto. If A and B are disjoint closed sets in Y, then $f^{-1}[A]$ and $f^{-1}[B]$ are disjoint closed sets in X. Pick U' and V' disjoint open sets in X with $f^{-1}[A] \subseteq U'$ and $f^{-1}[B] \subseteq V'$. Then $U = Y - f[X - U']$ and $V = Y - f[X - V']$ are open sets in Y.

If $y \in U$, then $y \notin f[X - U']$. Since f is onto, $y = f(x)$ for some $x \in U' \subseteq X - V'$. Therefore $y \in f[X - V']$ so $y \notin V$. Hence $U \cap V = \emptyset$.

If $y \in A$, then $f^{-1}[\{y\}] \subseteq f^{-1}[A] \subseteq U'$, so $f^{-1}[\{y\}] \cap (X - U') = \emptyset$. Therefore $y \notin f[X - U']$ so $y \in Y - f[X - U'] = U$. Therefore $A \subseteq U$ and, similarly, $B \subseteq V$ so Y is normal.

Since the T_1 property is hereditary and is preserved by closed onto maps, the statements in a) and b) hold for T_4 as well as normality. •

The next theorem gives us more examples of normal (and T_4) spaces.

Theorem 4.3 Every regular Lindelöf space X is normal (and therefore every Lindelöf T_3-space is T_4).

Proof Suppose A and B are disjoint closed sets in X. For each $x \in A$, use regularity to pick an open set U_x such that $x \in U_x \subseteq \text{cl}\, U_x \subseteq X - B$. Since the Lindelöf property is hereditary on closed subsets, a countable number of the U_x's cover A: relabel these as $U_1, U_2, ..., U_n, ...$. For each n, we have $\text{cl}\, U_n \cap B = \emptyset$. Similarly, choose a sequence of open sets $V_1, V_2, ..., V_n, ...$ covering B such that $\text{cl}\, V_n \cap A = \emptyset$ for each n.

We have that $\bigcup_{n=1}^{\infty} U_n \supseteq A$ and $\bigcup_{n=1}^{\infty} V_n \supseteq B$, but these unions may not be disjoint. So we define

$$\begin{array}{ll}
U_1^* = U_1 - \text{cl}\, V_1 & V_1^* = V_1 - \text{cl}\, U_1 \\
U_2^* = U_2 - (\text{cl}\, V_1 \cup \text{cl}\, V_2) & V_2^* = V_2 - (\text{cl}\, U_1 \cup \text{cl}\, U_2) \\
\quad\vdots & \quad\vdots \\
U_n^* = U_n - (\text{cl}\, V_1 \cup \text{cl}\, V_2 \cup ... \cup \text{cl}\, V_n) & V_n^* = V_n - (\text{cl}\, U_1 \cup \text{cl}\, U_2 \cup ... \cup \text{cl}\, U_n) \\
\quad\vdots & \quad\vdots
\end{array}$$

Let $U = \bigcup_{n=1}^{\infty} U_n^*$ and $V = \bigcup_{n=1}^{\infty} V_n^*$.

If $x \in A$, then $x \notin \text{cl}\ V_n$ for all n. But $x \in U_k$ for some k, so $x \in U_k^* \subseteq U$. Therefore $A \subseteq U$ and, similarly, $B \subseteq V$.

To complete the proof, we show that $U \cap V = \emptyset$. Suppose $x \in U$.

Then $x \in U_k^*$ for some k, so $x \notin \text{cl}\, V_1 \cup \text{cl}\, V_2 \cup ... \cup \text{cl}\, V_k$,
so $x \notin V_1 \cup V_2 \cup ... \cup V_k$
so $x \notin V_1^* \cup V_2^* \cup ... \cup V_k^*$
so $x \notin V_n^*$ for any $n \leq k$.

Since $x \in U_k^*$, then $x \in U_k$. So, if $n > k$, then $x \notin V_n^* = V_n - (\text{cl}\, U_1 \cup ... \cup \text{cl}\, U_k \cup ... \cup \text{cl}\, U_n)$

So $x \notin V_n^*$ for all n, so $x \notin V$ and therefore $U \cap V = \emptyset$. •

Example 4.4 The Sorgenfrey line S is regular because the sets $[a, b)$ form a base of closed neighborhoods at each point a. We proved in Example VI.3.2 that S is Lindelöf, so S is normal. Since S is T_1, we have that S is T_4.

5. Urysohn's Lemma and Tietze's Extension Theorem

We now turn our attention to the issue of "$T_4 \Rightarrow T_{3\frac{1}{2}}$". This is hard to prove because to show that a space X is $T_{3\frac{1}{2}}$, we need to prove that certain continuous functions exist; but the hypothesis "T_4" gives us no continuous functions to work with. As far as we know at this point, there could even be T_4 spaces on which every continuous real-valued function is constant! If T_4-spaces are going to have a rich supply of continuous real-valued functions, we will have to show that these functions can be "built from scratch" in a T_4-space. This will lead us to two of the most well-known classical theorems of general topology.

We begin with the following technical lemma. It gives a way to use a certain collection of open sets $\{U_r : r \in Q\}$ to construct a function $f \in C(X)$. The idea in the proof is quite straightforward, but I attribute its elegant presentation (and that of Urysohn's Lemma which follows) primarily to Leonard Gillman and Meyer Jerison.

Lemma 5.1 Suppose X is any topological space and let Q be any dense subset of $\mathbb{R}$. Suppose open sets $U_r \subseteq X$ have been defined, one for each $r \in Q$, in such a way that:

i) $X = \bigcup_{r \in Q} U_r$ and $\bigcap_{r \in Q} U_r = \emptyset$
ii) if $r, s \in Q$ and $r < s$, then $\text{cl}\, U_r \subseteq U_s$.

For $x \in X$, define $f(x) = \inf\{r \in Q : x \in U_r\}$. Then $f : X \to \mathbb{R}$ is continuous.

