The Gaussian Correlation Inequality

Luis Garcia German

Washington University in St. Louis

April 13, 2017

The Problem

A Gaussian measure μ on $\mathbb{R}^d$ with mean u and covariance matrix Σ is defined by

$$\mu(A) = (2\pi)^{n/2}|\Sigma|^{-1/2}\int_A \exp\left\{\frac{-1}{2}(x-u)^T\Sigma(x-u)\right\}dx.$$

The Problem

A Gaussian measure μ on $\mathbb{R}^d$ with mean u and covariance matrix Σ is defined by

$$\mu(A) = (2\pi)^{n/2}|\Sigma|^{-1/2}\int_A \exp\left\{\frac{-1}{2}(x-u)^T\Sigma(x-u)\right\}dx.$$

Theorem (Gaussian Correlation Inequality)

For a centered Gaussian measure μ,

$$\mu(K \cap L) \geq \mu(K)\mu(L),$$

for any symmetric convex sets K and L in $\mathbb{R}^d$.

Partial Results

1. First appears in 1959 in a paper by Olive Dunn.

Partial Results

1. First appears in 1959 in a paper by Olive Dunn.
2. 1967: Khatri and Ŝidák (independently) prove the GCI when K is a symmetric strip.

Partial Results

1. First appears in 1959 in a paper by Olive Dunn.
2. 1967: Khatri and Ŝidák (independently) prove the GCI when K is a symmetric strip.
3. 1977: L. D. Pitt provides proof for sets in $\mathbb{R}^2$.

Partial Results

1. First appears in 1959 in a paper by Olive Dunn.
2. 1967: Khatri and Ŝidák (independently) prove the GCI when K is a symmetric strip.
3. 1977: L. D. Pitt provides proof for sets in $\mathbb{R}^2$.
4. 1999: Gilles Hagré proves the GCI when K is a symmetric ellipsoid.

Partial Results

1. First appears in 1959 in a paper by Olive Dunn.
2. 1967: Khatri and Ŝidák (independently) prove the GCI when K is a symmetric strip.
3. 1977: L. D. Pitt provides proof for sets in $\mathbb{R}^2$.
4. 1999: Gilles Hagré proves the GCI when K is a symmetric ellipsoid.
5. 2014: Royen provides a proof of the GCI by extending it to multivariate gamma distributions.

Other Results

Theorem ([Wen99])

Let μ be a centered Gaussian measure on a separable Banach space E, then for any $0 < \lambda < 1$ and convex, symmetric sets K and L in E,

$$\mu(K \cap L)\mu(\lambda^2 K + (1-\lambda^2)L) \geq \mu(\lambda K)\mu((1-\lambda^2)^{1/2} L).$$

Other Results

Theorem ([Wen99])

Let μ be a centered Gaussian measure on a separable Banach space E, then for any $0 < \lambda < 1$ and convex, symmetric sets K and L in E,

$$\mu(K \cap L)\mu(\lambda^2 K + (1-\lambda^2)L) \geq \mu(\lambda K)\mu((1-\lambda^2)^{1/2}L).$$

In particular,

$$\mu(K \cap L) \geq \mu(\lambda K)\mu((1-\lambda^2)^{1/2}L).$$

Small Ball Probabilities

Small Ball Probabilities

The small ball probability (or small deviation) for the Gaussian measure studies the behavior of

$$\lim_{\epsilon \to 0} \log \mu(x : \|x\| \leq \epsilon).$$

Small Ball Probabilities

The small ball probability (or small deviation) for the Gaussian measure studies the behavior of

$$\lim_{\epsilon \to 0} \log \mu(x : \|x\| \leq \epsilon).$$

Equivalently, this is

$$\lim_{\epsilon \to 0} \log \mathbb{P}(\|X\| \leq \epsilon),$$

for a Gaussian vector X.

