Homework 1, Math 4111, due September 5

(1) Prove that for sets A, B, C, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
(2) If $f : A \to B$ and $g : B \to C$ are functions and $g \circ f : A \to C$ is injective, prove that f is injective.
(3) Let $f : A \to A$ be a function such that $(f \circ f)(a) = a$ for all $a \in A$. Prove that f is bijective.
(4) For any $n \in \mathbb{N}$, let $\Sigma_n = \{m \in \mathbb{N} | m \leq n\}$. If $f : \Sigma_{n+1} \to \Sigma_n$ is any function, prove that f is not injective. (This is usually called the 'Pigeon Hole principle'.)
(5) If $x \geq 0$ is a real number and $n \in \mathbb{N}$, prove that $(1+x)^n \geq 1+nx$.
(6) Let $n \in \mathbb{N}$. Prove that there exists a rational number x with $x^2 = n$ if and only if there is a natural number m with $m^2 = n$.
(7) Prove that given any prime number, there is a larger prime.
(8) Let $a \in \mathbb{Z}$ with p not dividing a for a prime p. Prove that there is a natural number n such that $p | a^n - 1$. (Hint: Look at remainders of $a^n - 1$ when divided by p and use the Pigeon Hole principle.)