Washington University in St. Louis
Math 5062 Final Exam
and Math 5061-5062 Qualifying Exam

Part I: Final Exam for Math 5062
Parts I and II: Qualifying Exam for Math 5061-5062

Name: ____________________________

Part I

Problem	Score
1	/10
2	/10
3	/10
4	/10
5	/10
6	/10
7	/10
8	/10
9	/10
Total	/80

Part II

Problem	Score
10	/10
11	/10
12	/10
13	/10
14	/10
Total	/40

Instructions: answer eight of questions 1-9 in Part I; if you are taking the qualifying exam, answer four of questions 10-14 in Part II.

1. (10 points) Consider a sample $X_1, \ldots, X_n$ of i.i.d. uniform random variables on $[0, \theta]$. Let $X_{(n)}$ denote the largest order statistic. Consider the nonparametric bootstrap estimator of the distribution of $n(\theta - X_{(n)})$ by resampling with replacement from $X_1, \ldots, X_n$. That is, pretend you don't know the population distribution and you want to estimate it using the empirical distribution function of the data. Show that the bootstrap estimate of the distribution of $n(\theta - X_{(n)})$ is not consistent.

2. (10 points) Suppose $Y_1, \ldots, Y_n \overset{i.i.d.}{\sim} N(\theta, 1)$. Choose as the prior π the degenerate prior with mass one at $\theta = 0$. Consider whether the posterior is consistent for θ in the set $A = \mathbb{R} - \{0\}$. Discuss how this relates to Doob's theorem.

3. (10 points) Consider a random sample (i.i.d.) of size n from a uniform distribution on the interval $[0, \theta]$. Let $f_\theta(\cdot)$ be the density of the largest order statistic, $X_{(n)}$. Show that

$$\frac{d}{d\theta} \int x f_\theta(x) dx \neq \int x \frac{d}{d\theta} f_\theta(x) dx.$$

How does this relate to the Cramer-Rao regularity conditions? In particular, consider the UMVU estimator of θ. Is it asymptotically efficient?

4. (10 points) Suppose the X_i are i.i.d. and $EX_1^4 < \infty$. Use the multivariate delta method to find the asymptotic distribution of

$$\sqrt{n}\begin{pmatrix} \bar{X}_n - \mu \\ S_n^2 - \sigma^2 \end{pmatrix}$$

where $S_n^2 = \sum_{i=1}^n (X_i - \bar{X}_n)^2/(n-1)$.

5. (10 points) Use the Continuity Theorem to prove the Weak Law of Large Numbers.
Hint: Use a Taylor expansion for both the real and imaginary parts of the characteristic function of $\bar{X}_n$.

6. (10 points) Recall that the sequence X_n converges in r-th mean to the random variable X, for $r \geq 0$, if $E(|X_n|^r)$ and $E(|X|^r)$ exist, and

$$\lim_{n \to \infty} E(|X_n - X|^r) = 0.$$

Prove that convergence in r-th mean, for $r \geq 1$, implies convergence in probability. Does this also hold for $r \geq 0$? Or do we need $r \geq 1$?

7. (10 points) State and sketch a proof of the univariate central limit theorem for i.i.d. sequences of random variables.

8. (10 points) State Schwartz's Theorem as precisely as you can. Make sure to define the Kullback-Leibler divergence as well. Is the Kullback-Leibler divergence a metric? You must justify your answer.

9. (10 points) State the Cramer-Rao regularity conditions and sketch the proof of the asymptotic normality of the MLE, i.e. let $\hat{\theta}_n$ denote a consistent root of the likelihood equation, sketch a proof that

$$\sqrt{n}(\hat{\theta}_n - \theta_0) \to_d N\left\{0, \frac{1}{I(\theta_0)}\right\}.$$

10. (10 points) Let $X_1, \ldots, X_n$ be i.i.d. discrete random variables with $p = \Pr(X_1 = 1)$ and $1 - p = \Pr(X_1 = 0)$. Suppose that $H_0 : p = p_0$ and $H_1 : p = p_1$ where $0 < p_0 < p_1 < 1$. What does the Neyman-Pearson lemma say about a UMP test of size α? Solve for any constants specific to the statistical model.

11. (10 points) Consider a Bayesian inference setting in which the posterior mean $\mathbb{E}[\Theta|X=x]$ is finite for each x. Show that under the loss function

$$L(\theta, a) = \begin{cases} k_1|\theta - a| & \text{if } a \leq \theta \\ k_2|\theta - a| & \text{otherwise} \end{cases}$$

with $k_1, k_2 > 0$ constant and for p an appropriate function of k_1 and k_2, every p-th quantile of the posterior distribution is a Bayes estimator.

12. (10 points) Let X follow a Poisson distribution with parameter θ. Thus the probability mass function is given by $f(x) = \Pr(X = x) = e^{-\theta}\theta^x/x!$, $x = 0, 1, \ldots$. What is the UMVUE for θ? What is the UMVUE for $g(\theta) = e^{-a\theta}$, for a known $a \in \mathbb{R}$? Is this estimator admissible? If not, what is it dominated by?

13. (10 points) Suppose that $X_1, \ldots, X_n$ are i.i.d. Poisson random variables with mean λ and that we aim to estimate $g(\lambda) = \exp(-\lambda) = \mathbb{P}_\lambda[X = 0]$.

(a) Show that $S_1 = \mathbb{I}(X_1 = 0)$ and $S_2 = \frac{1}{n}\sum_{i=1}^n \mathbb{I}(X_i = 0)$ are unbiased estimates of $g(\lambda)$, where $\mathbb{I}(A)$ is the indicator function for the event/set A, i.e. $\mathbb{I}(A) = 1$ if A occurs and zero otherwise.

(b) Show that $T = \sum_{i=1}^n X_i$ is sufficient.

(c) Compute the Rao-Blackwellized estimators $S_i^* = \mathbb{E}[S_i|T]$ for $i \in \{1, 2\}$. How does your answer relate to completeness?

14. (10 points) Suppose a test ϕ has α-Neyman structure (with respect to a statistic $T(X)$). Prove that ϕ is also α-similar. When is it true that every α-similar test has α-Neyman structure (with respect to $T(X)$)? You must justify (or prove) your answer.