FASCICOLO SPECIALE

RENDICONTI
DEL
SEMINARIO MATEMATICO

Convegno su
«DIFFERENTIAL GEOMETRY ON HOMOGENEOUS SPACES»

TORINO, 29 SETTEMBRE - 1 OTTOBRE 1983

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A CLASS OF HARMONIC MAPS FROM SURFACES INTO
REAL GRASSMANNIANS

This note refers to a previous paper of the authors [4] presented in Torino during the 1983 meeting on Differential Geometry held in the “Instituto Matematico” of the University; the reader is therefore referred to [4] for terminology and results.

Throughout this paper we shall call a map \( f : M \to S^{2n} \) pseudo-holomorphic if it is a linearly full, isotropic of order \( n \), minimal immersion of an oriented surface \( M \). In §1 we define the \( j^{th} \) associated maps

\[
f_j : M \to G_{2j+3} (R^{2n+1}) ,
\]

and the \( j^{th} \) Gauss maps

\[
\gamma_j : M \to G_2 (R^{2n+1}) ,
\]

for \( j = 0, 1, \ldots, n-1 \), where \( G_m (R^{2n+1}) \) is the Grassmannian of oriented \( m \)-dimensional subspaces of \( R^{2n+1} \), \( 1 \leq m \leq 2n+1 \). We then prove the following two theorems, (compare Barbosa [1], Obata [5] and Ruh-Vilms [6]).

Theorem 1. Let \( f : M \to S^{2n} \) be pseudo-holomorphic. Then its \( j^{th} \) associated map \( f_j, j = 0, 1, \ldots, n-1 \), is harmonic and it is regular and conformal outside the set of zeros (which is isolated) of the \( (j+1)^{th} \) contact invariant of \( f \).

Theorem 2. Let \( f : M \to S^{2n} \) be pseudo-holomorphic. Then the Gauss maps
of \( f \) satisfy:

\[ \begin{align*}
\text{i) } & \gamma_0 \text{ is regular and conformal on } M; \gamma_j \text{ is regular and conformal outside the set of common zeros (an isolated set) of the } \gamma_j^{1b} \text{ and } (j + 1)^{1b} \\
\text{contact invariants of } f, \text{ for } j = 1, \ldots, n - 2; \text{ and } \gamma_{n - 1} \text{ is regular and conformal outside the set of zeros (an isolated set) of the } (n - 1)^{1b} \\
\text{contact invariant of } f. \\
\text{ii) } & \gamma_j \text{ is harmonic, for } j = 0, 1, \ldots, n - 1. \\
\text{iii) } & \gamma_{n - 1} \text{ is anti-holomorphic.}
\end{align*} \]

In §2 the Riemannian geometry of the real Grassmannians is recalled, then Theorem 1 is proved in §3 and Theorem 2 is proved in §4.

§1. Let \( f : M \to S^{2n} \) be a pseudo-holomorphic map. In [4] it was shown that generalized \( n^{1b} \) order frame fields exist about any point of \( M \). We recall that this means that for any point of \( M \) there is a neighborhood \( U \) about that point on which there exists a map (a frame field along \( f \)) \( e : U \to 0(2n + 1) \) characterized by

\[
\begin{align*}
\phi^\alpha_0 &= 0, \quad \alpha \geqslant 3 \\
\phi^3_1 &= -k_1 \phi^2 + \mu_1 \phi^2, \quad \phi^3_2 = m_1 \phi^1 + k_1 \phi^2 \\
\phi^4_1 &= m_1 \phi^1 + k_1 \phi^2, \quad \phi^4_2 = k_1 \phi^1 - m_1 \phi^2 \\
\phi^\gamma_i &= 0, \quad \gamma \geqslant 5; \\
i (k_1 + im_1) (\phi^3_5 + i \phi^3_6) &= (k_2 + im_2) \phi \\
i (k_1 + im_1) (\phi^4_5 + i \phi^4_6) &= (ik_2 - m_2) \phi \\
\phi_{j+2}^\mu &= 0, \quad \mu \geqslant 7;
\end{align*}
\]

and the higher order analogues of (2), where

\[ e^* \Phi^A_B = \phi^A_B, \quad 0 \leqslant A, B \leqslant 2n \]

are the pull-backs by \( e \) of the Maurer-Cartan forms of \( 0(2n + 1) \).
These frame fields are determined up to changes \( \tilde{e} = eK \), where \( K \) takes values in

\[
(3) \quad \tilde{G}_n = \left\{ \begin{bmatrix} 0 & A_1 \\ \vdots & \ddots \\ & & A_n \end{bmatrix} : A_k \in SO(2), \quad k = 1, \ldots, n \right\} \subset O(2n+1)
\]