Small Ball Probabilities

Theorem ([KL93])

If X and Y are independent and

$$\lim_{\epsilon\to 0} \epsilon^{\gamma} \log \mathbb{P}(\|X\| \leq \epsilon) = -C_X, \qquad \lim_{\epsilon\to 0} \epsilon^{\gamma} \log \mathbb{P}(\|Y\| \leq \epsilon) = -C_Y,$$

for some $0 < \gamma < \infty$ and $0 \leq C_X, C_Y \leq \infty$,

Small Ball Probabilities

Theorem ([KL93])

If X and Y are independent and

$$\lim_{\epsilon\to 0} \epsilon^\gamma \log \mathbb{P}(\|X\| \le \epsilon) = -C_X, \qquad \lim_{\epsilon\to 0} \epsilon^\gamma \log \mathbb{P}(\|Y\| \le \epsilon) = -C_Y,$$

for some $0 < \gamma < \infty$ and $0 \le C_X, C_Y \le \infty$, then

$$\limsup_{\epsilon\to 0} \epsilon^\gamma \log \mathbb{P}(\|X+Y\| \le \epsilon) \le -\max\{C_X, C_Y\}$$

$$\liminf_{\epsilon\to 0} \epsilon^\gamma \log \mathbb{P}(\|X+Y\| \le \epsilon) \ge -\left(C_X^{1/1+\gamma} + C_Y^{1/1+\gamma}\right)^{1+\gamma}$$

Theorem ([Wen99])

For any joint Gaussian random vectors X and Y such that

$$\lim_{\epsilon\to 0} \epsilon^{\gamma} \log \mathbb{P}(\|X\| \leq \epsilon) = -C_X, \qquad \lim_{\epsilon\to 0} \epsilon^{\gamma} \log \mathbb{P}(\|Y\| \leq \epsilon) = 0,$$

for some $0 < \gamma < \infty$ and $0 < C_X < \infty$ we have

$$\lim_{\epsilon\to 0} \epsilon^{\gamma} \log \mathbb{P}(\|X + Y\| \leq \epsilon) = -C_X.$$

Theorem ([Wen99])

For any joint Gaussian random vectors X and Y such that

$$\lim_{\epsilon \to 0} \epsilon^{\gamma} \log \mathbb{P}(\|X\| \leq \epsilon) = -C_X, \qquad \lim_{\epsilon \to 0} \epsilon^{\gamma} \log \mathbb{P}(\|Y\| \leq \epsilon) = 0,$$

for some $0 < \gamma < \infty$ and $0 < C_X < \infty$ we have

$$\lim_{\epsilon \to 0} \epsilon^{\gamma} \log \mathbb{P}(\|X + Y\| \leq \epsilon) = -C_X.$$

Proof.

For a lower bound, choose any $0 < \delta, \lambda < 1$ so that

$$
\begin{aligned}
\mathbb{P}(\|X + Y\| \leq \epsilon) &\geq \mathbb{P}(\|X\| \leq (1-\delta)\epsilon,\ \|Y\| \leq \delta\epsilon) \\
&\geq \mathbb{P}(\|X\| \leq \lambda(1-\delta)\epsilon)\mathbb{P}(\|Y\| \leq (1-\lambda^2)^{1/2}\delta\epsilon)
\end{aligned} \tag{1}
$$

Proof.

For a lower bound, choose any $0 < \delta, \lambda < 1$ so that

$$
\begin{aligned}
\mathbb{P}(\|X + Y\| \leq \epsilon) &\geq \mathbb{P}(\|X\| \leq (1-\delta)\epsilon,\ \|Y\| \leq \delta\epsilon) \\
&\geq \mathbb{P}(\|X\| \leq \lambda(1-\delta)\epsilon)\mathbb{P}(\|Y\| \leq (1-\lambda^2)^{1/2}\delta\epsilon)
\end{aligned} \tag{1}
$$

$$
\begin{aligned}
\mathbb{P}(\|X\| \leq \epsilon/(1-\delta)\lambda) &\geq \mathbb{P}(\|X+Y\| \leq \epsilon/\lambda,\ \|Y\| \leq \delta\epsilon/(1-\delta)\lambda) \\
&\geq \mathbb{P}(\|X+Y\| \leq \epsilon)\mathbb{P}(\|Y\| \leq \frac{\epsilon\delta(1-\lambda^2)^{1/2}}{(1-\delta)\lambda})
\end{aligned} \tag{2}
$$

Proof.