We will use this Lemma only once, with $Q = \mathbb{Q}$. So if you like, there is no harm in assuming that $Q = \mathbb{Q}$ in the proof.

Proof Suppose $x \in X$. By i) we know that $x \in U_r$ for some r, so $\{r \in Q : x \in U_r\} \neq \emptyset$. And by ii), we know that $x \notin U_s$ for some s. For that s : if $x \in U_r$, then (by ii) $s \leq r$, so s is a lower bound for $\{r \in Q : x \in U_r\}$. Therefore $\{r \in Q : x \in U_r\}$ has a greatest lower bound, so the definition of f makes sense: $f(x) = \inf\{r \in Q : x \in U_r\} \in \mathbb{R}$.

From the definition of f, we get that for $r, s \in Q$,

a) if $x \in \text{cl}\, U_r$, then $x \in U_s$ for all $s > r$ so $f(x) \leq r$
b) if $f(x) < s$, then $x \in U_s$.

We want to prove f is continuous at each point $a \in X$. Since Q is dense in $\mathbb{R}$,

$$\{[r, s] : r, s \in Q \text{ and } r < f(a) < s\}$$

is a neighborhood base at $f(a)$ in $\mathbb{R}$. Therefore it is sufficient to show that whenever $r < f(a) < s$, then there is a neighborhood U of a such that $f[U] \subseteq [r, s]$.

Since $f(a) < s$, we have $a \in U_s$, and $f(a) > r$ gives us that $a \notin \text{cl}\, U_r$. Therefore $U = U_s - \text{cl}\, U_r$ is an open neighborhood of a. If $z \in U$, then $z \in U_s \subseteq \text{cl}\, U_s$, so $f(z) \leq s$; and $z \notin \text{cl}\, U_r$, so $z \notin U_r$ and $f(z) \geq r$. Therefore $f[U] \subseteq [r, s]$. •

Our first major theorem about normal spaces is still traditionally referred to as a "lemma" because it was a lemma in the paper where it originally appeared. Its author, Paul Urysohn, died at age 26, on the morning of 17 August 1924, while swimming off the coast of Brittany.

Theorem 5.2 (Urysohn's Lemma) A space X is normal iff whenever A, B are disjoint closed sets in X, there exists a function $f \in C(X)$ with $f|A = 0$ and $f|B = 1$. (*When such an f exists, we say that A and B are completely separated.*)

Note: Notice that if A and B happen to be disjoint zero sets, say $A = Z(g)$ and $B = Z(h)$, then the conclusion of the theorem is true in any space, without assuming normality: just let $f(x) = \frac{g^2(x)}{g^2(x) + h^2(x)}$. Then f is continuous, $f|A = 0$ and $f|B = 1$.

The conclusion of Urysohn's Lemma only says that $A \subseteq f^{-1}(0)$ and $B \subseteq f^{-1}(1)$: equality might not be true. In fact, if $A = f^{-1}(0)$ and $B = f^{-1}(1)$, then A and B were zero sets in the beginning, and the hypothesis of normality would have been unnecessary.

This shows again that zero sets are very special closed sets: in any space, disjoint zero sets are completely separated. Put another way: given Urysohn's Lemma, we can conclude that every nonnormal space must contain a closed set that is not a zero set.

Proof The proof of Urysohn's Lemma in one direction is almost trivial. If such a function f exists, then $U = \{x : f(x) < \frac{1}{2}\}$ and $V = \{x : f(x) > \frac{1}{2}\}$ are disjoint open sets (in fact, cozero sets) containing A and B respectively. It is the other half of Urysohn's Lemma for which Urysohn deserves credit.

Let A and B be disjoint closed sets in a normal space X. We will define sets open sets U_r ($r \in \mathbb{Q}$) in X in such a way that Lemma 5.1 applies. To start, let $U_r = \emptyset$ for $r < 0$ and $U_r = X$ for $r > 1$.

Enumerate the remaining rationals in $\mathbb{Q} \cap [0,1]$ as $r_1,\ r_2, ..., r_n, \dots,$ beginning the list with $r_1 = 1$ and $r_2 = 0$. We begin by defining $U_{r_1} = U_1 = X - B$. Then use normality to define $U_{r_2}(= U_0)$: since $A \subseteq U_{r_1} = X - B$, we can pick U_{r_2} so that

$$A \subseteq U_{r_2} \subseteq \text{cl}\, U_{r_2} \subseteq U_{r_1} = X - B$$

Then $0 = r_2 < r_3 < r_1 = 1$, and we use normality to pick an open set U_{r_3} so that

$$A \subseteq U_{r_2} \subseteq \text{cl}\, U_{r_2} \subseteq U_{r_3} \subseteq \text{cl}\, U_{r_3} \subseteq U_{r_1} = X - B$$

We continue by induction. Suppose $n \geq 3$ and that we have already defined open sets $U_{r_1}, U_{r_2}, ..., U_{r_n}$ in such a way that whenever $r_i < r_j < r_k \ (i,j,k \leq n)$, then

$$\text{cl}\, U_{r_i} \subseteq U_{r_j} \subseteq \text{cl}\, U_{r_j} \subseteq U_{r_k} \quad (*)$$

We need to define $U_{r_{n+1}}$ so that $(*)$ holds for $i,j,k \leq n+1$.

Since $r_1 = 1$ and $r_2 = 0$, and $r_{n+1} \in (0,1)$, it makes sense to define

$r_k =$ the largest among $r_1, r_2, ..., r_n$ that is smaller than r_{n+1}, and
$r_l =$ the smallest among $r_1, r_2, ..., r_n$ that is larger than r_{n+1}.