Thus with respect to the generalized \( n^{th} \) order frame fields \( e = (e_0, e_1, \ldots, e_{2n}) \) the vectors \( e_{2j+1}, e_{2j+2} \) are determined up to a rotation, for \( j = 1, \ldots, n-1 \). Furthermore

\[
(4) \quad N_j = \{ e_{2j+1}, e_{2j+2} \}, \quad \text{(i.e., the span),}
\]

defines a smooth rank 2 subbundle of the normal bundle \( TM^\perp \), which then decomposes into the orthogonal Whitney sum

\[
(5) \quad TM^\perp = N_1 \oplus \ldots \oplus N_{n-1}.
\]

**Definition 1:** The map, for \( j = 0, 1, \ldots, n-1 \),

\[
f_j : M \to G_{2j+3}(\mathbb{R}^{2n+1})
\]

into the Grassmannian of real oriented \((2j+3)\) dimensional subspaces of \( \mathbb{R}^{2n+1} \), defined by

\[
f_j(p) = \{ e(p), e_1(p), \ldots, e_{2j+2}(p) \}, \quad p \in M,
\]

will be called the \( j^{th} \) associated map of the pseudo-holomorphic \( f \).

Obata [5] called \( f_0 \) the Gauss map of \( f \).

**Definition 2:** The map, for \( j = 0, 1, \ldots, n-1 \),

\[
\gamma_j : M \to G_2(\mathbb{R}^{2n+1})
\]

defined by

\[
\gamma_j(p) = \{ e_{2j+1}(p), e_{2j+2}(p) \}, \quad p \in M.
\]
will be called the $j^{th}$ Gauss map of $f$.

§2. In this section we recall some of the geometry of $G_m(R^{2n+1}) = G_m(2n+1)$, $1 \leq m \leq 2n$. Let $e_A$, $A = 0, 1, \ldots, 2n$ denote the standard basis of $R^{2n+1}$. We choose the point $o = \{e_0, e_1, \ldots, e_{m-1}\}$ to be the origin of $G_m(2n+1)$. (Here $\{\}$ denotes the span with the orientation given by this ordered basis). The standard action of $0(2n+1)$ on $R^{2n+1}$ induces a transitive left action on $G_m(2n+1)$. The isotropy subgroup at the origin is

\begin{equation}
G_0 = SO(m) \times 0(2n + 1 - m),
\end{equation}

and thus

\begin{equation}
G_m(2n + 1) = 0(2n + 1)/G_0.
\end{equation}

We adopt the following indexing conventions:

$0 \leq A, B, C \leq 2n$, $1 \leq i, j, k \leq m - 1$, $0 \leq a, b \leq m - 1$, $m \leq \alpha, \beta \leq 2n$.

Then $\Phi_B^A$ are the matrix entries of $\Phi$. A basis of the annihilator $g \frac{1}{o}$ is given by $(\Phi^\alpha_\alpha)$. Thus the quadratic form

\begin{equation}
g = \sum_\alpha (\Phi^\alpha_0)^2 + \sum_{i, \alpha} (\Phi^\alpha_i)^2
\end{equation}

on $0(2n + 1)$ is $Ad(G_0)$-invariant. It defines an $0(2n + 1)$-invariant quadratic form $ds^2$ on $G_m(2n + 1)$, which is, in fact, the standard (up to homothety) invariant Riemannian metric on $G_m(2n + 1)$.

To understand the geometry of $ds^2$ it is convenient to digress a moment to an abstract setting. Let $N, ds^2$ be a Riemannian manifold and let

$F(N) \xrightarrow{\nu} N$

denote its principal bundle of orthonormal frames. Let $\theta = (\theta^p)$ denote the canonical form on $F(N)$, where we adopt for the moment the index convention $1 \leq p, q, r \leq \dim N = n$. Then the Levi-Civita connection form
\[ \omega = (\omega^p) \text{ of } ds^2 \text{ on } F(N) \text{ is characterized by the equations} \]

\[
\begin{align*}
  d\theta^p &= -\omega^p_q \wedge \theta^q \\
  \omega^p_p + \omega^q_q &= 0
\end{align*}
\]

plus the well-known properties involving right translations and the value on vertical vectors.

Returning to the geometry of \( G_m(2n+1) \), we choose and fix an orthonormal reference frame at \( o \) in \( G_m(2n+1) \). It is well known that fixing a reference frame at the origin gives rise to a bundle homomorphism

\[
\begin{array}{ccc}
0(2n+1) & \rightarrow & F(G_m(2n+1)) \\
\downarrow & & \downarrow \\
G_m(2n+1) & \rightarrow & G_m(2n+1)
\end{array}
\]

In this way each element of \( 0(2n+1) \) is an orthonormal frame of \( G_m(2n+1) \), namely the differential of this element applied to the reference frame.

