Homework 5, Math 4111, due October 3

Do not submit problems in blue, but at least do them.

(1) Prove that the only subsets of $\mathbb{R}^n$ which are both open and closed are $\mathbb{R}^n$ and the empty set. (Hint: If X is another such a set, pick $\mathbf{a} \in X, \mathbf{b} \notin X$ and consider $\sup\{t \in [0,1] | \mathbf{a} + t(\mathbf{b} - \mathbf{a}) \in X\}$. In other words, imitate what we did in class.)

(2) Let (M, d) be a metric space. If S, T are subsets of M, define $d(S,T) = \inf\{d(s,t) | s \in S, t \in T\}$, which makes sense, since this set is bounded below by zero. If $S = \{a\}$, a singleton set, we will write $d(a,T)$ instead of $d(\{a\},T)$.

   (a) Prove that if S is a closed subset and $a \in M$, then $d(a,S) = 0$ if and only if $a \in S$.

   (b) If S is compact and T is closed with $S \cap T = \emptyset$, prove that $d(S,T) > 0$.

   (c) Give an example of two closed subsets S, T of $\mathbf{R}^2$ with $S \cap T = \emptyset$, but $d(S,T) = 0$.

   (d) Prove that every closed subset of M is the intersection of countably many open sets.

(3) Prove that a collection of disjoint open sets in $\mathbb{R}^n$ is necessarily countable. Give an example of a collection of disjoint closed sets which is not countable.

(4) If X, Y are connected subsets of a metric space and $X \cap Y \neq \emptyset$, prove that $X \cup Y$ is connected.

(5) If S is a subset of $\mathbb{R}^n$ such that for every point $\mathbf{x} \in S$ has an open neighbourhood $B(\mathbf{x}, r)$ ($r > 0$ may depend on $\mathbf{x}$) which intersects S in a countable set, prove that S is countable.

(6) We say that a subset S of $\mathbb{R}^n$ is *convex*, if for any two points $\mathbf{a}, \mathbf{b} \in S$ and for any $t \in [0,1]$, $t\mathbf{a} + (1-t)\mathbf{b} \in S$.

   (a) Prove that any open ball in $\mathbb{R}^n$ is convex.

   (b) Prove that if S is convex, so is its closure. (Remember that the closure of S is its union with all its accumulation points.)

   (c) For a set S we define the *interior* of S, denoted by $\operatorname{int} S$ to be $\{\mathbf{x} \in S | B(\mathbf{x}, r) \subset S\}$ (again, $r > 0$ may depend on $\mathbf{x}$). Prove that if S is convex, so is $\operatorname{int} S$. (Hint: If U, V are open in $\mathbb{R}^n$, so is $U + V$ defined as $\{\mathbf{u} + \mathbf{v} | \mathbf{u} \in U, \mathbf{v} \in V\}$.)