Sample Problems

(1) Let $A, B \subset X$. Prove that $A \subset X - B$ if and only $B \subset X - A$.

(2) Prove that $\binom{n+1}{r} = \binom{n}{r} + \binom{n}{r-1}$. (Recall that $\binom{n}{r}$ is defined for integers $n \geq 0, 0 \leq r \leq n$ as $\frac{n!}{r!(n-r)!}$, where $n! = 1$ if $n = 0$ and if $n > 0$, $n! = 1 \cdot 2 \cdots n$).

(3) For any real numbers x, y and an integer $n \geq 0$, prove the binomial theorem, $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

(4) If $m, n \in \mathbb{N}$, let $r(m,n)$ to be the remainder of m when divided by n. Prove that for any $a, m, n \in \mathbb{N}$, $r(a^m - 1, a^n - 1) = a^{r(m,n)} - 1$.

(5) Consider the set $\{\frac{(a+1)^2}{a^2+1} | a \in \mathbb{R}\}$. Decide whether this set is bounded above, below or both and if so find the supremum (resp. infimum) of this set.

(6) Let $\{x_n\}$ be a convergent sequence, $x_n \in (M,d)$ with $\lim x_n = a$. Let $\{y_n\}$ be a subsequence of $\{x_n\}$. (This means the following. The sequence $\{x_n\}$ is given by a function $f : \mathbb{N} \to M$ and $f(n) = x_n$. Let $g : \mathbb{N} \to \mathbb{N}$ be any increasing function and if we define $y_n = f \circ g(n)$, $\{y_n\}$ is called a subsequence of $\{x_n\}$). Prove that $\lim y_n = a$.

(7) Prove that if $\{x_n\}$ is a sequence of real numbers with $\lim x_n = a \neq 0$, then the sequence $\{\frac{1}{x_n} | x_n \neq 0\}$ (why is this a sequence?) converges to $\frac{1}{a}$.

(8) Let M be a set with two metrics d_1, d_2 such that $d_1(x,y) \leq Cd_2(x,y)$ for a fixed positive constant C and for all $x, y \in M$. Prove that an open set in the d_1 metric is open in the d_2 metric.

(9) Consider $\mathbb{R}^n$ with the following three metrics (you do not have to check they are metrics). Let $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ be two points of $\mathbb{R}^n$. 1) $d_1(\mathbf{x}, \mathbf{y}) = \sup_{1 \leq i \leq n} |x_i - y_i|$; 2) $d_2(\mathbf{x}, \mathbf{y}) = \sqrt{\sum (x_i - y_i)^2}$; 3) $d_3(\mathbf{x}, \mathbf{y}) = \sum |x_i - y_i|$. Prove that $d_1(\mathbf{x}, \mathbf{y}) \leq d_2(\mathbf{x}, \mathbf{y}) \leq d_3(\mathbf{x}, \mathbf{y})$ and $d_2(\mathbf{x}, \mathbf{y}) \leq \sqrt{n} d_2(\mathbf{x}, \mathbf{y}) \leq d_1(\mathbf{x}, \mathbf{y})$.

(10) Prove that the function, $f : [0,1] \to [0,1]$ given by $f(x) = \sqrt{1 - x^2}$ is continuous.