It is an elementary exercise to verify that the canonical form restricted to \( 0(2n+1) \) is given by

\[
\theta = (\theta^a;\alpha) \quad \text{where} \quad \theta^a;\alpha = \Phi_a^\alpha
\]

The Maurer-Cartan structure equations of \( 0(2n+1) \) are

\[
\begin{align*}
  d\Phi^A_B &= -\Phi^A_C \Phi^C_B, \quad \text{and} \\
  \Phi^A_B &= -\Phi^B_A
\end{align*}
\]

Comparing (11) and (9) and using (10) together with the fact that these forms are left-invariant, we conclude that Levi-Civita connection forms \( \omega^{a,\alpha}_{\beta} \) restricted to \( 0(2n+1) \) are given by

\[
\begin{align*}
  \omega^{0,\alpha}_{\beta} &= \Phi^a_{\beta} \\
  \omega^{0,\alpha}_{\beta} &= \delta^\alpha_{\beta} \Phi^0 \\
  \omega^{i,\alpha}_{\beta} &= \delta^\alpha_{\beta} \Phi^i + \delta^i_{\beta} \Phi^a_{\beta}
\end{align*}
\]
In preparation for the next section we recall the definition of harmonic map in the abstract setting (cf. Chern-Goldberg [3]). Let \( M, db^2 \) be another Riemannian manifold, and let \( f : M \to N \) be a smooth map. Suppose that we have a locally defined smooth map \( e : M \to F(N) \) such that \( \pi \circ e = f \), \( (e) \) is a local orthonormal frame field along \( f \). We write

\[
e = (f ; E_1, ..., E_n)
\]

where, at each point \( x \) in the domain of \( f \), \( E_1(x), ..., E_n(x) \) is an orthonormal basis of \( T_{f(x)}N \). Then

\[
df = e^* \theta^\rho \otimes E_\rho
\]

Let \( \phi^\sigma, \sigma = 1, ..., \dim M = m \) be a local orthonormal coframe field in \( M \), so that

\[
db^2 = \sum (\phi^\sigma)^2.
\]

Let \( \phi^\sigma_\tau = - \phi^\tau_\sigma, 1 \leq \sigma, \tau \leq m \), denote the Levi-Civita connection forms of \( db^2 \) with respect to this coframe. Then

\[
e^* \theta^\rho = B_p^\sigma \phi^\sigma_\rho,
\]

for certain function \( B_p^\sigma \) in \( M \), and the covariant differential of \( df \) is (using (13) and (14))

\[
Ddf = dB_p^\sigma \otimes \phi^\sigma \otimes E_\rho - B_p^\sigma \otimes \phi^\sigma_\rho \otimes \phi^\tau \otimes E_\rho + B_p^\sigma \phi^\sigma \otimes e^* \om^q_\rho \otimes E_q
\]

\[
= (DB_p^\sigma) \otimes \phi^\sigma \otimes E_\rho,
\]

where

\[
DB_p^\sigma = dB_p^\sigma - B_p^\tau \phi^\tau_\sigma + B_q^\sigma e^* \om^\rho_\sigma \defeq B_{p,\tau} \phi^\tau
\]

is a 1-form in \( M \). The tension field \( \tau \) of \( f \) is the vector field along \( f \) given by

\[
\tau = \text{Trace } Ddf = \sum_{\sigma} B_{p,\sigma} E_\rho
\]
By definition $f$ is harmonic if $\tau = 0$.

§3. Proof of Theorem 1.

To make our computations we use local $n^{th}$ order frames $e$ along $f$. These are defined on some neighborhood of any point outside of some isolated set of points in $M$. They are characterized by

$$
e^*\psi = \begin{vmatrix}
0 & 0 & r_1\varphi^2 & -r_2\varphi^1 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \varphi^2_{2j-1} & 0 & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
r_j\varphi_2 & -r_j\varphi^1 & 0 & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
r_j\varphi^1 & r_j\varphi^2 & \varphi^2_{2j+2} & 0 & \ddots \\
0 & 0 & \ddots & \ddots & \ddots \\
\end{vmatrix}
$$

$\quad j = 1, \ldots , n = 1$

(Cf. equation (24) in [4]). The functions $r_j$ are the contact invariants of $f$. They are globally defined, have isolated zeros (precisely the isolated set referred to above), and $r_j^2 \in C^\infty (M)$. The forms $\varphi^1, \varphi^2$ define a local orthonormal coframe on $M$ for the induced metric $db^2$, and $\varphi^1_2$ is the Levi-Civita connection form of $db^2$ with respect to this coframe.

