Homework 7

Problem 1 (#5.11)

Consider the Lagrange multiplier method with

$$H(p_1, ..., p_n, \tau, \lambda) = \sum_{i=1}^{n} \log p_i + \tau(\sum_{i=1}^{n} p_i - 1) - n\lambda^\tau \sum_{i=1}^{n} p_i u_i.$$

Taking the derivatives of H and setting them to 0, we obtain that

$$\frac{1}{p_i} + \tau - n\lambda_\tau \mu_i = 0,\ i = 1, ..., n,\ \sum_{i=1}^{n} p_i \mu_i = 0$$

The solution to these equations is $\tau = -n$ and

$$\hat{p_i} = \frac{1}{n(1 + \lambda^\tau \mu_i)},\ i = 1, ..., n.$$

Substituting $\hat{p_i}$ into $\sum_{i=1}^{n} p_i \mu_i = 0$, we conclude that λ is the solution of

$$\sum_{i=1}^{n} \frac{\mu_i}{1 + \lambda^\tau \mu_i} = 0$$

Problem 2 (# 5.15)

i

Since

$$1 - q_i = 1 - \frac{p_i}{\sum_{j=i}^{n+1} p_j} = \frac{\sum_{j=i+1}^{n+1} p_j}{\sum_{j=i}^{n+1} p_j}$$

$$\prod_{i=1}^{n} q_i^{\delta_i}(1 - q_i)^{1-\delta_i} = \prod_{i=1}^{n} p_i^{\delta_i}(\sum_{j=i+1}^{n+1} p_j)^{1-\delta_i}(\sum_{j=i} p_j)^{-1}$$

From

$$\prod_{i=1}^{n}(1 - q_i)^{n-i} = (\frac{\sum_{j=2}^{n+1} p_j}{\sum_{j=1}^{n+1} p_j})^{n-1}(\frac{\sum_{j=3}^{n+1} p_j}{\sum_{j=2}^{n+1} p_j})^{n-2}...(\frac{\sum_{j=n}^{n+1} p_j}{\sum_{j=n-1}^{n+1} p_j})$$

$$= \sum_{j=2}^{n+1} p_j \sum_{j=3}^{n+1} p_j ... \sum_{j=n}^{n+1} p_j$$

$$= \prod_{i=1}^{n} \sum_{j=i}^{n+1} p_j$$

We obtain that

$$\prod_{i=1}^{n} q_i^{\delta_i}(1-q_i)^{n-i+1-\delta_i} = \prod_{i=1}^{n} p_i^{\delta_i}(\sum_{j=i+1}^{n+1} p_j)^{1-\delta_i}$$

The result follows since $q_1, ..., q_n$ are n free variables.

ii

From part(1),

$$\log \ell(p_1, ..., p_{n+1}) = \sum_{i=1}^{n}[\delta_i \log q_i + (n-i+1-\delta_i)\log 1-q_i]$$

Then

$$\frac{\partial \log \ell}{\partial q_i} = \frac{\delta_i}{q_i} - \frac{n-i+1-\delta_i}{1-q_i} = 0, i = 1, ..., n,$$

have the solution

$$\hat{q}_i = \frac{\delta_i}{n-i+1},\ i = 1, ..., n,$$

Which maximizes $\ell(p_1, ..., p_{n+1})$ since

$$\frac{\partial^2 \log \ell}{\partial {q_i}^2} = -\frac{\delta_i}{q_i^2} - \frac{n-i+1-\delta_i}{(1-q_i)^2} \leq 0,\ and\ \frac{\partial^2 \log \ell}{\partial q_i \partial q_k} = 0;$$

for any i and $k \neq i$. Since

$$\prod_{j=1}^{i}(1-q_j) = \frac{\sum_{k=2}^{n+1} p_k}{\sum_{k=1}^{n+1} p_k} \frac{\sum_{k=3}^{n+1} p_k}{\sum_{k=2}^{n+1} p_k} ... \frac{\sum_{k=i}^{n+1} p_k}{\sum_{k=i-1}^{n+1} p_k} = \sum_{k=i}^{n+1} p_k,$$

we obtain that

$$q_i \prod_{j=1}^{i}(1-q_j) = p_i,\ i = 1, ..., n.$$

Hence, by (i), $\ell(p_1, ..., p_{n+1})$ is maximized by

$$\hat{p_i} = \hat{q}_i \prod_{j=1}^{i}(1-\hat{q}_j) = \frac{\delta_i}{n-i+1} \prod_{j=1}^{i}(1-\frac{\delta_j}{n-j+1}), i = 1, ..., n$$

and $\hat{p_{n+1}} = 1 - \sum_{i=1}^{n} \hat{p_i}$

iii

Define $x_0 = 0$ and $x_{n+1} = \infty$. Let $t \in (x_i, x_{i+1}]$,
$i = 0, 1, ..., n$. Then

$$\prod_{x_i \leq t}(1-\frac{\delta_i}{n-i+1}) = \prod_{j=1}^{i}(1-\hat{q}_j) = \frac{\hat{p_i}}{\hat{q}_i} = 1 - \sum_{j=1}^{i-1} \hat{p_j}$$

