Homework 10, Math 4111, due 14 Nov 2013

Do not submit problems in blue, but at least attempt them.

(1) Prove that exp is differentiable at 0 and $\exp'(0) = 1$.
(2) Assuming the above, prove that exp is differentiable everywhere and $\exp'(x) = \exp(x)$ for all $x \in \mathbb{R}$. In particular $\exp \in C^\infty(\mathbb{R})$. (Hint: $\exp(x + y) = \exp x \cdot \exp y$)
(3) Prove that for $x < 0$, $\exp x < \sum_{k=0}^{2n} \frac{x^k}{k!}$ for any integer $n \geq 0$.
(4) Let $f : (-1, 1) \to \mathbb{R}$ be defined as $f(x) = \frac{1}{1-x}$. Prove that $f \in C^\infty(-1, 1)$.
(5) Letting f as above, we get the Taylor approximation for any $x \in (-1, 1)$, $f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)x^k}{k!} + \frac{f^{(n+1)}(x_1)x^{n+1}}{(n+1)!}$, with x_1 between 0 and x. Calculate a possible value of x_1 in terms of x, n.
(6) Let $f \in C^1([a, b])$ with $f(a) = f(b) = 0$. Prove that for any real number $r \in \mathbb{R}$, the equation $f'(x) = rf(x)$ has a solution in $[a, b]$.
(7) Let f be in $C^n([a, b])$ and assume that it has at least $n + 1$ zeroes in $[a, b]$. Prove that $f^{(n)}$ has at least one zero in $[a, b]$.