Now fix $j \in \{0, 1, \ldots , n-1\}$ and put $m = 2j + 3$. The following
diagram commutes:

\[
\begin{array}{c}
\pi_j \\
\downarrow \quad \downarrow \\
G_m \ (2n + 1) \\
\downarrow \quad \downarrow \\
S^{2n} \\
\uparrow \quad \uparrow \\
0 \ (2n + 1) \\
\pi \\
\pi_j \\
\end{array}
\]

where

\[
\pi : (e_0, e_1, \ldots, e_{2n}) \rightarrow e_0 ,
\]

\[
\pi_j : (e_0, e_1, \ldots, e_{2n}) \rightarrow (e_0, e_1, \ldots, e_{2j+2}) ,
\]

and \( f_j \) is the \( j^{th} \) associated map of \( f \) defined in §1. Hence \( e \) is also a local orthonormal frame field along \( f_j \).

Our index conventions are now, for \( m = 2j + 3 \) : \( 1 \leq i, k \leq 2j + 2 \); \( 0 \leq a, b \leq 2j + 2 \); \( 2j + 3 \leq \alpha, \beta \leq 2n \); \( 1 \leq \mu, \nu \leq 2 \).

We use (16) and (10) to compute the coefficients in (14):

\[
e^* \theta^a, \alpha = P^a, \alpha \phi^\mu
\]

For \( j > 0 \):

\[
e^* \phi^0_0 = B^0, \alpha \phi^\mu = 0
\]

\[
e^* \phi^2j+3_2j+1 = B^{2j+1, 2j+3} \phi^\mu = r_{j+1} \phi^2
\]

\[
e^* \phi^2j+3_2j+2 = B^{2j+2, 2j+3} \phi^\mu = -r_{j+1} \phi^1
\]

\[
e^* \phi^2j+4_2j+1 = B^{2j+1, 2j+4} \phi^\mu = r_{j+1} \phi^1
\]

\[
e^* \phi^2j+4_2j+2 = B^{2j+2, 2j+4} \phi^\mu = r_{j+1} \phi^2
\]

\[
e^* \phi^i_i = B^i, \alpha \phi^\mu = 0 , \text{ for } (i, \alpha) \notin S_j ,
\]
where \( S_j = \{(2j + 1, 2j + 3), (2j + 2, 2j + 3), (2j + 1, 2j + 4), (2j + 2, 2j + 4)\} \).

For \( j = 0 \):

\[
e^*\phi_0^0 = B^0,\alpha \phi^\mu = 0
\]

\[
e^*\phi_1^3 = B_{1,\mu}^1 \phi^\mu = -r_1 \phi^1
\]

\[
e^*\phi_2^3 = B_{2,\mu}^2 \phi^\mu = r_1 \phi^2
\]

\[
e^*\phi_4^1 = B_{1,\mu}^4 \phi^\mu = r_1 \phi^2
\]

\[
e^*\phi_5^4 = B_{2,\mu}^1 \phi^\mu = r_1 \phi^4
\]

\[
e^*\phi_i^i = B^i,\alpha \phi^\mu = 0, \text{ for } (i, \alpha) \notin S_0,
\]

where \( S_0 = \{(1, 3), (2, 3), (1, 4), (2, 4)\} \).

In (17) and (18) the functions \( B_{\mu}^{a,\alpha} \) are defined by the last equation in each line.

From (17), (18) and (8) we have for any \( j \geq 0 \)

\[
f_j^* d\tau - e^* g = 2r_{j+1}^2 ((\phi^1)^2 + (\phi^2)^2) = 2r_{j+1}^2 db^2.
\]

Hence the \( f_j \) associated map \( f_j \) is regular and conformal outside of the set of zeros of the \( (j + 1)^{st} \) order contact invariant \( r_{j+1} \) of \( f \).

To show now that \( f_j \) is harmonic we must compute its tension field \( \tau \), and this requires that we compute the covariant derivatives of the coefficients \( B^{a,\alpha} \), (cf. §2).

Using equations (15), (16), (10), (12), (17) and (18) we obtain

\[
DB_{\mu}^{a,\alpha} = B_{\mu}^{2j+1,2j+3} \delta_{2j+3}^a \phi_{2j+1}^0 + B_{\mu}^{2j+3,2j+3} \delta_{2j+3}^a \phi_{2j+2}^0
\]

\[
+ B_{\mu}^{2j+1,2j+4} \delta_{2j+4}^a \phi_{2j+1}^0 + B_{\mu}^{2j+3,2j+4} \delta_{2j+4}^a \phi_{2j+2}^0,
\]

where we write \( \phi_B^A = e^{A} \phi_B^A \).

