Polynomials with no zeros on a face of the bidisk

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$p \in \mathbb{C}[z, w]$ WITH NO ZEROS

ON

$\mathbb{T} \times \overline{\mathbb{D}}$ "FACE" OF

$\mathbb{D}^2 = \mathbb{D} \times \mathbb{D}$

Unit CIRCLE

CLOSED UNIT DISK

BIDISK

MOTIVATION: FEJÉR-RIESZ FACTORIZATION

$$t(z) = \sum_{j=-N}^{N} t_j z^j > 0 \quad \text{on } \mathbb{T}$$

$$\Longrightarrow t(z) = |p(z)|^2$$

$p \in \mathbb{C}[z]$ WITH NO ZEROS ON $\overline{\mathbb{D}}$ and degree at most N.

GERONIMO-WOERDEMAN: CHARACTERIZE

WHEN $t(z,w) = \sum_{\substack{|j| \le N \\ |k| \le M}} t_{j,k} z^j w^k > 0$ FOR $|z| = |w| = 1$

CAN BE FACTORED AS

$|p(z,w)|^2$ WHERE $p \in \mathbb{C}[z,w]$

HAS NO ZEROS IN $\overline{\mathbb{D}^2}$

CHARACTERIZATION DONE IN TERMS OF MOMENTS OF

$$\frac{1}{t(z,w)} \frac{|dz|}{2\pi} \frac{|dw|}{2\pi} \text{ ON } \mathbb{T}^2$$

<u>NOTE</u>: MEASURES

$$\frac{1}{|p(z,w)|^2} \frac{|dz|}{2\pi} \frac{|dw|}{2\pi}$$

ARE CALLED BERNSTEIN–SZEGŐ

FACT: IF $t = \sum_{\substack{|j| \le n \\ |k| \le m}} t_{j,k} z^j w^k$, $s = \sum_{\substack{|j| \le n \\ |k| \le m}} s_{j,k} z^j w^k$

ARE > 0 ON $\mathbb{T}^2$

AND $\int_{\mathbb{T}^2} \frac{z^j w^k}{t} = \int_{\mathbb{T}^2} \frac{z^j w^k}{s}$ $\quad |j| \le n$, $|k| \le m$

THEN: $t = s$ ON $\mathbb{T}^2$

POSITIVE TOEPLITZ FORM:

Let $\mathcal{L}_{n,m} = \text{span}\{z^j w^k : |j| \le n, |k| \le m\}$

$$P_{n,m} = \text{span}\left\{z^j w^k : \begin{array}{l} 0 \le j \le n \\ 0 \le k \le m \end{array}\right\}$$

$T : \mathcal{L}_{n,m} \to \mathbb{C}$ LINEAR IS A PTF

IF $T\left(f(z,w)\overline{f\left(\frac{1}{\bar{z}}, \frac{1}{\bar{w}}\right)}\right) > 0$ FOR

ALL NONZERO $f \in P_{n,m}$

ASSOCIATED INNER PRODUCT

$$\langle f, g \rangle_{\mathcal{T}} = \mathcal{T}\left(f(z,w)\, \overline{g\left(\frac{1}{\bar{z}}, \frac{1}{\bar{w}}\right)} \right)$$

$$f, g \in \mathcal{P}_{n,m}$$

ASSOCIATED HILBERT SPACE

$$\mathcal{H}_{\mathcal{T}} = \left(\mathcal{P}_{n,m}, \langle \cdot, \cdot \rangle_{\mathcal{T}} \right)$$

GERONIMO - WOERDEMAN

GIVEN A P.T.F. T

G-W CHARACTERIZE WHEN

$\exists p \in \mathbb{C}[z,w]$, NO ZEROS ON $\overline{\mathbb{D}}^2$

S.T. T MATCHES A BERNSTEIN-SZEGŐ MEASURE

$$T(z^j w^k) = \int_{\mathbb{T}^2} \frac{z^j w^k}{|p(z,w)|^2} \frac{|dz||dw|}{(2\pi)^2}$$

REPRESENTATION HOLDS IFF

$$\left(P_{n,m-1} \ominus P_{n-1,m-1}\right) \perp \left(P_{n-1,m} \ominus P_{n-1,m-1}\right)$$

USING $\mathcal{H}_J$

- CLOSELY RELATED TO AUTOREGRESSIVE MODELS FOR "SPATIAL STATIONARY PROCESSES"
- INTERESTING BY-PRODUCT IS COLE-WERMER SOS FORMULA (EQUIVALENT TO ANDÔ'S INEQUALITY AND AGLER'S PICK THM ON $\mathbb{D}^2$)

GERONIMO-ILIEV: CHARACTERIZE WHEN A P.T.F. $\mathcal{T}$ CAN BE REPRESENTED AS

$$\mathcal{T}(z^j w^k) = \int_{\mathbb{T}^2} \frac{z^j w^k}{|p(z,w)|^2} \frac{|dz|\,|dw|}{(2\pi)^2}$$

WHERE $p \in \mathbb{C}[z,w]$ HAS NO ZEROS ON $\overline{\mathbb{T} \times \mathbb{D}}$ AND DEGREE AT MOST (n,m)

CHARACTERIZATION DIFFICULT TO STATE.

