Homework 8, Math 4111, due October 31

(1) Let $\mathcal{C}$ be the set of continuous functions on the closed interval $[0,1]$ with sup norm. Define a function $\mathrm{ev} : \mathcal{C} \to \mathbb{R}$ (usually called the evaluation map) by $\mathrm{ev}(f) = f(0)$ for any $f \in \mathcal{C}$. Prove that ev is uniformly continuous.

(2) Let (M, d) be a compact metric space and let $\{U_\alpha\}$ be an open cover of M. Show that there exists a positive number δ (called the Lebesgue number for the covering) such that for any point $a \in M$, $B(a, \delta) \subset U_\alpha$ for some α.

(3) Use the above to deduce the theorem proved in class: If $f : (M, d) \to (N, d')$ is continuous and M is compact, then f is uniformly continuous.

(4) Define functions (polynomials) $P_n : \mathbb{R} \to \mathbb{R}$ by, $P_n(x) = \sum_{k=0}^{n} \frac{(-1)^k x^{2k}}{(2k)!}$ for non-negative integers n. You may quote results from class or previous homework if necessary.
   (a) For any $a \in \mathbb{R}$, prove that $\lim P_n(a)$ exists. We call this $\cos a$, the cosine function.
   (b) Prove that on any closed bounded interval, $\{P_n\}$ form a CS with respect to the sup norm. Deduce that cos is continuous.

(5) For any $a > 0$, we can define (after what we did in class) a function $f_a : \mathbb{R} \to \mathbb{R}$ as $f_a(x) = \exp(x \log a)$ (in calculus, the familiar notation being a^x). Decide whether f_a is monotonic. (It is clearly continuous).

(6) A function $f : \mathbb{R} \to \mathbb{R}$ (or an open set, closed set etc.) is said to satisfy *Lipschitz condition* of order $\alpha \in \mathbb{R}$ at $c \in \mathbb{R}$, if there is a positive constant $M(c)$ and a $\delta > 0$ such that for all $x \in (c - \delta, c + \delta)$, $|f(x) - f(c)| < M(c)|x - c|^\alpha$. Prove that if $\alpha > 0$, then f is continuous at c and if $\alpha > 1$, then f is differentiable at c.