Hence, if \( \alpha \neq 2j + 3 \) or \( 2j + 4 \), then

\[
B_{\mu}^{a,\alpha} = 0
\]
where these coefficients are defined by the second equation in (15). If \( a = 2j + 3 \)

\[
DB^0_{\mu} = - B^{2j+1,2j+3}_{\mu} \phi_0^{2j+1} - B^{2j+2,2j+3}_{\mu} \phi_0^{2j+2},
\]

and if \( \alpha = 2j + 4 \)

\[
DB^0_{\mu} = - B^{2j+1,2j+4}_{\mu} \phi_0^{2j+1} - B^{2j+2,2j+4}_{\mu} \phi_0^{2j+2}.
\]

In either case, by (16),

\[
(22) \quad B^0_{\mu} = 0 \quad \text{for} \quad j > 0.
\]

If \( j = 0 \), then

\[
(23) \quad B^{0,3}_{\mu} = B^{\nu,3}_{\mu}, \quad B^{0,4}_{\mu} = - B^{\nu,4}_{\mu}.
\]

Thus for \( 0 \leq j \leq n - 1 \), we have from (18), (21), (22), and (23) and the definition of the tension field

\[
(24) \quad \tau^0,\alpha = \sum_{\mu} B^0_{\mu} E_{0,\alpha} = 0.
\]

Again using (15), (16), (10), (12), (17) and (18) we have

\[
(25) \quad DB^i_{\mu} = dB^i_{\mu} - B^i_{\mu} \phi_\mu^i + B^{2j+1,2j+3}_{\mu} (\delta^i_{2j+3} \phi_{2j+1}^i + \delta^i_{2j+1} \phi_{2j+3}^i) \\
+ B^{2j+2,2j+3}_{\mu} (\delta^i_{2j+3} \phi_{2j+2}^i + \delta^i_{2j+2} \phi_{2j+3}^i) \\
+ B^{2j+1,2j+4}_{\mu} (\delta^i_{2j+4} \phi_{2j+1}^i + \delta^i_{2j+1} \phi_{2j+4}^i) \\
+ B^{2j+2,2j+4}_{\mu} (\delta^i_{2j+4} \phi_{2j+2}^i + \delta^i_{2j+2} \phi_{2j+4}^i).
\]

We begin by considering the case \((i, \alpha) \in S_j\), where \( S_j \) was defined in (17) and (18). We recall the following formula from [4] (equation (30)_j)

\[
(26) \quad *d \log r_{j+1} = \phi_2^1 - \phi_{2j+2}^{2j+1} + \phi_{2j+4}^{2j+3}.
\]
where \(*\) is the Hodge star operator on \(M\). We consider the case \((i, \alpha) = (2j + 1, 2j + 3)\). The remaining cases are completely similar and we will simply state the results.

If \(j > 0\), then from (25), (17) and (26) we obtain

\[
DB_1^{i, \alpha} = r_{j+1} (\ast d \log r_{j+1}) = \ast d r_{j+1},
\]

and from (25) and (17) we have

\[
DB_2^{i, \alpha} = dr_{j+1}.
\]

Hence, if \(dr_{j+1} = r_{j+1, \mu} \phi^\mu\), then from (27) and (28)

\[
B_{11}^{i, \alpha} = -r_{j+1, 2}, \quad B_{22}^{i, \alpha} = r_{j+1, 2},
\]

and thus

\[
r^{2j+1, 2j+3} = \sum_{\mu} B_{\mu \mu}^{i, \alpha} = 0.
\]

In the same way

\[
r^{i, \alpha} = 0, \quad \text{for} \quad j = 0, 1, ..., n-1, \ (i, \alpha) \in S_j.
\]

We are left with the case \((i, \alpha) \notin S_j\). We do here the cases \(j > 0\). The results are the same when \(j = 0\). Now, from (17),

\[
B_{\mu}^{i, \alpha} = 0.
\]

From (31) and (25) we deduce that if \(i \neq 2j + 1\) or \(2j + 2\), and \(\alpha \neq 2j + 3\) or \(2j + 4\), then

\[
DB_{\mu}^{i, \alpha} = 0, \quad \text{and}
\]

\[
B_{\mu \nu}^{i, \alpha} = 0.
\]

The remaining cases are similar to the case

\(i = 2j + 1\) and \(\alpha \neq 2j + 3\) or \(2j + 4\),
which we shall show here. From (25) and (31)

\[(33) \quad DB_{\mu}^{2j+1,\alpha} = B_{\mu}^{2j+1,2j+3} \phi_{2j+3}^{\alpha} + B_{\mu}^{2j+1,2j+4} \phi_{2j+4}^{\alpha} \]

Thus, from (16), if \( \alpha > 2j + 6 \), then

\[DB_{\mu}^{2j+1,\alpha} = 0\]

and hence

\[(34) \quad B_{\mu}^{2j+1,\alpha} = 0 \]

The cases \( \alpha = 2j + 5 \) and \( \alpha = 2j + 6 \) remain. From (33), (16) and (17)

\[(35) \quad B_{11}^{2j+1,2j+5} = r_{j+1}^{j+2}, \quad B_{22}^{2j+1,2j+5} = r_{j+1}^{j+2} \]

\[(36) \quad B_{11}^{2j+1,2j+6} = 0 = B_{22}^{2j+1,2j+6} \]

Thus from (36), (35), (34) and (32) we deduce

\[(37) \quad r^{i,\alpha} = 0, \quad \text{for} \quad (i, \alpha) \notin S_j, \quad j \geq 0.\]

Equations (24), (30) and (37) complete the proof of Theorem 1.