By the induction hypothesis, we already have $\text{cl}\, U_{r_k} \subseteq U_{r_l}$. Then use normality to pick an open set $U_{r_{n+1}}$ so that

$$\text{cl}\, U_{r_k} \subseteq U_{r_{n+1}} \subseteq \text{cl}\, U_{r_{n+1}} \subseteq U_{r_l}.$$

The U_r's defined in this way satisfy the conditions of Lemma 5.1, so the function $f : X \to \mathbb{R}$ defined by $f(x) = \inf\{r \in \mathbb{Q} : x \in U_r\}$ is continuous. If $x \in A$, then $x \in U_{r_2} = U_0$ and $x \notin U_r$ if $r < 0$, so $f(x) = 0$. If $x \in B$ then $x \notin U_1$, but $x \in U_r = X$ for $r > 1$, so $f(x) = 1$. •

Once we have the function f we can replace it, if we like, by $g = (0 \vee f) \wedge 1$ so that A and B are completely separated by a function $g \in C^*(X)$. It is also clear that we can modify g further to get an $h \in C^*(X)$ for which $h|A = a$ and $h|B = b$ where a and b are any two real numbers.

With Urysohn's Lemma, the proof of the following corollary is obvious.

Corollary 5.3 $T_4 \Rightarrow T_{3\frac{1}{2}}$.

There is another famous characterization of normal spaces in terms of $C(X)$. It is a result about "extending" continuous real-valued functions defined on closed subspaces.

We begin with the following two lemmas. Lemma 5.4, called the "Weierstrass M-Test" is a slight generalization of a theorem with the same name in advanced calculus. It can be useful in "piecing together" infinitely many real-valued continuous functions to get a new one. Lemma 5.5 will be used in the proof of Tietze's Extension Theorem (Theorem 5.6).

Lemma 5.4 (Weierstrass M-Test) Let X be a topological space. Suppose $f_n : X \to \mathbb{R}$ is continuous for each $n \in \mathbb{N}$ and that $|f_n(x)| \le M_n$ for all $x \in X$. If $\sum_{n=1}^{\infty} M_n < \infty$, then $f(x) = \sum_{n=1}^{\infty} f_n(x)$ converges (absolutely) for all x and $f : X \to \mathbb{R}$ is continuous.

Proof For each x, $\sum_{n=1}^{\infty}|f_n(x)| \le \sum_{n=1}^{\infty} M_n < \infty$, so $\sum_{n=1}^{\infty} f_n(x)$ converges (absolutely) by the Comparison Test.

Suppose $a \in X$ and $\epsilon > 0$. Choose N so that $\sum_{n=N+1}^{\infty} M_n < \frac{\epsilon}{4}$. Each f_n is continuous, so for $n = 1, ..., N$ we can pick a neighborhood U_n of a such that for $x \in U_n$, $|f_n(x) - f_n(a)| < \frac{\epsilon}{2N}$. Then $U = \bigcap_{n=1}^{N} U_n$ is a neighborhood of a, and for $x \in U$ we get $|f(x) - f(a)|$

$= |\sum_{n=1}^{N}(f_n(x) - f_n(a)) + \sum_{n=N+1}^{\infty}(f_n(x) - f_n(a))| \le \sum_{n=1}^{N}|f_n(x) - f_n(a)| + \sum_{n=N+1}^{\infty}|f_n(x) - f_n(a)|$

$\le \sum_{n=1}^{N}|f_n(x) - f_n(a)| + \sum_{n=N+1}^{\infty}|f_n(x)| + |f_n(a)| < N \cdot \frac{\epsilon}{2N} + \sum_{n=N+1}^{\infty} 2M_n < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$

Therefore f is continuous at a. •

Lemma 5.5 Let A be a closed set in a normal space X and let a be a positive real number. Suppose $h : A \to [-r, r]$ is continuous. Then there exists a continuous $\phi : X \to [-\frac{r}{3}, \frac{r}{3}]$ such that $|h(x) - \phi(x)| \le \frac{2r}{3}$ for each $x \in A$.

Proof Let $A_1 = \{x \in A : h(x) \le -\frac{r}{3}\}$ and $B_1 = \{x \in A : h(x) \ge \frac{r}{3}\}$. A is closed, and A_1 and B_1 are disjoint closed sets in A, so A_1 and B_1 are closed in X. By Urysohn's Lemma, there exists a continuous function $\phi : X \to [-\frac{r}{3}, \frac{r}{3}]$ such that $\phi|A_1 = -\frac{r}{3}$ and $\phi|B_1 = \frac{r}{3}$.

If $x \in A_1$, then $-r \le h(x) \le -\frac{r}{3}$ and $\phi(x) = -\frac{r}{3}$, so $|h(x) - \phi(x)| \le |-r - (-\frac{r}{3})| = \frac{2r}{3}$; and similarly if $x \in B_1$, $|h(x) - \phi(x)| \le \frac{2r}{3}$. If $x \in A - (A_1 \cup B_1)$, then $h(x)$ and $\phi(x)$ are both in $[-\frac{r}{3}, \frac{r}{3}]$ so $|h(x) - \phi(x)| \le \frac{2r}{3}$. •

Theorem 5.6 (Tietze's Extension Theorem) A space X is normal iff whenever A is a closed set in X and $f \in C(A)$, then there exists a function $g \in C(X)$ such that $g|A = f$.

> Note: if A is a closed subset of $X = \mathbb{R}$, then it is quite easy to prove the theorem. In that case, $\mathbb{R} - A$ is open and can be written as a countable union of disjoint open intervals I, where each $I = (a, b)$ or $(-\infty, b)$ or (a, ∞) (*see Theorem II.3.4*). For each of these intervals I, the endpoints are in A, where f is already defined. If $I = (a, b)$ then extend the definition of f over I by using a straight line segment to join $(a, f(a))$ and $(b, f(b))$ on the graph of f. If $I = (a, \infty)$. then extend the graph of f over I using a horizontal right ray at height $f(a)$; if $I = (-\infty, b)$, then extend the graph of f over I using a horizontal left ray at height $f(b)$.