For a lower bound, choose any $0 < \delta, \lambda < 1$ so that

$$\begin{aligned}\mathbb{P}(\|X+Y\| \le \epsilon) &\ge \mathbb{P}(\|X\| \le (1-\delta)\epsilon,\ \|Y\| \le \delta\epsilon) \\ &\ge \mathbb{P}(\|X\| \le \lambda(1-\delta)\epsilon)\mathbb{P}(\|Y\| \le (1-\lambda^2)^{1/2}\delta\epsilon)\end{aligned} \tag{1}$$

$$\begin{aligned}\mathbb{P}(\|X\| \le \epsilon/(1-\delta)\lambda) &\ge \mathbb{P}(\|X+Y\| \le \epsilon/\lambda,\ \|Y\| \le \delta\epsilon/(1-\delta)\lambda) \\ &\ge \mathbb{P}(\|X+Y\| \le \epsilon)\mathbb{P}(\|Y\| \le \frac{\epsilon\delta(1-\lambda^2)^{1/2}}{(1-\delta)\lambda})\end{aligned} \tag{2}$$

Inequalities (1) and (2) then (resp.) give us the bounds

$$\liminf_{\epsilon\to 0} \epsilon^{\gamma} \log \mathbb{P}(\|X+Y\| \le \epsilon) \ge -(\lambda(1-\delta))^{-\gamma} C_X.$$

$$\limsup_{\epsilon\to 0} \epsilon^{\gamma} \log \mathbb{P}(\|X+Y\| \le \epsilon) \le -(\lambda(1-\delta))^{-\gamma} C_X.$$

The result follows by letting $\delta \to 0$, $\lambda \to 1$. □

Theorem ([LM15])

For a centered Gaussian measure μ *and any symmetric convex sets* K *and* L *in* $\mathbb{R}^d$*,*

$$\mu(K \cap L) \geq \mu(K)\mu(L).$$

Theorem ([LM15])

For a centered Gaussian measure μ and any symmetric convex sets K and L in $\mathbb{R}^d$,

$$\mu(K \cap L) \geq \mu(K)\mu(L).$$

As any symmetric closed set is a countable intersection of symmetric strips, it is enough to show the GCI in the case when K and L are symmetric strips,

Theorem ([LM15])

For a centered Gaussian measure μ and any symmetric convex sets K and L in $\mathbb{R}^d$,

$$\mu(K \cap L) \geq \mu(K)\mu(L).$$

As any symmetric closed set is a countable intersection of symmetric strips, it is enough to show the GCI in the case when K and L are symmetric strips,i.e. when for some vectors v_i and $t_i \geq 0$,

$$K = \{x \in \mathbb{R}^d \ : \ |\langle x, v_i \rangle| \leq t_i, \ \forall 1 \leq i \leq n_1\}$$
$$L = \{x \in \mathbb{R}^d \ : \ |\langle x, v_i \rangle| \leq t_i, \ \forall n_1 + 1 \leq i \leq n_1 + n_2\}$$

Theorem ([LM15])

For a centered Gaussian measure μ and any symmetric convex sets K and L in $\mathbb{R}^d$,

$$\mu(K \cap L) \geq \mu(K)\mu(L).$$

As any symmetric closed set is a countable intersection of symmetric strips, it is enough to show the GCI in the case when K and L are symmetric strips,i.e. when for some vectors v_i and $t_i \geq 0$,

$$K = \{x \in \mathbb{R}^d \ : \ |\langle x, v_i \rangle| \leq t_i, \ \forall 1 \leq i \leq n_1\}$$
$$L = \{x \in \mathbb{R}^d \ : \ |\langle x, v_i \rangle| \leq t_i, \ \forall n_1 + 1 \leq i \leq n_1 + n_2\}$$

Set $n := n_1 + n_2$ and $X_i := \langle G, v_i \rangle$, for $i = 1, \ldots, n$, where G is the Gaussian r.v. distributed according to μ.