§4. Proof of Theorem 2.

We begin by repeating much of §2 for the case of \( G_2 (2n + 1) \). It is convenient to reformulate everything for different choices of origin of \( G_2 (2n + 1) \).

Fix \( j \in \{0, \ldots, n - 1\} \). As in §2 we let \( e_A, A = 0, 1, \ldots, 2n \) denote the standard ordered basis of \( R^{2n+1} \), but now we choose the point

\[o_j = \{e_{2j+1}, e_{2j+2}\}\]

as the origin of \( G_2 (2n + 1) \). Let \( G_j \) denote the isotropy subgroup at \( o_j \)
of $0 \ (2n+1)$, and let $g_j$ denote its Lie algebra.

We adopt the following indexing convention: $1 \leq M, R \leq 2j \ ; \ 2j+1 \leq s, t \leq 2j+2 \ ; \ 2j+3 \leq \alpha, \beta \leq 2n \ ;$ where if $j = 0$ then $M, R$ are vacuous, and if $j = n-1$ then $\alpha, \beta$ are vacuous. We continue the convention $0 \leq A, B \leq 2n$ of $\mathfrak{g}_2$.

A basis of the annihilator $g_j^\perp$ is $\{\Phi^s_0, \Phi^s_M, \Phi^s_s\}$ where $\Phi = (\Phi^s_\beta)$ is the Maurer-Cartan form of $0 \ (2n+1)$.

The quadratic form

$$g_j = \sum_s (\Phi^s_0)^2 + \sum_{s,M} (\Phi^s_M)^2 + \sum_{\alpha,s} (\Phi^s_\alpha)^2$$

on $0 \ (2n+1)$ is $Ad (G_j)$-invariant, and thus defines an $0 \ (2n+1)$-invariant quadratic tensor $dS^2$ on $G_2 \ (2n+1)$, which is the standard invariant Riemannian metric there, up to homothety.

As we saw in (10), we see that the components of the canonical form of $G_2 \ (2n+1)$ restricted to $0 \ (2n+1)$ are given by

$$\theta^0_s = \Phi^s_0, \ \theta^M_s = \Phi^s_M, \ \theta^s_\alpha = \Phi^s_\alpha.$$

Using the structure equations (11) we again deduce, as we did in (12), that the Levi-Civita connection forms of $g_j$ on $0 \ (2n+1)$ are given by:

$$\omega^0_{0,s} = \Phi^s_t$$

$$\omega^0_{M,t} = \delta^s_t \Phi^0_M$$

$$\omega^0_{s,\alpha} = -\delta^s_t \Phi^0_\alpha$$

$$\omega^M_{R,t} = \delta^M_R \Phi^s_t + \delta^s_t \Phi^M_R$$

$$\omega^M_{s,\alpha} = -\delta^s_t \Phi^M_\alpha$$

$$\omega^s_{t,\beta} = \delta^s_t \Phi^s_\beta + \delta^s_\beta \Phi^s_t$$

Fix $j \in \{0, 1, \ldots, n-1\}$ and let $e = (f; e_1, \ldots, e_{2n})$ be a local $n$th order frame field along $f$. The following diagram commutes, as can be seen
from the definition of \( \gamma_j \) in §1:

\[
\begin{array}{c}
\pi : A = (e_0, \ldots, e_{2n}) \rightarrow e_0 \\
\tilde{\pi}_j : A \rightarrow (e_{2j+1}, e_{2j+2})
\end{array}
\]

where

\[
\pi : A = (e_0, \ldots, e_{2n}) \rightarrow e_0
\]

\[
\tilde{\pi}_j : A \rightarrow (e_{2j+1}, e_{2j+2})
\]

and \( \gamma_j \) is the \( j^{th} \) Gauss map of \( f \).