As with Urysohn's Lemma, half of the proof is easy. The significant part of theorem is proving the existence of the extension g when X is normal.

Proof ($\Leftarrow$) Suppose A and B are disjoint closed sets in X. A and B are clopen in the subspace $A\cup B$ so the function $f: A\cup B \to [0,1]$ defined by $f|A = 0$ and $f|B = 1$ is continuous. Since $A\cup B$ is closed in X, there is a function $g \in C(X)$ such that $g|(A\cup B) = f$. Then $U = \{x : g(x) < \frac{1}{2}\}$ and $V = \{x : g(x) > \frac{1}{2}\}$ are disjoint open sets (cozero sets, in fact) that contain A and B respectively. Therefore X is normal.

($\Rightarrow$) The idea is to find a sequence of functions $g_i \in C(X)$ such that $|f(x) - \sum_{i=1}^{n} g_i(x)| \to 0$ as $n \to \infty$ for each $x \in A$ (where f is defined). The sums $\sum_{i=1}^{n} g_i(x)$ are defined on all of X and as $n \to \infty$ we can think of them as giving better and better approximations to the extension g that we want. Then we can let $g(x) = \lim_{n\to\infty} \sum_{i=1}^{n} g_i(x) = \sum_{i=1}^{\infty} g_i(x)$. The details follow. We proceed in three steps, but the heart of the argument is in Case I.

Case I Suppose f is continuous and that $f: A \to [-1,1]$. We claim there is a continuous function $g: X \to [-1,1]$ with $g|A = f$.

Using Lemma 5.5 (with $h = f$, $r = 1$) we get a function $g_1 = \phi: X \to [-\frac{1}{3}, \frac{1}{3}]$ such that for $x \in A$, $|f(x) - g_1(x)| \leq \frac{2}{3}$. Therefore $f - g_1 : A \to [-\frac{2}{3}, \frac{2}{3}]$.

Using Lemma 5.5 again (with $h = f - g_1, r = \frac{2}{3}$), we get a function $g_2 : X \to [-\frac{2}{9}, \frac{2}{9}]$ such that for $x \in A$, $|f(x) - g_1(x) - g_2(x)| \leq \frac{4}{9} = (\frac{2}{3})^2$. So $f - (g_1 + g_2) : A \to [-\frac{4}{9}, \frac{4}{9}]$

Using Lemma 5.5 again (with $h = f - g_1 - g_2, r = \frac{4}{9}$), we get a function $g_3 : X \to [-\frac{4}{27}, \frac{4}{27}]$ such that for $x \in A$, $|f(x) - g_1(x) - g_2(x) - g_3(x)| \leq \frac{8}{27} = (\frac{2}{3})^3$. So $f - (g_1 + g_2 + g_3) : A \to [-\frac{8}{27}, \frac{8}{27}]$.

We continue, using induction, to find for each i a continuous function $g_i : X \to [-\frac{2^{i-1}}{3^i}, \frac{2^{i-1}}{3^i}]$ such that $|f(x) - \sum_{i=1}^{n} g_i(x)| \leq (2/3)^n$ for $x \in A$.

Since $\sum_{i=1}^{\infty} |g_i(x)| \leq \sum_{i=1}^{\infty} \frac{2^{i-1}}{3^i} < \infty$, the series $g(x) = \sum_{i=1}^{\infty} f_i(x)$ converges (absolutely) for every $x \in X$, and g is continuous by the Weierstrass M-Test. Since $|g(x)| = |\sum_{i=1}^{\infty} g_i(x)|$ $\leq \sum_{i=1}^{\infty} |g_i(x)| \leq \sum_{i=1}^{\infty} \frac{2^{i-1}}{3^i} = 1$, we have $g : X \to [-1,1]$.

Finally, for $x \in A$, $|f(x) - g(x)| = \lim_{n\to\infty} |f(x) - \sum_{i=1}^{n} g_i(x)| \leq \lim_{n\to\infty} (\frac{2}{3})^n = 0$, so $g|A = f$ and the proof for Step I is complete.

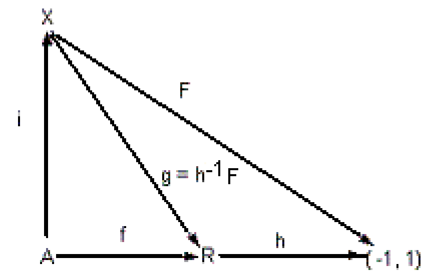
Case II Suppose $f : A \to (-1,1)$ is continuous. We claim there is a continuous function $g : X \to (-1,1)$ with $g|A = f$.

Since $f : A \to (-1,1) \subseteq [-1,1]$, we can apply Case I to find a continuous function $F : X \to [-1,1]$ with $F|A = f$. To get g, we merely make a slight modification to F to get a g that still extends f but where g has all its values in $(-1,1)$.

Let $B = \{x \in X : F(x) = \pm 1\}$. A and B are disjoint closed sets in X, so by Urysohn's Lemma there is a continuous $h : X \to [-1, 1]$ such that $h|B = 0$ and $h|A = 1$. If we let $g(x) = F(x)h(x)$, then $g : X \to (-1, 1)$ and $g|A = f$, completing the proof of Case II.

Case III (the full theorem) Suppose $f : A \to \mathbb{R}$ is continuous. We claim there is a continuous function $g : X \to \mathbb{R}$ with $g|A = f$.

Let $h : \mathbb{R} \to (-1, 1)$ be a homeomorphism. Then $h \circ f : A \to (-1, 1)$ and, by Step II, there is a continuous $F : X \to (-1, 1)$ with $F|A = h \circ f$.



Let $g = h^{-1} \circ F : X \to \mathbb{R}$. Then for $x \in A$ we have $g(x) = h^{-1}(F(x))$ $= h^{-1}((h \circ f))(x) = f(x)$. •

It is easy to see that $C^*(X)$ can replace $C(X)$ in the statement of Tietze's Extension Theorem.