Theorem ([LM15])

For a centered Gaussian measure μ and any symmetric convex sets K and L in $\mathbb{R}^d$,

$$\mu(K \cap L) \geq \mu(K)\mu(L).$$

As any symmetric closed set is a countable intersection of symmetric strips, it is enough to show the GCI in the case when K and L are symmetric strips,i.e. when for some vectors v_i and $t_i \geq 0$,

$$K = \{x \in \mathbb{R}^d \ : \ |\langle x, v_i \rangle| \leq t_i, \ \forall 1 \leq i \leq n_1\}$$
$$L = \{x \in \mathbb{R}^d \ : \ |\langle x, v_i \rangle| \leq t_i, \ \forall n_1 + 1 \leq i \leq n_1 + n_2\}$$

Set $n := n_1 + n_2$ and $X_i := \langle G, v_i \rangle$, for $i = 1, \ldots, n$, where G is the Gaussian r.v. distributed according to μ. Note that each X_i is Gaussian.

Theorem ([Roy14])

Let $n = n_1 + n_2$ and X be an n-dimensional centered Gaussian vector. Then for any $t_1, \dots, t_n > 0$,

$$\mathbb{P}(|X_1| \leq t_1, \cdots, |X_n| \leq t_n) \\ \geq \mathbb{P}(|X_1| \leq t_1, \cdots, |X_{n_1}| \leq t_{n_1})\mathbb{P}(|X_{n_1+1}| \leq t_{n_1+1}, \cdots, |X_n| \leq t_n)$$

Theorem ([Roy14])

Let $n = n_1 + n_2$ and X be an n-dimensional centered Gaussian vector. Then for any $t_1, \ldots, t_n > 0$,

$$\mathbb{P}(|X_1| \le t_1, \cdots, |X_n| \le t_n) \ge \mathbb{P}(|X_1| \le t_1, \cdots, |X_{n_1}| \le t_{n_1})\mathbb{P}(|X_{n_1+1}| \le t_{n_1+1}, \cdots, |X_n| \le t_n)$$

Let Σ denote the covariance matrix of X, and define $\Sigma(\tau)$ as

$$\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix},$$

Theorem ([Roy14])

Let $n = n_1 + n_2$ and X be an n-dimensional centered Gaussian vector. Then for any $t_1, \dots, t_n > 0$,

$$\mathbb{P}(|X_1| \leq t_1, \cdots, |X_n| \leq t_n) \\ \geq \mathbb{P}(|X_1| \leq t_1, \cdots, |X_{n_1}| \leq t_{n_1}) \mathbb{P}(|X_{n_1+1}| \leq t_{n_1+1}, \cdots, |X_n| \leq t_n)$$

Let Σ denote the covariance matrix of X, and define $\Sigma(\tau)$ as

$$\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}, \quad \text{and} \quad \Sigma(\tau) = \begin{pmatrix} \Sigma_{11} & \tau\Sigma_{12} \\ \tau\Sigma_{21} & \Sigma_{22} \end{pmatrix}, \forall 0 \leq \tau \leq 1$$

where Σ_{ij} is the smaller $n_i \times n_j$ covariate matrix.

Theorem ([Roy14])

Let $n = n_1 + n_2$ and X be an n-dimensional centered Gaussian vector. Then for any $t_1, \ldots, t_n > 0$,

$$\mathbb{P}(|X_1| \le t_1, \cdots, |X_n| \le t_n) \\ \ge \mathbb{P}(|X_1| \le t_1, \cdots, |X_{n_1}| \le t_{n_1})\mathbb{P}(|X_{n_1+1}| \le t_{n_1+1}, \cdots, |X_n| \le t_n)$$

Let Σ denote the covariance matrix of X, and define $\Sigma(\tau)$ as

$$\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}, \quad \text{and} \quad \Sigma(\tau) = \begin{pmatrix} \Sigma_{11} & \tau\Sigma_{12} \\ \tau\Sigma_{21} & \Sigma_{22} \end{pmatrix}, \forall 0 \le \tau \le 1$$

where Σ_{ij} is the smaller $n_i \times n_j$ covariate matrix. Set $Z_i(\tau) := \frac{1}{2}X_i(\tau)^2$, where $X(\tau) \sim N(0, \Sigma(\tau))$.