As in §3 we use the convention \( 1 \leq \mu, \nu \leq 2 \). We use (16) and (39) to compute the coefficients in (14):

\[
e^* \vartheta^{0,s} = B_{\mu}^{0,s} \phi_{\mu} \quad e^* \vartheta^{M,s} = B_{\mu}^{M,s} \phi_{\mu} \quad e^* \vartheta^{s,\alpha} = B_{\mu}^{s,\alpha} \phi_{\mu}
\]

These are:

For \( j = 2, \ldots, n-2 \):

\[
B_{\mu}^{0,s} = 0
\]

\[
B_{\mu}^{2j-1,2j+1} = r_j \delta_{\mu}^2, \quad B_{\mu}^{2j,2j+1} = -r_j \delta_{\mu}^1
\]

\[
B_{\mu}^{2j-1,2j+2} = r_j \delta_{\mu}^1, \quad B_{\mu}^{2j,2j+2} = r_j \delta_{\mu}^2
\]

(41)

\[
B_{\mu}^{M,s} = 0, \quad \text{for} \quad (M,s) \notin H_j
\]

\[
B_{\mu}^{2j+1,2j+3} = r_{j+1} \delta_{\mu}^2, \quad B_{\mu}^{2j+2,2j+3} = -r_{j+1} \delta_{\mu}^1
\]

\[
B_{\mu}^{2j+1,2j+4} = r_{j+1} \delta_{\mu}^1, \quad B_{\mu}^{2j+2,2j+4} = r_{j+1} \delta_{\mu}^2
\]

\[
B_{\mu}^{s,\alpha} = 0, \quad \text{for} \quad (s, \alpha) \notin S_j
\]
where

\[ H_j = \{(2j - 1, 2j + 1), (2j, 2j + 1), (2j - 1, 2j + 2), (2j, 2j + 2)\}, \]

and \( S_j \) is defined in (17).

For \( j = 1 \):

\[ B^{0,s}_{\mu} = 0 \]

\[ B^{1,3}_{\mu} = -r_1 \delta^1_{\mu}, \quad B^{2,3}_{\mu} = r_1 \delta^2_{\mu} \]

\[ B^{1,4}_{\mu} = r_1 \delta^2_{\mu}, \quad B^{2,4}_{\mu} = r_1 \delta^1_{\mu} \]

(41)\]

\[ B^{M,s}_{\mu} = 0, \quad \text{for } (M, s) \notin H_1 \]

\[ B^{3,5}_{\mu} = r_2 \delta^2_{\mu}, \quad B^{4,5}_{\mu} = -r_2 \delta^1_{\mu} \]

\[ B^{3,6}_{\mu} = r_2 \delta^1_{\mu}, \quad B^{4,6}_{\mu} = r_2 \delta^2_{\mu} \]

\[ B^{s,\alpha}_{\mu} = 0, \quad \text{for } (s, \alpha) \notin S_1 \]

where

\[ H_1 = \{(1,3), (2,3), (1,4), (2,4)\}, \]

and \( S_1 \) is defined in (17).

For \( j = 0 \), (so the \( M \) index is vacuous):

\[ B^{0,1}_{\mu} = \delta^1_{\mu}, \quad B^{0,2}_{\mu} = \delta^2_{\mu} \]

(41)\]

\[ B^{1,3}_{\mu} = -r_1 \delta^1_{\mu}, \quad B^{2,3}_{\mu} = r_1 \delta^2_{\mu} \]

\[ B^{1,4}_{\mu} = r_1 \delta^2_{\mu}, \quad B^{2,4}_{\mu} = r_1 \delta^1_{\mu} \]

\[ B^{s,\alpha}_{\mu} = 0, \quad \text{if } \alpha \geq 5 \]

For \( j = n - 1 \), when \( n = 2 \), the coefficients are given by the first three
114

rows of (41) and when \( n \geq 3 \):

\[ B^{0,s}_{\mu} = 0 \]

\[ B^{M,s}_{\mu} = 0 \text{ if } M \leq 2n - 4 \]

\[ B^{2n-3,2n-1}_{\mu} = r_{n-1}^2 \delta^2_{\mu} \]

\( B^{2n-2,2n-1}_{\mu} = -r_{n-1} \delta^1_{\mu} \)

\[ B^{2n-3,2n-1}_{\mu} = r_{n-1}^2 \delta^1_{\mu} \]

\[ B^{2n-2,2n}_{\mu} = r_{n-1}^2 \delta^2_{\mu} \]

(which are all the cases since here the \( \alpha \) index is vacuous)

From (38) and (41) if follows that

\[ \gamma_0^* ds^2 = e^s g_0 = (1 + 2r_1^2) db^2 \]

\[ \gamma_j^* ds^2 = e^s g_j = 2(r_j^2 + r_{j+1}^2) db^2, \quad j = 1, ..., n-2 \]

\[ \gamma_{n-1}^* ds^2 = e^s g_{n-1} = 2r_{n-1}^2 db^2 \]

where, as above, \( db^2 = (\phi_1^1)^2 + (\phi_2^2)^2 \) is the metric on \( M \). This proves (i) of Theorem 2.

