Homework 6, Math 4111, due October 10

Do not submit problems in blue, but at least attempt them.

(1) Prove that any continuous map from $\mathbb{R}$ to $\mathbb{Q}$ (with the usual metrics) is constant.

(2) Let $f : \mathbb{R} \to \mathbb{R}$ be a function and assume that it is continuous at a point $p \in \mathbb{R}$. Does that mean there is a small interval $(p-r, p+r)$ $(r > 0)$ where f is continuous?

(3) Prove that the function $f : \mathbb{R} - \{0\} \to \mathbb{R} - \{0\}$ give by $f(x) = x^{-1}$ is continuous. Deduce that if $g : M \to \mathbb{R}$ is a continuous function (M, as usual a metric space) with $g(x) \neq 0$ for all $x \in M$, then the function $h : M \to \mathbb{R}$ given by $h(x) = \frac{1}{g(x)}$ is continuous.

(4) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with the property $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Prove that there exists an $a \in \mathbb{R}$ such that $f(x) = ax$ for all $x \in \mathbb{R}$. (Hint: What is a? What is $f(x/n)$ for $n \in \mathbb{N}, x \in \mathbb{R}$?)

(5) Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given as $f(x,y) = \frac{xy^2}{x^2+y^4}$ if $(x,y) \neq (0,0)$ and $f(0,0) = 0$. Show that f is not continuous at $(0,0)$. Decide whether the function $g : \mathbb{R}^2 \to \mathbb{R}$ given by $g(x,y) = \frac{x^2y^2}{x^2+y^2}$ if $(x,y) \neq (0,0)$ and $g(0,0) = 0$ is continuous at $(0,0)$.

(6) Let $S \subset \mathbb{R}^3$ consist of the three unit line segments starting from the origin along the three axes. Algebraically,

$$S = \{(x,0,0) | x \in [0,1]\} \cup \{(0,y,0) | y \in [0,1]\} \cup \{(0,0,z) | z \in [0,1]\}.$$

If $f : S \to \mathbb{R}$ is any continuous function, show that there exists $a \neq b$ in S such that $f(a) = f(b)$.

(7) (a) Let $\mathcal{B}$ be the set of bounded functions from $[0,1] \to \mathbb{R}$ with the sup norm $||f|| = \sup_{x\in[0,1]} |f(x)|$. So, the metric $d(f,g) = ||f - g||$ also makes sense, since $f - g \in \mathcal{B}$ if $f, g \in \mathcal{B}$. Prove that $\mathcal{B}$ is complete with respect to this metric. (Hint: Prove that for any CS $\{f_n\} \in \mathcal{B}$ and any $x \in [0,1]$, $\{f_n(x)\}$ is a CS.)

(b) Prove that $\mathcal{C}$, the set of continuous functions on $[0,1]$ is a subset of $\mathcal{B}$. Deduce that $\mathcal{C}$ is complete.