Hence,

$$\sum_{i=1}^{n+1} \hat{p_i} I_{(0,t]}(x_i) = 1 - \prod_{x_i \leq t}(1-\frac{\delta_i}{n-i+1}).$$

iv

When $\delta_i = 1$ for all i,

$$\hat{p_i} = \frac{1}{n-i+1}\prod_{j=1}^{i-1}\frac{n-j}{n-j+1}$$
$$= \frac{1}{n-i+1}\frac{n-1}{n}\frac{n-2}{n-1}...\frac{n-i+1}{n-i+2} = \frac{1}{n}$$

Problem 3

(# 5.20)

Suppose that ξ satisfies $\ell(\theta, \xi) = sup_\xi \ell(\theta, \xi)$ for any θ. Then the profile likelihood function is $\ell_P(\theta) = \ell(\theta, \hat{\xi})$. If $\hat{\theta}$ satisfies $\ell_P(\hat{\theta}) = sup_\theta \ell_P(\theta)$, then $\ell(\hat{\theta}, \hat{\xi}) = \ell_P(\hat{\theta}) \geq \ell_P(\theta) = \ell(\theta, \hat{\xi}) \geq \ell(\theta, \xi)$ for any θ and ξ. Hence, $(\hat{\theta}, \hat{\xi})$ is an MLE.

5.21)

The likelihood function is

$$\ell(\mu, \sigma^2) = (2\pi)^{-n/2}(\sigma^2)^{-n/2}exp(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(X_i - \mu)^2)$$

For fixed σ^2, $\ell(\mu, \sigma^2) \leq \ell(\bar{X}, \sigma^2)$, since $\sum_{i=1}^{n}(X_i - \mu)^2 \geq \sum_{i=1}^{n}(X_i - \bar{X})^2$. Hence the maximum does not depend on σ^2 and the profile likelihood function is

$$\ell\bar{X}, \sigma^2 = (2\pi)^{-n/2}(\sigma^2)^{-n/2}exp(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(X_i - \bar{X})^2).$$

By the result in the previous exercise, the profile MLE of σ^2 is the same as the MLE of σ^2. This can also be shown by directly verifying that $\ell(\bar{X}, \sigma^2)$ is maximized at $\hat{\sigma^2} = n^{-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$. For fixed μ, $\ell(\mu, \sigma^2)$ is maximized at $\sigma^2(\mu) = n^{-1}\sum_{i=1}^{n}(X_i - \mu)^2$. Then the profile likelihood function is

$$(2\pi)^{-n/2}e^{-n/2}[\frac{n}{\sum_{i=1}^{n}(X_i - \mu)^2}]^{n/2}.$$

Since $\sum_{i=1}^{n}(X_i - \mu)^2 \geq \sum_{i=1}^{n}(X_i - \bar{X})^2$, $\ell(\mu, \sigma^2(\mu))$ is maximized at $\bar{X}$.

Problem 4 # 5.23)

i

If $\pi(x) \equiv \pi$, then X_i and δ_i are independent. Hence, $F_1(x) = P(X_i \leq x|\delta_i = 1) = P(X_i \leq x) = F(x)$ for any x. If $F_1(x) = F(x)$ for any x, then $P(X_i \leq x, \delta_i = 1) = P(X_i \leq x)P(\delta_i = 1)$ for any x and, hence X_i and δ_i are independent. Thus, $\pi(x) \equiv \pi$.

ii

Note that

$$\hat{F}(x) = \frac{\sum_{i=1}^{n} \delta_i I_{(-\infty,x]}(X_i)}{\sum_{i=1}^{n} \delta_i}$$

Since $E[\delta_i I_{(-\infty,x]}(X_i)|\delta_i] = \delta_i F_1(x)$, we obtain that

$$E[F\hat{(}x)] = E[E[\hat{F}(x)|\delta_1, ..., \delta_n]]$$

$$= E[\frac{\sum_{i=1}^{n} E[\delta_i I_{(-\infty,x]}(X_i)|\delta_i]}{\sum_{i=1}^{n} \delta_i}]$$

$$= E[\frac{\sum_{i=1}^{n} \delta_i F_1(x)}{\sum_{i=1}^{n} \delta_i}] = F_1(x)$$

. From the law of large numbers,

$$\frac{1}{n}\sum_{i=1}^{n} \delta_i I_{(-\infty,x]}(X_i) \to_p E[\delta_1 I_{(-\infty,x]}(X_1)] = E[\delta_1 F_1(x)] = \pi F_1(x)$$

and

$$\frac{1}{n}\sum_{i=1}^{n} \delta_i \to_p E(\delta_1) = \pi$$

Hence, $\hat{F}(x) \to_p F_1(x)$

iii

When $\pi(x) = \pi$, $F(x) = F_1(x)$. Hence, $\hat{F}(x)$ is unbiased and consistent for $F(x)$. When $\pi(x)$ is not constant, $F(x) \neq F_1(x)$ for some x. Since $\hat{F}(x)$ is unbiased and consistent for $F_1(x)$, it is biased and inconsistent for $F(x)$ for x at which $F(x) \neq F_1(x)$.