GOAL: PRESENT A NEW AND SIMPLER HILBERT SPACE GEOMETRIC CHARACTERIZATION AS WELL AS APPLICATIONS AND EXTENSIONS.

THM: GIVEN A POS. TOEPLITZ FORM $\mathcal{T}$ ON $P_{n,m}$. TFAE.

(BS) $\mathcal{T}$ CAN BE REPRESENTED USING A BERNSTEIN-SZEGŐ MEASURE $\frac{1}{|p|^2} d\sigma$ WHERE $p \in \mathbb{C}[z, w]$ HAS NO ZEROS IN $\mathbb{T} \times \overline{\mathbb{D}}$

(SPLIT-SHIFT ORTHOGONALITY CONDITION)

(MATRIX CONDITION) ← (CLOSE TO G-I THM OMITTED TODAY)

WHAT IS THE SPLIT-SHIFT CONDITION?

$$\mathcal{E}'_{n,m} = P_{n,m} \ominus w P_{n,m-1} =$$

$$\mathcal{E}'_{n-1,m} = P_{n-1,m} \ominus w P_{n-1,m-1} =$$

Let $\mathcal{E}'_{n,m} = P_{n,m} \ominus w P_{n,m-1}$, $\mathcal{E}'_{n-1,m} = P_{n-1,m} \ominus w P_{n-1,m-1}$

Def: A P.T.F. $\mathcal{J}$ satisfies the **split-shift cond** if $\exists\, K_1, K_2 \leq \mathcal{E}'_{n-1,m}$

s.t.

(1) $K_1 \oplus K_2 = \mathcal{E}'_{n-1,m}$

(2) $K_1 \perp z K_2$

(3) $K_1, z K_2 \leq \mathcal{E}'_{n,m}$

$$\mathcal{E}'_{n,m} = P_{n,m} \ominus w P_{n,m-1}, \quad \mathcal{E}'_{n-1,m} = P_{n-1,m} \ominus w P_{n-1,m-1}$$

SPLIT-SHIFT COND $\Longrightarrow$

$$\mathcal{E}'_{n-1,m} = K_1 \oplus K_2$$

$$\mathcal{E}'_{n,m} = K_1 \oplus z K_2 \oplus \mathbb{C}\rho$$

(G-W THM $\Longleftrightarrow$ $\uparrow$ w/ $K_1 = \{0\}$)

THM: Let J be a P.T.F on $P_{n,m}$ REPRESENTABLE AS A B.-S. MEASURE $\frac{1}{|p|^2} d\sigma$ WHERE $p \in \mathbb{C}[z,w]$

- HAS NO ZEROS IN $\mathbb{T} \times \overline{\mathbb{D}}$
- WE WRITE $p(z,0) = a(z)b(z)$

  a – NO ZEROS IN $\overline{\mathbb{D}}$

  b – ALL ZEROS IN $\mathbb{D}$

THEN, J SATISFIES THE SPLIT-SHIFT CONDITION USING...

$$K_1 = P_{\mathcal{E}'_{n-1,m}} \left\{ g(z)\,a(z) : \begin{array}{l} g \in \mathbb{C}[z] \\ \deg g < \deg b \end{array} \right\}$$

$$K_2 = P_{\mathcal{E}'_{n-1,m}} \left\{ g(z)\,b(z) : \begin{array}{l} g \in \mathbb{C}[z] \\ \deg g < n - \deg b \end{array} \right\}$$

IN THE DEF. OF SPLIT-SHIFT.

I.E.

$$\mathcal{E}'_{n-1,m} = K_1 \oplus K_2$$

$$\mathcal{E}'_{n,m} = K_1 \oplus zK_2 \oplus \mathbb{C}p$$

WHY SHOULD WE CARE ABOUT SUCH AN ABSTRACT CONDITION?