Example 5.7 We now know enough about normality to see some of its bad behavior. The Sorgenfrey line S is normal (*Example 4.4*) but the Sorgenfrey plane $S \times S$ is not normal.

To see this, let $D = \mathbb{Q} \times \mathbb{Q}$, a countable dense set in $S \times S$. Every continuous real-valued function on $S \times S$ is completely determined by its values on D. (*See Theorem II.5.12. The theorem is stated for the case of functions defined on a pseudometric space, but the proof is written in a way that applies just as well to functions with any space X as domain.*) Therefore the mapping $C(S \times S) \to C(D)$ given by $f \mapsto f|D$ is one-to-one, so $|C(S \times S)| \leq |C(D) \leq |\mathbb{R}^D| = c^{\aleph_0} = c$.

$A = \{(x, y) \in S \times S : x + y = 1\}$ is closed and discrete in the subspace topology, so every function defined on A is continuous, that is, $\mathbb{R}^A = C(A)$ and so $|C(A)| = c^c = 2^c$. If $S \times S$ were normal, then each $f \in C(A)$ could be extended (by Tietze's Theorem) to a continuous function in $C(S \times S)$. This would mean that $|C(S \times S)| \geq |\mathbb{R}^A| = c^c = 2^c > c$. which is false. Therefore normality is not even finitely productive.

The comments following the statement of Urysohn's Lemma imply that $S \times S$ must contain closed sets that are not zero sets.

A completely similar argument "counting continuous real-valued functions" shows that the Moore plane Γ (Example III.5.6) is not normal: use that Γ is separable and the x-axis in Γ is an uncountable closed discrete subspace.

Questions about the normality of products are difficult. For example, it was an open question for a long time whether the product of a normal space X with such a nice, well-behaved space as $[0,1]$ must be normal. In the 1950's, Dowker proved that $X \times [0,1]$ is normal iff X is normal and "countably paracompact."

However, this result was unsatisfying – because no one knew whether a normal space was automatically "countably paracompact." In the 1960's, Mary Ellen Rudin constructed a normal space X which was not countably paracompact. But this example was still unsatisfying because the construction assumed the existence of a space called a "Souslin line" – and whether a Souslin line exists cannot be decided in the ZFC set theory! In other words, the space X she constructed required adding a new axiom to ZFC.

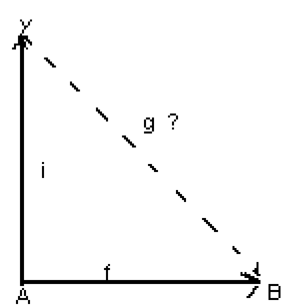
Things were finally settled in 1971 when Mary Ellen Rudin constructed a "real" example of a normal space X whose product with $[0,1]$ is not normal. By "real," we mean that X was constructed in ZFC, with no additional set theoretic assumptions. Among other things, this complicated example made use of the box topology on a product.

Example 5.8 The Sorgenfrey line S is T_4, so S is $T_{3\frac{1}{2}}$ and therefore the Sorgenfrey plane $S \times S$ is also $T_{3\frac{1}{2}}$. So $S \times S$ is an example showing that $T_{3\frac{1}{2}}$ does not imply T_4.

Extension theorems such as Tietze's are an important topic in mathematics. In general, an "extension theorem" has the following form:

$A \subseteq X$ and $f : A \to B$, then there is a function $g : X \to B$ such that $g|A = f$.

For example, in algebra one might ask: if A is a subgroup of X and $f : A \to B$ is an isomorphism, can f be extended to a homomorphism $g : X \to B$?



If we let $i : A \to X$ be the injection $i(a) = a$, then the condition "$g|A = f$" can be rewritten as $g \circ i = f$. In the language of algebra, we are asking whether there is a suitable function g which "makes the diagram commute."

Specific extension theorems impose conditions on A and X, and usually we want g to share some property of f such as continuity. Here are some illustrations, without further details.

1) Extension theorems that generalize of Tietze's Theorem: by putting stronger hypotheses on X, we can relax the hypotheses on B.

Suppose A is closed in X and $f : A \to B$ is continuous.

If $\begin{cases} X \text{ is normal} & \text{and } B = \mathbb{R} \text{ (Tietze's Theorem)} \\ X \text{ is normal} & \text{and } B = \mathbb{R}^n \\ X \text{ is collectionwise normal** } & \text{and } B \text{ is a separable Banach space*} \\ X \text{ is paracompact**} & \text{and } B \text{ is a Banach space*} \end{cases}$

then f has a continuous extension $g : X \to B$.

The statement that $\mathbb{R}^n$ can replace $\mathbb{R}$ in Tietze's Theorem is easy to prove:

If X is normal and $f : A \to \mathbb{R}^n$ is continuous, write $f(x) = (f_1(x), f_2(x), ..., f_n(x))$ where each $f_i : A \to \mathbb{R}$. By Tietze's Theorem, there exists for each i a continuous extension $g_i : X \to \mathbb{R}$ with $g_i|A = f_i$. If we let $g(x) = (g_1(x), ..., g_n(x))$, then $g : X \to \mathbb{R}^n$ and $g|A = f$. In other words, we separately extend the coordinate functions in order to extend f. And in this example, n could even be an infinite cardinal.

** A normed linear space is a vector space V with a norm $|v|$ (= "absolute value") that defines the "length" of each vector. Of course, a norm must satisfy certain axioms – for example, $|v_1 + v_2| \le |v_1| + |v_2|$. These properties guarantee that a norm can be used to define a metric: $d(v_1, v_2) = |v_1 - v_2|$. A Banach space is a normed linear space which is complete in this metric d.*

For example, $\mathbb{R}^n$: the usual norm $|(x_1, x_2, ..., x_n)| = \sqrt{x_1^2 + x_2^2 + ... + x_n^2}$ produces the usual metric, which is complete. So $\mathbb{R}^n$ is a separable Banach space.

