Sample Problems

(1) Let \( A, B \subset X \). Prove that \( A \subset X - B \) if and only if \( B \subset X - A \).

(2) Prove that \( \binom{n+1}{r} = \binom{n}{r} + \binom{n}{r-1} \). (Recall that \( \binom{n}{r} \) is defined for integers \( n \geq 0, 0 \leq r \leq n \) as \( \frac{n!}{r!(n-r)!} \), where \( n! = 1 \) if \( n = 0 \) and if \( n > 0, n! = 1 \cdot 2 \cdots n \)).

(3) For any real numbers \( x, y \) and an integer \( n \geq 0 \), prove the binomial theorem, \((x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k}y^k\).

(4) If \( m, n \in \mathbb{N} \), let \( r(m, n) \) to be the remainder of \( m \) when divided by \( n \). Prove that for any \( a, m, n \in \mathbb{N} \), \( r(a^m - 1, a^n - 1) = a \cdot r(m, n) - 1 \).

(5) Consider the set \( \{ (a+1)^2 | a \in \mathbb{R} \} \). Decide whether this set is bounded above, below or both and if so find the supremum (resp. infimum) of this set.

(6) Let \( \{ x_n \} \) be a convergent sequence, \( x_n \in (M, d) \) with \( \lim x_n = a \). Let \( \{ y_n \} \) be a subsequence of \( \{ x_n \} \). (This means the following. The sequence \( \{ x_n \} \) is given by a function \( f : \mathbb{N} \rightarrow M \) and \( f(n) = x_n \). Let \( g : \mathbb{N} \rightarrow \mathbb{N} \) be any increasing function and if we define \( y_n = f \circ g(n) \), \( \{ y_n \} \) is called a subsequence of \( \{ x_n \} \)).

Prove that \( \lim y_n = a \).

(7) Prove that if \( \{ x_n \} \) is a sequence of real numbers with \( \lim x_n = a \neq 0 \), then the sequence \( \{ \frac{1}{x_n} | x_n \neq 0 \} \) (why is this a sequence?) converges to \( \frac{1}{a} \).

(8) Let \( M \) be a set with two metrics \( d_1, d_2 \) such that \( d_1(x, y) \leq C d_2(x, y) \) for a fixed positive constant \( C \) and for all \( x, y \in M \). Prove that an open set in the \( d_1 \) metric is open in the \( d_2 \) metric.

(9) Consider \( \mathbb{R}^n \) with the following three metrics (you do not have to check they are metrics). Let \( \mathbf{x} = (x_1, \ldots, x_n) \) and \( \mathbf{y} = (y_1, \ldots, y_n) \) be two points of \( \mathbb{R}^n \). 1) \( d_1(\mathbf{x}, \mathbf{y}) = \sup_{1 \leq i \leq n} |x_i - y_i| \); 2) \( d_2(\mathbf{x}, \mathbf{y}) = \sqrt{\sum (x_i - y_i)^2} \); 3) \( d_3(\mathbf{x}, \mathbf{y}) = \sum |x_i - y_i| \). Prove that \( d_1(\mathbf{x}, \mathbf{y}) \leq d_2(\mathbf{x}, \mathbf{y}) \leq d_3(\mathbf{x}, \mathbf{y}) \) and \( d_2(\mathbf{x}, \mathbf{y}) \leq \sqrt{n} d_2(\mathbf{x}, \mathbf{y}) \leq d_1(\mathbf{x}, \mathbf{y}) \).

(10) Prove that the function, \( f : [0, 1] \rightarrow [0, 1] \) given by \( f(x) = \sqrt{1 - x^2} \) is continuous.