Theorem ([Roy14])

Let $n = n_1 + n_2$ and X be an n-dimensional centered Gaussian vector. Then for any $t_1, \ldots, t_n > 0$,

$$\mathbb{P}(|X_1| \le t_1, \cdots, |X_n| \le t_n) \\ \ge \mathbb{P}(|X_1| \le t_1, \cdots, |X_{n_1}| \le t_{n_1})\mathbb{P}(|X_{n_1+1}| \le t_{n_1+1}, \cdots, |X_n| \le t_n)$$

Let Σ denote the covariance matrix of X, and define $\Sigma(\tau)$ as

$$\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}, \quad \text{and} \quad \Sigma(\tau) = \begin{pmatrix} \Sigma_{11} & \tau\Sigma_{12} \\ \tau\Sigma_{21} & \Sigma_{22} \end{pmatrix}, \forall 0 \le \tau \le 1$$

where Σ_{ij} is the smaller $n_i \times n_j$ covariate matrix. Set $Z_i(\tau) := \frac{1}{2}X_i(\tau)^2$, where $X(\tau) \sim N(0, \Sigma(\tau))$. Setting $s_i := t_i^2/2$, the Theorem above is then equivalent to

$$\mathbb{P}(Z_1(1) \le s_1, \ldots, Z_n(1) \le s_n) \ge \mathbb{P}(Z_1(0) \le s_1, \ldots, Z_n(0) \le s_n).$$

Theorem ([Roy14])

Let $n = n_1 + n_2$ and X be an n-dimensional centered Gaussian vector. Then for any $t_1, \dots, t_n > 0$,

$$\mathbb{P}(|X_1| \le t_1, \cdots, |X_n| \le t_n) \ge \mathbb{P}(|X_1| \le t_1, \cdots, |X_{n_1}| \le t_{n_1})\mathbb{P}(|X_{n_1+1}| \le t_{n_1+1}, \cdots, |X_n| \le t_n)$$

Let Σ denote the covariance matrix of X, and define $\Sigma(\tau)$ as

$$\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}, \quad \text{and} \quad \Sigma(\tau) = \begin{pmatrix} \Sigma_{11} & \tau\Sigma_{12} \\ \tau\Sigma_{21} & \Sigma_{22} \end{pmatrix}, \forall 0 \le \tau \le 1$$

where Σ_{ij} is the smaller $n_i \times n_j$ covariate matrix. Set $Z_i(\tau) := \frac{1}{2}X_i(\tau)^2$, where $X(\tau) \sim N(0, \Sigma(\tau))$. Setting $s_i := t_i^2/2$, the Theorem above is then equivalent to

$$\mathbb{P}(Z_1(1) \le s_1, \dots, Z_n(1) \le s_n) \ge \mathbb{P}(Z_1(0) \le s_1, \dots, Z_n(0) \le s_n).$$

For this, it is enough to show that the function $\tau \mapsto \mathbb{P}(Z_1(\tau) \le s_1, \dots, Z_n(\tau) \le s_n)$ is nondecreasing on $[0, 1]$.

References

- James Kuelbs and Wenbo V Li, *Metric entropy and the small ball problem for gaussian measures*, Journal of Functional Analysis **116** (1993), no. 1, 133–157.
- Rafał Latała and Dariusz Matlak, *Royen's proof of the gaussian correlation inequality*, arXiv preprint arXiv:1512.08776 (2015).
- Thomas Royen, *A simple proof of the gaussian correlation conjecture extended to multivariate gamma distributions*, arXiv preprint arXiv:1408.1028 (2014).
- V Wenbo, *A gaussian correlation inequality and its applications to small ball probabilities*.