Homework 11, Math 4111, due 21 Nov 2013

(1) Let $f \in C^\infty([a,b])$ such that $|f^{(n)}(x)| \leq M$ for all $x \in [a,b]$ and for all $n \geq 0$. Prove that there exists polynomials P_d for $d \geq 0$ such that $\lim P_d = f$ in the sup norm.

(2) Let $f = x^n + 1$. Calculate $V_f(a,b)$, the total variation for $a < b$. (Hint: There are several possibilities).

(3) Calculate $V_f(a,b)$ for any polynomial f, if you know the roots of f'.

(4) Let $f : (a,b) \to \mathbb{R}$ be a function (where a may be $-\infty$ and/or b may be $+\infty$). We say that f is of bounded variation, if for any closed and bounded interval $[c,d] \subset (a,b)$, f is of bounded variation on $[c,d]$ and $V_f(c,d)$ for all possible choices of c,d are bounded above. As usual in this case, define $V_f(a,b)$ to be the supremum of $V_f(c,d)$ as $[c,d] \subset (a,b)$ varies. Most of the results we proved in class are true in this set up.

Prove that if f is of bounded variation in (a,b), then f is bounded. Similarly, if f,g are of bounded variation on (a,b), prove that so is $f+g$ and $V_{f+g}(a,b) \leq V_f(a,b) + V_g(a,b)$. (Imitate proofs done in class.)

(5) Let $f : [a,b] \to [c,d]$ be a non-decreasing function with $a < c < d < b$. Prove that there exists an $x \in [a,b]$ with $f(x) = x$.

(6) Let f be defined on $[0,1]$ as follows. $f(0) = 0$. If $0 < x \leq 1$, there is a unique non-negative integer n such that $\frac{1}{2^{n+1}} < x \leq \frac{1}{2^n}$. Define $f(x) = 2^{-n}$. Prove that f is Riemann integrable.