*** Roughly, a "collectionwise normal" space is one in which certain infinite collections of disjoint closed sets can be enclosed in disjoint open sets. We will not give definitions for "collectionwise normal" (or the stronger condition, "paracompactness") here, but is true that*

$$\begin{Bmatrix} \text{metric} \\ \text{or} \\ \text{compact } T_2 \end{Bmatrix} \Rightarrow \text{paracompact} \Rightarrow \text{collectionwise normal} \Rightarrow \text{normal}$$

Therefore, in the theorems cited above, a continuous map f defined on a closed subset of a metric space (or, compact T_2 space) and valued in a Banach space B can be continuously extended a function $g : X \to B$.

2) The Hahn-Banach Theorem is another example, taken from functional analysis, of an extension theorem. A special case states:

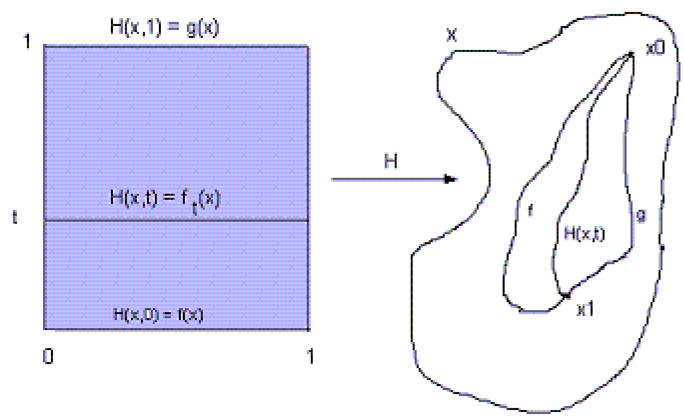
> Suppose M is a linear subspace of a real normed linear space X and that $f : M \to \mathbb{R}$ is linear and satisfies $f(x) \leq ||x||$ for all $x \in M$. There there is a linear $F : X \to \mathbb{R}$ such that $F|M = f$ and $F(x) \leq ||x||$ for all $x \in X$.

3) Homotopy is usually not discussed in terms of extension theorems, but extensions are really at the heart of the idea.

> Let $f, g : [0,1] \to X$ be continuous and suppose that $f(0) = g(0) = x_0$ and $f(1) = g(1) = x_1$. Then f and g are paths in X that start at x_0 and end at x_1. Let B be the boundary of the square $[0,1]^2 \subseteq \mathbb{R}^2$ and define $F : B \to X$ by
>
> $F(x,0) = f(x) \qquad F(x,1) = g(x)$
> $F(0,t) = x_0 \qquad F(1,t) = x_1$

Thus F agrees with f on the bottom edge of B and with g on the top edge. F is constant $(= x_0)$ on the left edge of B and constant $(= x_1)$on the right edge of B. We ask whether F can be extended to a continuous map defined on the whole square, $H : [0,1]^2 \to X$.

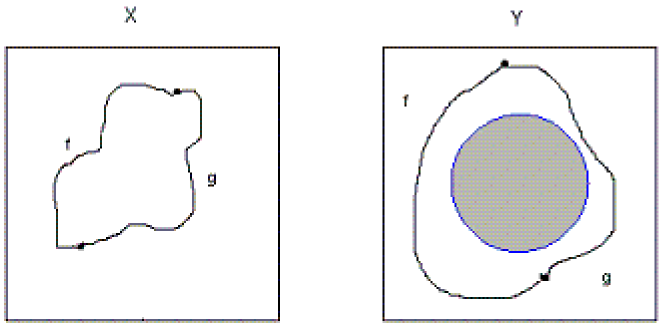
If such an extension H does exist, then we have



For each $t \in [0,1]$, restrict H to the line segment at height t to define $f_t(x) = H(x,t)$. Then for each $t \in [0,1]$, f_t is also a path in X from x_0 to x_1. As t varies from 0 to 1, we can think of the f_t's as a family of paths in X that continuously deform $f_0 = f$ into $f_1 = g$.

The continuous extension H (if it exists) is called a homotopy between f and g with fixed endpoints, and we say that the paths f and g are homotopic with fixed endpoints.

In the space X on the left, below, it seems intuitively clear that f can be continuously deformed (with endpoints held fixed) into g – in other words, that H exists.



However in the space Y pictured on the right, f and g together form a loop that surrounds a "hole" in Y, and it seems intuitively clear that the path f cannot be continuously deformed into the path g inside the space Y – that is, the extension H does not exist.

In some sense, homotopy can be used to detect the presence of certain "holes" in a space, and is one important part of algebraic topology.

The next theorem shows us where compact Hausdorff spaces stand in the discussion of separation properties.

Theorem 5.9 A compact T_2 space X is T_4.

Proof X is Lindelöf, and a regular Lindelöf space is normal (Theorem 4.3). Therefore it is sufficient to show that X is regular. Suppose F is a closed set in X and $x \notin F$. For each $y \in F$ we can pick disjoint open sets U_y and V_y with $x \in U_y$ and $y \in V_y$. F is compact so a finite number of the V_y's cover F – say $V_{y_1}, V_{y_2}, \ldots, V_{y_n}$. Then $x \in \bigcap_{i=1}^{n} U_{y_i} = U$, $F \subseteq \bigcup_{i=1}^{n} V_{y_i} = V$, and U, V are disjoint open sets. ●

Therefore, our results line up as:

$$(*) \quad \text{compact metric} \Rightarrow \left\{ \begin{array}{c} \text{compact } T_2 \\ \underline{\qquad} \\ \text{metric} \end{array} \right. \Rightarrow T_4 \Rightarrow T_{3\frac{1}{2}} \Rightarrow T_3 \Rightarrow T_2 \Rightarrow T_1 \Rightarrow T_0$$

In particular, Urysohn's Lemma and Tietze's Extension Theorem hold in metric spaces and in compact T_2 spaces.