To compute the tension field \( \tau \) of \( \gamma_j \) we proceed exactly as we did in §3 to prove Theorem 1. The result is that \( \tau = 0 \). We will illustrate the computations, which all use (40), (41), (16) and (26), in just one case. For \( 2 \leq j \leq n-2 \):

\[ DB^{2j-1,2j+2}_{\mu} = dB^{2j-1,2j+2}_{\mu} - B^{2j-1,2j+2}_{\nu} \phi^{\nu}_{\mu} \]

\[ + B^{M,i}_{\mu} \omega^{2j-1,2j+2}_{M,i} + B^{t,\alpha}_{\mu} \omega^{2j-1,2j+2}_{t,\alpha} \]

\[ = dB^{2j-1,2j+2}_{\mu} - B^{2j-1,2j+2}_{\nu} \phi^{\nu}_{\mu} + B^{2j-1,2j+1}_{\mu} \omega^{2j-1,2j+2}_{2j-1,2j+1} \]

\[ + B^{2j+2}_{2j+2} \omega^{2j-1,2j+2}_{2j+2} + B^{2j+2,2j+3}_{\mu} \omega^{2j-1,2j+2}_{2j+2,2j+3} + B^{2j+2,2j+3}_{\mu} \omega^{2j-1,2j+2}_{2j+2,2j+3} \]

\[ = dB^{2j-1,2j+2}_{\mu} - B^{2j-1,2j+2}_{\nu} \phi^{\nu}_{\mu} + B^{2j-1,2j+1}_{\mu} \phi^{2j+2}_{2j+1} + B^{2j+2}_{2j+2} \phi^{2j+2}_{2j+2} \]

Therefore

\[ DB^{2j-1,2j+2}_{1} = dr_j \]
(44) \[ DB_2^{2j-1,2j+2} = - (\ast d r_j) \ . \]

From (43) and (44) we deduce

(45) \[ \tau^{2j-1,2j+2} = 0 \ . \]

The remaining cases go in the same way. This proves (ii) of Theorem 2.

To prove (iii) of Theorem 2 we need to examine the complex structure of \( G_2 (2n+1) \). This is easily understood in terms of its standard imbedding into \( CP^{2n} \): the oriented plane \( \{ v_1, v_2 \} \) with oriented orthonormal basis \( v_1, v_2 \) goes to \( [v_1 + iv_2] \in CP^{2n} \). Here we let

\[ p : C^{2n+1} \setminus \{0\} \to CP^{2n} \]

denote the standard projection and write

\[ p (v) = [v] \ . \]

Let \( e = (e_0, e_1, \ldots, e_{2n}) \) be a local \( n^{th} \) order frame field along \( f \).

As a mapping into \( CP^{2n} \) the \( j^{th} \) Gauss map \( \gamma_j \) is given by

\[ \gamma_j = [e_{2j+1} + ie_{2j+2}] \ , \ j = 0, 1, \ldots, n-1 \ . \]

Consider the locally defined map

(46) \[ E_j : M \to C^{2n+1} \]

\[ E_j = e_{2j+1} + ie_{2j+2} \ . \]

Then, on the domain of \( e \), \( \gamma_j = p \circ E_j \ . \)

Using (16) and the structure equations (11) we obtain what Chern calls the Frenet-Boruvka formulas for \( f \)

\[ dE_0 = - \phi e_0 + i \phi_1 e_0 - r_1 \overline{\phi E_1} \]

\[ dE_j = - ir_j \phi E_{j-1} + i \phi_{j+1} E_j - ir_{j+1} \phi E_{j+1} \ , \]

(47) \[ \text{for } j = 2, \ldots, n-1 \text{ if } n \geq 3 \ . \]
\[ dE_1 = r_1 \bar{\phi}E_0 + i\phi_1E_1 - ir_2\phi E_2, \quad \text{if} \quad n \geqslant 3, \]
\[ dE_{n-1} = i\phi_2^{n-1}E_{n-1} - ir_{n-1}\bar{\phi}E_{n-2}, \quad \text{for} \quad n \geqslant 2. \]

Then
\[ (48) \quad d\gamma_j = dp_{E_j} \circ dE_j \]

where \( dp_{E_j} \) denotes the differential of \( p \) at the point \( E_j \in \mathbb{C}^{2n+1} \).

Using the fact that
\[ dp_v \nu = 0, \quad \text{for any} \quad \nu \in \mathbb{C}^{2n+1} \setminus \{0\}, \]

we conclude from (47) and (48) that \( \gamma_{n-1} \) is anti-holomorphic, while the \( \gamma_j \), for \( j \leq n-2 \) are neither holomorphic nor anti-holomorphic. This completes the proof of Theorem 2.

REFERENCES


