Homework 5, Math 4111, due October 3

Do not submit problems in blue, but at least do them.

(1) Prove that the only subsets of $\mathbb{R}^n$ which are both open and closed are $\mathbb{R}^n$ and the empty set. (Hint: If $X$ is another such a set, pick $a \in X, b \notin X$ and consider $\sup \{t \in [0, 1] | a + t(b - a) \in X\}$. In other words, imitate what we did in class.)

(2) Let $(M, d)$ be a metric space. If $S, T$ are subsets of $M$, define $d(S, T) = \inf \{d(s, t) | s \in S, t \in T\}$, which makes sense, since this set is bounded below by zero. If $S = \{a\}$, a singleton set, we will write $d(a, T)$ instead of $d(\{a\}, T)$.

(a) Prove that if $S$ is a closed subset and $a \in M$, then $d(a, S) = 0$ if and only if $a \in S$.

(b) If $S$ is compact and $T$ is closed with $S \cap T = \emptyset$, prove that $d(S, T) > 0$.

(c) Give an example of two closed subsets $S, T$ of $\mathbb{R}^2$ with $S \cap T = \emptyset$, but $d(S, T) = 0$.

(d) Prove that every closed subset of $M$ is the intersection of countably many open sets.

(3) Prove that a collection of disjoint open sets in $\mathbb{R}^n$ is necessarily countable. Give an example of a collection of disjoint closed sets which is not countable.

(4) If $X, Y$ are connected subsets of a metric space and $X \cap Y \neq \emptyset$, prove that $X \cup Y$ is connected.

(5) If $S$ is a subset of $\mathbb{R}^n$ such that for every point $x \in S$ has an open neighbourhood $B(x, r)$ ($r > 0$ may depend on $x$) which intersects $S$ in a countable set, prove that $S$ is countable.

(6) We say that a subset $S$ of $\mathbb{R}^n$ is convex, if for any two points $a, b \in S$ and for any $t \in [0, 1]$, $ta + (1 - t)b \in S$.

(a) Prove that any open ball in $\mathbb{R}^n$ is convex.

(b) Prove that if $S$ is convex, so is its closure. (Remember that the closure of $S$ is its union with all its accumulation points.)

(c) For a set $S$ we define the interior of $S$, denoted by $\text{int} S$ to be $\{x \in S | B(x, r) \subset S\}$ (again, $r > 0$ may depend on $x$). Prove that if $S$ is convex, so is $\text{int} S$. (Hint: If $U, V$ are open in $\mathbb{R}^n$, so is $U + V$ defined as $\{u + v | u \in U, v \in V\}$.)