Notice that

i) the space $(0,1)$ is T_4 but not compact T_2

ii) the Sorgenfrey line S is T_4 (see example 5.4) but not metrizable. (*If S were metrizable, then $S \times S$ would be metrizable and therefore T_4 – which is false : see Example 5.7*).

iii) $[0,1]^c$ is compact T_2 (*assuming, for now, the Tychonoff Product Theorem VI.3.10*) but not metrizable (*why?*)

iv) $(0,1)$ is metrizable but not compact.

Combining these observations with earlier examples, we see that none of the implications in (*) is reversible.

Example 5.10 (*See Example 5.7*) The Sorgenfrey plane $S \times S$ is $T_{3\frac{1}{2}}$, so $S \times S$ can be embedded in a cube $[0,1]^m$ and $[0,1]^m$ is compact T_2 (*assuming the Tychonoff Product Theorem*). Since $S \times S$ is not normal, we see now that a normal space can have nonnormal subspaces. This example, admittedly, is not terribly satisfying since we can't visualize how $S \times S$ "sits" inside $[0,1]^m$. In Chapter VIII (Example 8.10), we will look at an example of a T_4 space in which it's easy to "see" why a certain subspace isn't normal.

6. Some Metrization Results

Now we have enough information to completely characterize separable metric spaces topologically.

Theorem 6.1 (Urysohn's Metrization Theorem) A second countable T_3-space is metrizable.
Note: We proved a similar metrization theorem in Corollary 3.18, but there the separation hypothesis was $T_{3\frac{1}{2}}$ rather than T_3.

Proof X is second countable so X is Lindelöf, and Theorem 4.3 tells us that a Lindelöf T_3-space is T_4. Therefore X is $T_{3\frac{1}{2}}$. So by Corollary 3.18, X is metrizable. •

Because a separable metrizable space is second countable and T_3, we have a complete characterization: X is a separable metrizable space iff X is a second countable T_3-space. So, with hindsight, we now see that the hypothesis "$T_{3\frac{1}{2}}$" in Corollary 3.18 was unnecessarily strong. In fact, we see that T_3 and $T_{3\frac{1}{2}}$ are equivalent in a space that is second countable.

Further developments in metrization theory hinged on work of Arthur H. Stone in the late 1940's – in particular, his result that metric spaces have a property called "paracompactness." This led quickly to a complete characterization of metrizable spaces that came roughly a quarter century after Urysohn's work. We state this characterization here without a proof.

A family of sets $\mathcal{B}$ in $(X, \mathcal{T})$ is called locally finite if each point $x \in X$ has a neighborhood N that has nonempty intersection witth only finitely many sets in $\mathcal{B}$. The family $\mathcal{B}$ is called σ-locally finite if we can write $\mathcal{B} = \bigcup_{n \in \mathbb{N}} \mathcal{B}_n$ where each subfamily $\mathcal{B}_n$ is locally finite.

Theorem 6.2 (The Bing-Smirnov-Nagata Metrization Theorem) $(X, \mathcal{T})$ is metrizable iff X is T_3 and has a σ-locally finite base $\mathcal{B}$.

Note: If X is second countable, a countable base $\mathcal{B} = \{O_1, O_2, ..., O_n, ...\}$ is σ-locally finite – because we can write $\mathcal{B} = \bigcup \mathcal{B}_n$, where $\mathcal{B}_n = \{O_n\}$. Therefore this Metrization Theorem includes Urysohn's Metrization Theorem as a special case.

The Bing-Smirnov-Nagata Theorem has the typical form of most metrization theorems: X is metrizable iff "X has enough separation" and "X has a nice enough base."

Exercises

E15. Let (X, d) be a metric space and $S \subseteq X$. Prove that if each continuous $f\colon S \to \mathbb{R}$ extends to a continuous $g\colon X \to \mathbb{R}$, then S is closed. (*The converse, of course, follows from Tietze's Extension Theorem.)*

E16. Urysohn's Lemma says that in a T_4-space disjoint closed sets are completely separated. Part a) shows that this is also true in a Tychonoff space if one of the closed sets is compact.

a) Suppose X is Tychonoff and $F, K \subseteq X$ where F is closed, K is compact. X and $F \cap K = \emptyset$. Prove that there is an $f \in C(X)$ such that $f \mid K = 0$ and $f \mid F = 1$. (*This is another example of the rule of thumb that "compact spaces act like finite spaces." If necessary, try proving the result first for a finite set K.*)

b) Suppose X is Tychonoff and that $p \in U$, where U is open in X. Prove $\{p\}$ is a G_δ set in X iff there exists a continuous function $f : X \to [0, 1]$ such that $f^{-1}(1) = \{p\}$ and $f|X - U = 0$.

E17. Suppose Y is a Hausdorff space. Define $x \sim y$ in Y iff there does not exist a continuous function $f : X \to [0, 1]$ such that $f(x) \neq f(y)$. Prove or disprove: $Y/\sim$ is a Tychonoff space.

E18. Prove that a Hausdorff space X is normal iff for each finite open cover $\mathcal{U} = \{U_1, \ldots, U_n\}$ of X, there exist continuous functions $f_i\colon X \to [0, 1]$ $(i = 1, \ldots, n)$ such that $\sum_{i=1}^{n} f_i(x) = 1$ for each $x \in X$ and such that, for each i, $f_i \mid X - U_i \equiv 0$. (*Such a set of functions is called a partition of unity subordinate to the finite cover $\mathcal{U}$.*)

Hint ($\Rightarrow$) First build a new open cover $\mathcal{V} = \{V_1, \ldots, V_n\}$ that "shrinks" $\mathcal{U}$ in the sense that, $V_i \subseteq cl\, V_i \subseteq U_i$ for each i. To begin the construction, let $F_1 = X - \bigcup_{i>1} U_i$. Pick an open V_1 so that $F_1 \subseteq V_1 \subseteq cl\, V_1 \subseteq U_1$. Then $\{V_1, U_2, \ldots, U_n\}$ still covers X. Continue by looking at $F_2 = X - (V_1 \cup \bigcup_{i>2} U_i)$ and defining V_2 so that $\{V_1, V_2, U_3, \ldots, U_n\}$ is still a cover and $V_2 \subseteq cl\, V_2 \subseteq U_2$. Continue in this way to replace the U_i's one by one. Then use Urysohn's lemma to get functions g_i which can then be used to define the f_i's. .

