Uchiyama's Lemma and the John-Nirenberg Inequality

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Bounded Mean Oscillation

$$\|f\|_* = \sup_{I \subset \mathbb{T}} \fint_I |f - f_I| \, d\sigma$$

$$\fint_I f$$

$$\|f\|_{*,2}^2 = \sup_{I \subset \mathbb{T}} \fint_I |f - f_I|^2 \, d\sigma$$

Bounded Mean Oscillation

$$\|f\|_{BMO_1} = \sup_{z \in \mathbb{D}} \int |f - f(z)| P_z \, d\sigma$$

$$\|f\|_{BMO_2}^2 = \sup_{z \in \mathbb{D}} \int |f - f(z)|^2 P_z \, d\sigma$$

P_z = Poisson Kernel, $f(z) = \int f P_z \, d\sigma$

John-Nirenberg

$$\exists\, C, c > 0 : \|f\|_* \leq 1 \Longrightarrow$$

$$\left|\left\{\zeta \in I : |f(\zeta) - f_I| > \lambda\right\}\right| \leq C e^{-c\lambda} |I| \quad , I \subset \mathbb{T}$$

STRONG JOHN-NIRENBERG

$\exists C, c > 0 : \|f\|_* \leq 1 \Longrightarrow$

$$⨍_I \exp(c|f - f_I|)\, d\sigma < C$$

$I \subset \mathbb{T}$

CONFORMALLY INVARIANT VERSION

$$\exists C, c > 0 : \|f\|_{BMO_2} \leq 1 \Longrightarrow$$

$$\int_{\mathbb{T}} \exp(c|f - f(z)|) P_z < C$$

$$z \in \mathbb{D}$$

STANDARD APPROACH

J-N $\Longrightarrow$ BMO NORMS

$\Longleftarrow$ EQUIVALENT

$\Longrightarrow$ BMO_2 NORM CONVENIENT

$\Longleftarrow$ FOR $BMOC(\mathrm{Re}\,H^1)^*$

OUR APPROACH

METHOD: GREEN'S THM

IN THE SPIRIT OF

- T. WOLFF / CORONA THM
- NAZAROV-TREIL / MUCKENHOUPT-WHEEDEN THM
- PETERMICHL-TREIL-WICK / RK THESIS FOR CARLESON MEASURES

Green's Thm

$f \in C^2(\overline{\mathbb{D}})$

$$\int_{\mathbb{T}} f P_z \, d\sigma - f(z) = \int_{\mathbb{D}} \Delta f \, g_z \, dA$$

$P_z = $ Poisson , $g_z = $ Green's fcn

Hardy-Stein identities

$f \in H^p(\mathbb{D}) \quad 0 < p < \infty$

$$\int_{\mathbb{T}} |f|^p P_z \, d\sigma - |f(z)|^p = p^2 \int_{\mathbb{D}} |f'|^2 |f|^{p-2} g_z \, dA$$

P_z = Poisson, g_z = Green's fcn

UCHIYAMA

$z \in \mathbb{D}$

$\phi \in C^2(\overline{\mathbb{D}})$, $f \in Hol(\overline{\mathbb{D}})$

$$\int_{\mathbb{T}} |f| e^{\phi} P_z \, d\sigma \geq \int_{\mathbb{D}} \Delta\phi \, e^{\phi} |f| g_z \, dA$$

Lemma: $F \in BMOA$, $f \in H^1$

$$\int_{\mathbb{D}} |F'|^2 |f| g_z \, dA \leq \frac{e}{4} \|F\|_{BMO_2}^2 \int_{\mathbb{T}} |f| P_z \, d\sigma$$

Proof: $\|F\|_{BMO_2} = 1$ **UCHIYAMA**

$$\phi(z) = -\int_{\mathbb{T}} |F - F(z)|^2 P_z \, d\sigma = |F(z)|^2 - |F|^2(z)$$

$$\Delta \phi = 4|F'|^2$$

Corollary: $F \in BMOA$, $h \in H^2$

$$\left| \int_{\mathbb{T}} (F - F(z)) \overline{h} \, P_z \, d\sigma \right| \le 2\sqrt{e} \, \|F\|_{BMO_2} \times \int |h| P_z \, d\sigma$$

Proof: Hardy-Stein + Uchiyama

$$\left| 4 \int_{\mathbb{D}} F' \, \overline{h'} \, g_z \, dA \right|^2 \le 4 \int_{\mathbb{D}} |F'|^2 |h| g_z \, dA \times \int_{\mathbb{D}} \frac{|h'|^2}{|h|} g_z \, dA$$

Corollary: $F \in BMOA$

$$\int_{\mathbb{T}} |F - F(z)|^2 P_z \, d\sigma \leq 2\sqrt{e} \, \|F\|_{BMO_2} \int_{\mathbb{T}} |F - F(z)| P_z \, d\sigma$$

$$\|F\|_{BMO_2} \leq 2\sqrt{e} \, \|F\|_{BMO_1}$$

COROLLARY: $F \in BMOA$, $k \geq 2$

$$\int |F - F(z)|^k P_z \, d\sigma \leq \frac{e}{4} k^2 \, \|F\|^2_{BMO_2} \times \int |F - F(z)|^{k-2} P_z \, d\sigma$$

PROOF: HARDY-STEIN

$$k^2 \int_{\mathbb{D}} |F'|^2 |F - F(z)|^{k-2} g_z \, dA$$

APPLY LEMMA WITH

$$f = (F - F(z))^{k-2}$$

Iterate!

$$\int |F - F(z)|^{2k} P_z \, d\sigma \le e^{k-1} (k!)^2 \|F\|_{BMO_2}^{2k}$$

$$\int |F - F(z)|^{2k+1} P_z \, d\sigma$$

$$\le \left(\frac{e}{4}\right)^k \left(\frac{(2k+1)!}{2^k k!}\right)^2 \|F\|_{BMO_2}^{2k+1}$$

STRONG J-N

$F \in BMOA$

$$\varepsilon < \frac{2}{\sqrt{e}\,\|F\|_{BMO_2}} \Longrightarrow$$

$$\int_{\mathbb{T}} \exp\left(\varepsilon |F - F(z)|\right) P_z \, d\sigma < \frac{3}{\left(1 - \frac{\varepsilon\sqrt{e}}{2}\|F\|_{BMO_2}\right)^{3/2}}$$

FURTHER DEVELOPMENTS

UCHIYAMA LEMMA $\rightsquigarrow$ WORKS FOR ANALYTIC FCNS

HARDY-STEIN $\rightsquigarrow$ ANALOGUE FOR $1 < p < \infty$

STILL POSSIBLE TO WORK OUT REAL/HARMONIC CASE.

REFERENCE

UCHIYAMA'S LEMMA AND THE JOHN-NIRENBERG INEQUALITY

TO APPEAR IN BULL. LOND. MATH. SOC.

ON arXiv