Rational inner functions in the Schur-Agler class

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Goal is to understand:

- Function theory on the polydisk $\mathbb{D}^n$.
- von Neumann inequalities
- Positive polynomials and sums of squares decompositions.

Rational inner functions

Rational inner functions are generalizations of finite Blaschke products.

$$\phi(z) = \frac{\prod_{j=1}^{d} z - a_j}{\prod_{j=1}^{d} 1 - \bar{a}_j z} = \frac{\tilde{p}(z)}{p(z)}$$

where $p \in \mathbb{C}[z]$ has no zeros on $\overline{\mathbb{D}}$, degree d, and

$$\tilde{p}(z) = z^d \overline{p(1/\bar{z})}$$

The same works in several variables:

$$\phi(z_1, \ldots, z_n) = \frac{\tilde{p}(z_1, \ldots, z_n)}{p(z_1, \ldots, z_n)}$$

where $p \in \mathbb{C}[z_1, \ldots, z_n]$ has no zeros on $\overline{\mathbb{D}}^n$, multidegree $d = (d_1, \ldots, d_n)$, and

$$\tilde{p}(z_1, \ldots, z_n) = z^d \overline{p(1/\bar{z}_1, \ldots, 1/\bar{z}_n)}$$

von Neumann inequalities and the Schur-Agler class

- (von Neumann) For any holomorphic $f : \mathbb{D} \to \mathbb{D}$ and any contractive operator T

$$\|f(T)\| \leq 1$$

- *Schur class*: Holomorphic $f : \mathbb{D}^n \to \mathbb{D}$
- *Schur-Agler class*: f in Schur class satisfying

$$\|f(T_1, \ldots, T_n)\| \leq 1$$

  for all commuting, contractive n-tuples $(T_1, \ldots, T_n)$.
- Schur-Agler class $\subset$ Schur class, with equality for $n = 1, 2$.

Sums of squares

Let $p \in \mathbb{C}[z_1, \dots, z_n]$ have no zeros in $\overline{\mathbb{D}}^n$, assume p has multidegree $d = (d_1, \dots, d_n)$. Define

$$\tilde{p}(z) = z^d \overline{p(1/\bar{z}_1, 1/\bar{z}_2, \dots, 1/\bar{z}_n)}$$

$$\left| \frac{\tilde{p}(z)}{p(z)} \right| = 1 \text{ on } \mathbb{T}^n, \quad \leq 1 \text{ on } \overline{\mathbb{D}}^n$$

So,

$$|p(z)|^2 - |\tilde{p}(z)|^2 \geq 0 \text{ on } \overline{\mathbb{D}}^n$$

Does the left hand side equal

$$\sum_{j=1}^{n} (1 - |z_j|^2) SOS_j?$$

Rational inner functions and the Schur-Agler class

Answer:

$$\frac{\tilde{p}}{p} \text{ is in the Schur-Agler class}$$

if and only if

$$|p(z)|^2 - |\tilde{p}(z)|^2 = \sum_{j=1}^{n}(1-|z_j|^2)SOS_j$$

Why? Possible to plug in $T = (T_1, \ldots, T_n)$

$$I - \frac{\tilde{p}}{p}(T)\frac{\tilde{p}}{p}(T)^* = \sum A_{j,k}(T)(I - T_j T_j^*)A_{j,k}(T)^*$$

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Philosophy

- Rational inner functions form a natural (dense) subclass of the Schur class. (More natural than normalized polynomials?)
- Try to understand the Schur-Agler class (and hence von Neumann inequalities) by understanding rational inner functions in the Schur-Agler class.

Questions

- How do you tell if $\tilde{p}/p$ is in the Schur-Agler class?
- If $\tilde{p}/p$ is in the Schur-Agler class, how do you write down a sums of squares decomposition?
- How many squares are required in the sums of squares?

Two variables $z = (z_1, z_2)$

Suppose $p \in \mathbb{C}[z_1, z_2]$ has no zeros on $\mathbb{D}^2$ and degree (d_1, d_2). There exist 2 *canonical* sums of squares decompositions.

$$|p(z)|^2 - |\tilde{p}(z)|^2 = (1 - |z_1|^2) SOS_1 + (1 - |z_2|^2) SOS_2$$

It is possible to...

- choose SOS_1 and SOS_2 to have d_1 and d_2 squares.
- choose SOS_1 maximal and SOS_2 minimal (or vice versa).
- express SOS_1, SOS_2 using orthonormal bases of certain subspaces of polynomials obtained using the measure $\frac{1}{|p|^2} |dz_1||dz_2|$.
- construct SOS_1, SOS_2 using the one variable matrix Fejér-Riesz decomposition.
- characterize when SOS_1 and SOS_2 are unique.

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Recent work for more than two variables

- General facts
- Multi-affine symmetric polynomials
- Three variables

General facts

Take $p \in \mathbb{C}[z_1, \dots, z_n]$, degree $d = (d_1, \dots, d_n)$.
Suppose

$$|p(z)|^2 - |\tilde{p}(z)|^2 = \sum_{j=1}^{n} (1 - |z_j|^2) SOS_j.$$

- Cannot choose SOS_j to be a sum of d_j squares.
- Example: $p(z) = 3 - z_1 - z_2 - z_3$.
- Can choose SOS_j to be sum of at most $d_j \prod_{k \neq j} (d_k + 1)$ squares.

Multi-affine symmetric case

Take $p \in \mathbb{C}[z_1, \ldots, z_n]$, no zeros on $\overline{\mathbb{D}}^n$, symmetric, degree $d = (1, \ldots, 1)$.

- Can give a concrete necessary and sufficient condition for

$$|p(z)|^2 - |\tilde{p}(z)|^2 = \sum_{j=1}^{n} (1 - |z_j|^2) SOS_j$$

  and can construct SOS_j explicitly.
- Holds for $p_r(z) := p(rz)$ for $0 < r < 1$ small enough.
- Question:is $\tilde{p}/p$ automatically in the Schur-Agler class? Have found no counterexamples!

Three variables

Take $p \in \mathbb{C}[z_1, z_2, z_3]$, no zeros on $\overline{\mathbb{D}}^3$, degree $d = (d_1, d_2, d_3)$.
When is $\tilde{p}/p$ in the Schur-Agler class? i.e.

$$|p(z)|^2 - |\tilde{p}(z)|^2 = \sum_{j=1}^{3} (1 - |z_j|^2) SOS_j?$$

- A. Kummert 1989: if $d = (1, 1, 1)$.
- GK: if $d = (d_1, 1, 1)$.
- GK: if $d = (d_1, d_2, 1)$, for large enough r, s

  $$z_1^r z_2^s \frac{\tilde{p}(z)}{p(z)}$$

  is in the Schur-Agler class.
- Closely related to positive trig polynomials and sums of squares decompositions.

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Final questions

- Is "the multiplication by a monomial" property on previous page more general?
- The orthogonal polynomials viewpoint is very useful in two variables. Not as useful yet in three or more variables.
- Can one characterize p with $\tilde{p}/p$ Schur-Agler in terms of orthogonality relations in $L^2(\frac{1}{|p|^2} d\sigma)$?
- If so, can one build "canonical" sums of squares decompositions using subspaces of polynomials in $L^2(\frac{1}{|p|^2} d\sigma)$?

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