E19. Suppose X is a compact, countable Hausdorff space. Prove that X is completely metrizable.

Hint: 1) For each pair of points $x_n \neq x_m$ in X pick disjoint open sets $U_{m,n}$ and $V_{m,n}$ containing these points. Consider the collection of all finite intersections of such sets.

2) Or: Since X is, countable, every singleton $\{p\}$ is a G_δ set. Use regularity to find a descending sequence of open sets V_n containing p such that $\bigcap_{n=1}^{\infty} cl\, V_n = \{p\}$. Prove that the V_n's are a neighborhood base at p.

E20. A space X is called completely normal if every subspace of X is normal. (*For example, every metric space is completely normal).*

a) Prove that X is completely normal if and only if the following condition holds:

whenever $A, B \subseteq X$ and $(\text{cl}\, A \cap B) \cup (A \cap \text{cl}\, B) = \emptyset$ (that is, each of A, B is disjoint from the closure of the other), then there exist disjoint open sets U and V with $A \subseteq U$ and $B \subseteq V$.

b) Recall that the "scattered line" (*Exercise IIIE.10*) consist of the set $X = \mathbb{R}$ with the topology $\mathcal{T} = \{U \cup V : U$ is open in the usual topology on $\mathbb{R}$ and $V \subseteq \mathbb{P}\}$. Prove that the scattered line is completely normal and therefore T_4.

E21. A T_1-space X is called perfectly normal if whenever A and B are disjoint nonempty closed sets in X, there is an $f \in C(X)$ with $f^{-1}(0) = A$ and $f^{-1}(1) = B$.

a) Prove that every metric space (X, d) is perfectly normal.

b) Prove that X is perfectly normal iff X is T_4 and every closed set in X is a G_δ-set.
Note: Example 3.10 shows a T_4. space X that is not perfectly normal.

c) Show that the scattered line (see *Exercise E20*) is not perfectly normal, even though every singleton set $\{p\}$ is a G_δ-set.

d) Show that the scattered line is T_4.
Hint: Use the fact that $\mathbb{R}$, with the usual topology, is normal. Nothing deeper than Urysohn's Lemma is required but the problem is a bit tricky.

E22. Prove that a T_3 space $(X, \mathcal{T})$ has a locally finite base $\mathcal{B}$ iff $\mathcal{T}$ is the discrete topology.
(*Compare to Theorem 6.2.*)

Chapter VII Review

Explain why each statement is true, or provide a counterexample.

1. Suppose $(X, \mathcal{T})$ is a topological space and let $\mathcal{T}_w$ be the weak topology on X generated by $C(X)$. Then $\mathcal{T} \subseteq \mathcal{T}_w$.

2. If X is regular and $x \in \text{cl}\,\{y\}$, then $y \in \text{cl}\,\{x\}$.

3. Every separable Tychonoff space can be embedded in $[0,1]^{\aleph_0}$.

4. If $f \in C(\mathbb{Q})$, we say g is a square root of f if $g \in C(\mathbb{Q})$ and $g^2 = f$. If a function f in $C(\mathbb{Q})$ has more than one square root, then it has c square roots.

5. In a Tychonoff space, every closed set is an intersection of zero sets.

6. A subspace of a separable space need not be separable, but every subspace of the Sorgenfrey line is separable.

7. Suppose $\mathbb{N}$ has the cofinite topology. If A is closed in $\mathbb{N}$, then every $f \in C(A)$ can be extended to a function $g \in C(\mathbb{N})$.

8. For $n = 1, 2, ...$, let $f_n : \mathbb{R} \to \mathbb{R}$ be given by $f_n(x) = x + n$ and let $\mathcal{T}$ be the weak topology on $\mathbb{R}$ generated by the f_n's. Then the evaluation map $e : \mathbb{R} \to \mathbb{R}^{\aleph_0}$ given by $e(x)(n) = f_n(x)$ is an embedding.

9. Let C be the <u>set</u> of points in the Cantor set with the subspace topology from the Sorgenfrey line S. Every continuous function $f : C \to \mathbb{R}$ can be extended to a continuous function $g : S \to \mathbb{R}$.

10. If (X, d) is a metric space, then X is homeomorphic to a dense subspace of some compact Hausdorff space.

11. Suppose $F \subseteq \mathbb{R}^2$. F is closed iff F is a zero set.

12. Suppose K is a compact subset of the Hausdorff space $X \times Y$. Let $A = \pi_X[K]$. Then A is T_4.

13. Every space is the continuous image of a metrizable space.

14. Let $\mathcal{F} = \{f \in \mathbb{R}^{\mathbb{R}} : f \text{ is not continuous}\}$. The weak topology on $\mathbb{R}$ generated by $\mathcal{F}$ is the discrete topology.

55. A compact T_2 space is metrizable if and only if it is second countable.

16. Suppose F and K are disjoint subsets of a Tychonoff space X, where F is closed and K is compact. There are disjoint <u>cozero</u> sets U and V with $F \subseteq U$ and $K \subseteq V$.

17. Every T_4 space is homeomorphic to a subspace of some cube $[0,1]^m$.

18.. Suppose X is a T_4-space with nonempty pairwise disjoint closed subspaces $F_1, ..., F_n$. There is an $f \in C(X)$ such that $f|F_i = i$ for all $i = 1, ..., n$.