THM: Let $p \in \mathbb{C}[z,w]$, $\deg p = (n,m)$,

- p HAS NO ZEROS ON $\mathbb{T} \times \overline{\mathbb{D}}$
- $p(z,0)$ HAS n_2 ZEROS IN $\mathbb{D}$. SET $n_1 = n - n_2$
- $\overleftarrow{p}(z,w) := z^n w^m \overline{p\left(\frac{1}{\bar{z}}, \frac{1}{\bar{w}}\right)}$

THEN: $|p(z,w)|^2 - |\overleftarrow{p}(z,w)|^2$

HAS THE FOLLOWING (HERMITIAN) SUM OF SQUARES REP.

$$|p(z,w)|^2 - |\overleftarrow{p}(z,w)|^2$$

$$= (1-|z|^2)\left(\sum_1^{n_1} |A_j(z,w)|^2 - \sum_1^{n_2} |B_j(z,w)|^2\right)$$

$$+ (1-|w|^2)\sum_1^{m} |C_j(z,w)|^2$$

WHERE $\{A_j\}$ FORM O.N. BASIS FOR $\overleftarrow{K_2}$

$\{B_j\}$ FORM O.N. BASIS FOR K_1

$\{C_j\}$ FORM O.N. BASIS FOR $\Sigma^2_{n,m-1}$

COR: FOR $p \in \mathbb{C}[z,w]$ W/ NO ZEROS ON $\overline{\mathbb{D}} \times \overline{\mathbb{D}}$

AND $\overleftarrow{p}(z,w) = z^n w^m \overline{p\left(\frac{1}{\bar z}, \frac{1}{\bar w}\right)}$,

$$\frac{\overleftarrow{p}}{p} = A + B\begin{pmatrix} zI & & \\ & wI & \\ & & I \end{pmatrix}\left(\begin{pmatrix} I & & \\ & I & \\ & & zI \end{pmatrix} - D\begin{pmatrix} zI & & \\ & wI & \\ & & I \end{pmatrix}\right)^{-1} C$$

FOR SOME UNITARY $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$

"TRANSFER FUNCTION" REPRESENTATION

TWO SPECIAL CASES

- $n_2 = 0 \Rightarrow p$ NO ZEROS ON $\overline{\mathbb{D}^2}$
  $\rightarrow$ COLE-WERMER SOS FORMULA
- $m = 0 \Rightarrow p(z, w) = p(z)$
  $\rightarrow$ SCHUR-COHN METHOD FOR COUNTING ROOTS IN AND OUTSIDE $\overline{\mathbb{D}}$.

Q: IF $p \in \mathbb{C}[z,w]$ HAS NO ZEROS MERELY ON $\mathbb{T}^2$, IS THERE A FORMULA

$$|p(z,w)|^2 - |\overleftarrow{p}(z,w)|^2 = (1-|z|^2)\left(\sum |A_j|^2 - \sum |B_j|^2\right) + (1-|w|^2)\left(\sum |C_j|^2 - \sum |D_j|^2\right)$$

?

APPENDIX:

MATRIX CONDITION

WHAT IS THE MATRIX CONDITION?

- EASIER TO CHECK
- REVEALS LESS ABOUT HILBERT SPACE GEOMETRY THAN SPLIT-SHIFT CONDITION
- CLOSE TO ORIGINAL GERONIMO-ILIEV THM.

$$\mathcal{E}^1_{n-1,m} = P_{n-1,m} \ominus w P_{n-1,m-1}, \quad \mathcal{E}^2_{n,m-1} = P_{n,m-1} \ominus z P_{n-1,m-1}$$

$$\mathcal{F}^2_{n,m-1} = P_{n,m-1} \ominus P_{n-1,m-1}$$

projection

DEFINE

$$A : \mathcal{E}^1_{n-1,m} \to w\,\mathcal{E}^2_{n,m-1}, \quad A = P_{w\mathcal{E}^2_{n,m-1}} M_z$$

$$B : w\,\mathcal{F}^2_{n,m-1} \to \mathcal{E}^1_{n-1,m}, \quad B = P_{\mathcal{E}^1_{n-1,m}}$$

$$T : \mathcal{E}^1_{n-1,m} \to \mathcal{E}^1_{n-1,m}, \quad T = P_{\mathcal{E}^1_{n-1,m}} M_z$$

DEF: A P.T.F $\mathcal{J}$ on $P_{n,m}$ satisfies the "MATRIX COND" IF

- THE INVARIANT SUBSPACE OF T GENERATED BY THE RANGE OF B is IN THE KERNEL OF A.

- $AT^jB = 0$, $j = 0, \dots, n-1$

(CLOSE TO ORIGINAL GERONIMO-ILIEV CONDITION)